Dr Oliver Mathematics
Advance Level Mathematics
Core Mathematics 3: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.

1. Given $y = 2x(3x - 1)^5$,

   (a) find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, giving your answer as a single fully factorised expression. (4)

   (b) Hence find the set of values of x for which $\dfrac{\mathrm{d}y}{\mathrm{d}x} \leqslant 0$. (2)

2. The function f is defined by

$$\mathrm{f}(x) = \frac{6}{2x+5} + \frac{2}{2x-5} + \frac{60}{4x^2-25}, \; x > 4.$$

   (a) Show that (4)

$$\mathrm{f}(x) = \frac{A}{Bx + C},$$

   where A, B, and C are constants to be found.

   (b) Find $\mathrm{f}^{-1}(x)$ and state its domain. (3)

3. The value of a car is modelled by the formula

$$V = 16\,000\mathrm{e}^{-kt} + A, \; t \geqslant 0, \; t \in \mathbb{R},$$

   where V is the value of the car in pounds, t is the age of the car in years, and k and A are positive constants.

   Given that the value of the car is £17 500 when new and £13 500 two years later,

   (a) find the value of A, (1)

   (b) show that (4)

$$k = \ln\left(\frac{2}{\sqrt{3}}\right).$$

   (c) Find the age of the car, in years, when the value of the car is £6 000. (4)
   Give your answer to 2 decimal places.

4. Figure 1 shows a sketch of part of the curve C with equation

$$y = \mathrm{e}^{-2x} + x^2 - 3.$$

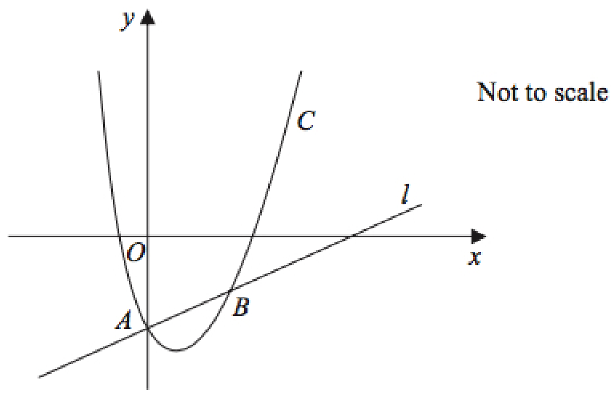

Figure 1: $y = \mathrm{e}^{-2x} + x^2 - 3$

The curve C crosses the y-axis at the point A. The line l is the normal to C at the point A.

(a) Find the equation of l, writing your answer in the form $y = mx + c$, where m and c are constants. (5)

The line l meets C again at the point B, as shown in Figure 1.

(b) Show that the x-coordinate of B is a solution of (2)

$$x = \sqrt{1 + \tfrac{1}{2}x - \mathrm{e}^{-2x}}.$$

Using the iterative formula

$$x_{n+1} = \sqrt{1 + \tfrac{1}{2}x_n - \mathrm{e}^{-2x_n}},$$

with $x_1 = 1$,

(c) find x_2 and x_3 to 3 decimal places. (2)

5. Figure 2 shows part of the graph with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 2|5 - x| + 3, \; x \geqslant 0.$$

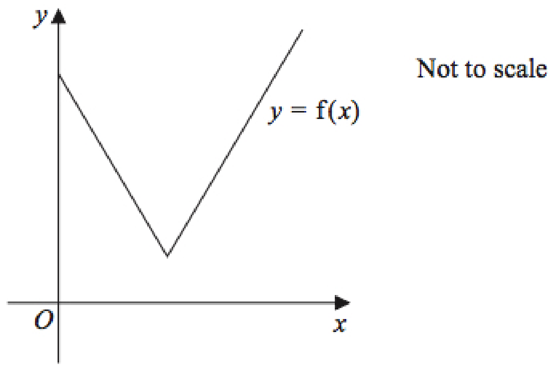

Figure 2: $\mathrm{f}(x) = 2|5 - x| + 3$

Given that the equation $\mathrm{f}(x) = k$, where k is a constant, has exactly one root,

(a) state the set of possible values of k. (2)

(b) Solve the equation (4)

$$\mathrm{f}(x) = \tfrac{1}{2}x + 10.$$

The graph with equation

$$y = \mathrm{f}(x)$$

is transformed onto the graph with equation

$$y = 4\,\mathrm{f}(x - 1).$$

The vertex on the graph with equation $y = 4\,\mathrm{f}(x - 1)$ has coordinates (p, q).

(c) State the value of p and the value of q. (2)

6. (a) Using the identity for $\tan(A \pm B)$, solve, for $-90^\circ < x < 90^\circ$, (4)

$$\frac{\tan 2x + \tan 32^\circ}{1 - \tan 2x \tan 32^\circ} = 5.$$

Give your answers, in degrees, to 2 decimal places.

(b) (i) Using the identity for $\tan(A \pm B)$, show that (2)

$$\tan(3\theta - 45^\circ) \equiv \frac{\tan 3\theta - 1}{1 + \tan 3\theta},\ \theta \neq (60n + 45)^\circ,\ n \in \mathbb{Z}.$$

(ii) Hence solve, for $0^\circ < \theta < 180^\circ$, (5)

$$(1 + \tan 3\theta)\tan(\theta + 28^\circ) = \tan 3\theta - 1.$$

7. The curve C has equation

$$y = \frac{\ln(x^2+1)}{x^2+1},\ x \in \mathbb{R}.$$

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a single fraction, simplifying your answer. (3)

(b) Hence find the exact coordinates of the stationary points of C. (6)

8. (a) By writing (2)

$$\sec\theta = \frac{1}{\cos\theta},$$

show that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\sec\theta) = \sec\theta\tan\theta.$$

(b) Given that (5)

$$x = \mathrm{e}^{\sec y},\ x > \mathrm{e},\ 0 < y < \tfrac{1}{2}\pi,$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x\sqrt{\mathrm{g}(x)}},\ x > \mathrm{e},$$

where $\mathrm{g}(x)$ is a function of $\ln x$.

9. *Solutions based entirely on graphical or numerical methods are not acceptable.*

(a) Express (3)

$$\sin\theta - 2\cos\theta$$

in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$.

Give the exact value of R and the value of α, in radians, to 3 decimal places.

$$M(\theta) = 40 + (3\sin\theta - 6\cos\theta)^2.$$

(b) Find (3)

(i) the maximum value of $M(\theta)$,

(ii) the smallest value of θ, in the range $0 < \theta \leqslant 2\pi$, at which the maximum value of $M(\theta)$ occurs.

$$N(\theta) = \frac{30}{5 + 2(\sin 2\theta - 2\cos 2\theta)^2}.$$

(c) Find (3)

(i) the maximum value of $N(\theta)$,

(ii) the largest value of θ, in the range $0 < \theta \leqslant 2\pi$, at which the maximum value of $N(\theta)$ occurs.