# Dr Oliver Mathematics
# Further Pure Mathematics
# Maclaurin and Taylor Series
# Past Examination Questions

This booklet consists of 23 questions across a variety of examination topics. The total number of marks available is 203.

1. (a) Write down the expansion of $\sin 2x$ is ascending powers of $x$, up to and including the termin $x^5$. (1)

**Solution**

$$\begin{aligned}\sin 2x &= (2x) - \tfrac{1}{3!}(2x)^3 + \tfrac{1}{5!}(2x)^5 + \dots \\ &= \underline{\underline{2x - \tfrac{4}{3}x^3 + \tfrac{4}{15}x^5 + \dots}}\end{aligned}$$

(b) Show that, for some value of $k$, (4)

$$\lim_{x \to 0} \left[ \frac{2x - \sin 2x}{x^2 \ln(1 + kx)} \right] = 16$$

and state this value of $k$.

**Solution**

$$2x - \sin 2x = 2x - \left(2x - \tfrac{4}{3}x^3 + \tfrac{4}{15}x^5 + \dots\right) = \tfrac{4}{3}x^3 - \tfrac{4}{15}x^5 + \dots$$

$$x^2 \ln(1 + kx) = x^2 \left[(kx) - \tfrac{1}{2}(kx)^2 + \dots\right] = kx^3 - \tfrac{1}{2}k^2x^5 + \dots$$

So

$$\begin{aligned}\lim_{x \to 0} \left[ \frac{2x - \sin 2x}{x^2 \ln(1 + kx)} \right] &= \lim_{x \to 0} \left[ \frac{\frac{4}{3}x^3 - \frac{4}{15}x^5 + \dots}{kx^3 - \frac{1}{2}k^2x^5 + \dots} \right] \\ &= \lim_{x \to 0} \left[ \frac{\frac{4}{3} - \frac{4}{15}x^2 + \dots}{k - \frac{1}{2}k^2x^2 + \dots} \right] \\ &= \frac{4}{3k}.\end{aligned}$$

Hence

$$\frac{4}{3k} = 16 \Rightarrow \underline{\underline{k = \tfrac{1}{12}}}.$$

2. It is given that

$$y = \sqrt{4 + \sin x}.$$

(a) Express $y\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $\cos x$. (2)

**Solution**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{2}(4 + \sin x)^{-\frac{1}{2}} \cos x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{2}y^{-1} \cos x$$
$$\Rightarrow y\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{2}\cos x$$

(b) Find the value of $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$ when $x = 0$. (5)

**Solution**

Differentiate the expression from part (a):

$$y\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{2}\cos x \Rightarrow \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\tfrac{1}{2}\sin x$$
$$\Rightarrow 2\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\tfrac{1}{2}\cos x.$$

Now, when $x = 0$, $y = 2$,

$$2\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{4},$$

$$(\tfrac{1}{4})^2 + 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\tfrac{1}{32},$$

and

$$2 \times \tfrac{1}{4} \times (-\tfrac{1}{32}) + 2\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \tfrac{1}{4} \times (-\tfrac{1}{32}) = -\tfrac{1}{2} \Rightarrow \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -\tfrac{61}{256}.$$

(c) Hence, by using Maclaurin's theorem, find the first four terms in the expansion, is ascending powers of $x$ of $\sqrt{4 + \sin x}$. (2)

**Solution**

Hence

$$\begin{aligned} y &= 2 + \tfrac{1}{4}x + \frac{-\frac{1}{32}}{2!}x^2 + \frac{-\frac{61}{256}}{3!}x^3 + \ldots \\ &= \underline{\underline{2 + \tfrac{1}{4}x - \tfrac{1}{64}x^2 - \tfrac{61}{1536}x^3 + \ldots}} \end{aligned}$$

3. (a)

$$y = \ln(\cos x + \sin x).$$

(i) Show that (4)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2}{1 + \sin 2x}.$$

**Solution**

$$\begin{aligned} & y = \ln(\cos x + \sin x) \\ \Rightarrow\quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\sin x + \cos x}{\cos x + \sin x} \\ \Rightarrow\quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-(\cos x + \sin x)^2 - (-\sin x + \cos x)^2}{(\cos x + \sin x)^2} \\ \Rightarrow\quad & \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-(\cos^2 x + 2\sin x\cos x + \sin^2 x) - (\sin^2 x - 2\sin x\cos x + \cos^2 x)}{(\cos x + \sin x)^2} \\ \Rightarrow\quad & \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-2(\sin^2 x + \cos^2 x)}{(1 + 2\sin x\cos x)} \\ \Rightarrow\quad & \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2}{1 + \sin 2x}}}, \end{aligned}$$

as required.

(ii) Find $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$. (1)

**Solution**

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2(1 + \sin 2x)^{-1} \Rightarrow \underline{\underline{\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 4\cos 2x(1 + \sin 2x)^{-2}}}.$$

(b) (i) Hence use Maclaurin's theorem to show that the first three non-zero terms in the expansion, in ascending powers of $x$, of $\ln(\cos x + \sin x)$, are (3)

$$x - x^2 + \tfrac{2}{3}x^3.$$

**Solution**
When $x = 0$,

$$\begin{aligned} y &= \ln(\cos 0 + \sin 0) = \ln 1 = 0, \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-\sin 0 + \cos 0}{\cos 0 + \sin 0} = 1, \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\frac{2}{1 + \sin 0} = -2, \\ \frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= 4\cos 0(1 + \sin 0)^{-2} = 4. \end{aligned}$$

Hence

$$\begin{aligned} y &= 0 + (1)x + \frac{2}{2!}x^2 + \frac{4}{3!}x^3 + \ldots \\ &= \underline{\underline{x - x^2 + \tfrac{2}{3}x^3 + \ldots}}, \end{aligned}$$

as required.

(ii) Write down the first three non-zero terms in the expansion, in ascending powers of $x$, of $\ln(\cos x - \sin x)$. (1)

**Solution**

$$\begin{aligned} \ln(\cos x - \sin x) &= \ln(\cos(-x) + \sin(-x)) \\ &= (-x) - (-x)^2 + \tfrac{2}{3}(-x)^3 + \ldots \\ &= \underline{\underline{-x - x^2 - \tfrac{2}{3}x^3 + \ldots}} \end{aligned}$$

(c) Hence find the first three non-zero terms in the expansion, in ascending powers of $x$, of (4)

$$\ln\left(\frac{\cos 2x}{\mathrm{e}^{3x-1}}\right).$$

**Solution**

$$\begin{aligned}
\ln\left(\frac{\cos 2x}{e^{3x-1}}\right) &= \ln\cos 2x - \ln e^{3x-1} \\
&= \ln(\cos^2 x - \sin^2 x) - (3x-1) \\
&= \ln\left[(\cos x + \sin x)(\cos x - \sin x)\right] - (3x-1) \\
&= \ln(\cos x + \sin x) + \ln(\cos x - \sin x) - (3x-1) \\
&= \left(x - x^2 + \tfrac{2}{3}x^3 + \ldots\right) + \left(-x - x^2 - \tfrac{2}{3}x^3 + \ldots\right) - (3x-1) \\
&= \underline{\underline{1 - 3x - 2x^2 + \ldots}}
\end{aligned}$$

4. (a) It is given that

$$y = \ln(e^{3x}\cos x).$$

(i) Show that (3)

$$\frac{dy}{dx} = 3 - \tan x.$$

**Solution**

$$\begin{aligned}
\frac{d}{dx}(e^{3x}\cos x) &= 3e^{3x}\cos x - e^{3x}\sin x \\
&= e^{3x}(3\cos x - \sin x)
\end{aligned}$$

and

$$\begin{aligned}
\frac{dy}{dx} &= \frac{e^{3x}(3\cos x - \sin x)}{e^{3x}\cos x} \\
&= \frac{3\cos x - \sin x}{\cos x} \\
&= \underline{\underline{3 - \tan x}},
\end{aligned}$$

as required.

(ii) Find $\dfrac{d^4y}{dx^4}$. (3)

**Solution**

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = 3 - \tan x \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\sec^2 x \\
\Rightarrow \frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= -2\sec x(\sec x \tan x) \\
\Rightarrow \frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= -2\sec^2 x \tan x \\
\Rightarrow \frac{\mathrm{d}^4y}{\mathrm{d}x^4} &= -2\sec^2 x(\sec^2 x) - 4\sec x(\sec x \tan x)\tan x \\
\Rightarrow \frac{\mathrm{d}^4y}{\mathrm{d}x^4} &= \underline{\underline{-2\sec^4 x - 4\sec^2 x \tan^2 x}}.
\end{aligned}
$$

(b) Hence use Maclaurin's theorem to show that the first three non-zero terms in the expansion, in ascending powers of $x$, of $y = \ln(\mathrm{e}^{3x} \cos x)$ are (3)

$$3x - \tfrac{1}{2}x^2 - \tfrac{1}{12}x^4.$$

**Solution**

$$
\begin{aligned}
y\big|_{x=0} &= 0, \\
\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} &= 3, \\
\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} &= -1, \\
\left.\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right|_{x=0} &= 0, \\
\left.\frac{\mathrm{d}^4y}{\mathrm{d}x^4}\right|_{x=0} &= -2,
\end{aligned}
$$

and hence

$$
\begin{aligned}
\ln(\mathrm{e}^{3x}\cos x) &= 0 + 3x + \frac{1}{2!}(-1)x^2 + 0 + \frac{1}{4!}(-2)x^4 + \dots \\
&= \underline{\underline{3x - \tfrac{1}{2}x^2 - \tfrac{1}{12}x^4 + \dots}}.
\end{aligned}
$$

(c) Write down the expansion of $\ln(1 + px)$, where $p$ is a constant, in ascending powers of $x$ up to and including the term in $x^2$. (1)

**Solution**

$$\ln(1+px) = px - \tfrac{1}{2}(px)^2 + \ldots = \underline{\underline{px - \tfrac{1}{2}p^2x^2 + \ldots}}.$$

(d) (i) Find the value of $p$ for which (2)

$$\lim_{x\to 0}\left[\frac{1}{x^2}\ln\left(\frac{e^{3x}\cos x}{1+px}\right)\right]$$

exists.

**Solution**

$$\begin{aligned} & \lim_{x\to 0}\left[\frac{1}{x^2}\ln\left(\frac{e^{3x}\cos x}{1+px}\right)\right] \\ = & \lim_{x\to 0}\left[\frac{\ln(e^{3x}\cos x) - \ln(1+px)}{x^2}\right] \\ = & \lim_{x\to 0}\left[\frac{\left(3x - \frac{1}{2}x^2 - \frac{1}{12}x^4 + \ldots\right) - \left(px - \frac{1}{2}p^2x^2 + \ldots\right)}{x^2}\right] \\ = & \lim_{x\to 0}\left[\frac{(3-p)x + \frac{1}{2}(p^2-1)x^2 + \ldots}{x^2}\right] \\ = & \lim_{x\to 0}\left[\frac{3-p}{x} + \tfrac{1}{2}(p^2-1) + \ldots\right], \end{aligned}$$

and so $\underline{\underline{p=3}}$.

(ii) Hence find the value of (2)

$$\lim_{x\to 0}\left[\frac{1}{x^2}\ln\left(\frac{e^{3x}\cos x}{1+px}\right)\right]$$

when $p$ takes the value found in part (d) (i).

**Solution**

$$\begin{aligned} \lim_{x\to 0} &= \tfrac{1}{2}(3^2-1) \\ &= \underline{\underline{4}}. \end{aligned}$$

5. Given that $y = \tan x$,

(a) find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$, and $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$. (3)

**Solution**

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \underline{\underline{\sec^2 x}},\\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 2\sec x(\sec x\tan x)\\ &= \underline{\underline{2\sec^2 x\tan x}}, \text{ and}\\ \frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= 2\sec^2 x(\sec^2 x) + 4\sec x(\sec x\tan x)\tan x\\ &= \underline{\underline{2\sec^4 x + 4\sec^2 x\tan^2 x}}.\end{aligned}$$

(b) Find the Taylor series expansion of $\tan x$ in ascending powers of $(x - \frac{\pi}{4})$ up to and including the term in $(x - \frac{\pi}{4})^3$. (3)

**Solution**

$$\begin{aligned} y\big|_{x=\frac{\pi}{4}} &= 1,\\ \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0\frac{\pi}{4}} &= 2,\\ \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=\frac{\pi}{4}} &= 4,\\ \left.\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right|_{x=\frac{\pi}{4}} &= 16,\end{aligned}$$

and hence

$$\begin{aligned}\tan x &= 1 + 2(x - \tfrac{\pi}{4}) + \frac{1}{2!}(4)(x - \tfrac{\pi}{4})^2 + \frac{1}{3!}(16)(x - \tfrac{\pi}{4})^3 + \dots\\ &= \underline{\underline{1 + 2(x - \tfrac{\pi}{4}) + 2(x - \tfrac{\pi}{4})^2 + \tfrac{8}{3}(x - \tfrac{\pi}{4})^3 + \dots}}.\end{aligned}$$

(c) Hence show that (2)

$$\tan\frac{3\pi}{10} \approx 1 + \frac{\pi}{10} + \frac{\pi^2}{200} + \frac{\pi^3}{3000}.$$

**Solution**

$$\begin{aligned}\tan\frac{3\pi}{10} &= 1 + 2(\tfrac{3\pi}{10} - \tfrac{\pi}{4}) + 2(\tfrac{3\pi}{10} - \tfrac{\pi}{4})^2 + \tfrac{8}{3}(\tfrac{3\pi}{10} - \tfrac{\pi}{4})^3 + \dots \\ &= 1 + 2(\tfrac{\pi}{20}) + 2(\tfrac{\pi}{20})^2 + \tfrac{8}{3}(\tfrac{\pi}{20})^3 + \dots \\ &= \underline{\underline{1 + \tfrac{\pi}{10} + \tfrac{\pi^2}{200} + \tfrac{\pi^3}{3000} + \dots}}\end{aligned}$$

6. (a) Prove by induction that (8)

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^x \cos x) = 2^{\frac{1}{2}n}\,\mathrm{e}^x \cos(x + \tfrac{1}{4}n\pi),\ n \geqslant 1.$$

**Solution**

$\underline{n = 1}$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x \cos x) &= \mathrm{e}^x \cos x - \mathrm{e}^x \sin x \\ &= \mathrm{e}^x(\cos x - \sin x)\end{aligned}$$

and

$$\begin{aligned}2^{\frac{1}{2}}\,\mathrm{e}^x \cos(x + \tfrac{1}{4}\pi) &= 2^{\frac{1}{2}}\,\mathrm{e}^x(\cos x \cos \tfrac{1}{4}\pi - \sin x \sin \tfrac{1}{4}\pi) \\ &= \sqrt{2}\,\mathrm{e}^x(\tfrac{1}{\sqrt{2}}\cos x - \tfrac{1}{\sqrt{2}}\sin x) \\ &= \mathrm{e}^x(\cos x - \sin x),\end{aligned}$$

and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}(\mathrm{e}^x \cos x) = 2^{\frac{1}{2}k}\,\mathrm{e}^x \cos(x + \tfrac{1}{4}k\pi)$$

Then,

$$\begin{aligned}\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(\mathrm{e}^x \cos x) &= \frac{\mathrm{d}}{\mathrm{d}x}(2^{\frac{1}{2}k}\,\mathrm{e}^x \cos(x + \tfrac{1}{4}k\pi)) \\ &= 2^{\frac{1}{2}k}(\mathrm{e}^x \cos(x + \tfrac{1}{4}k\pi) - \mathrm{e}^x \sin(x + \tfrac{1}{4}k\pi)) \\ &= 2^{\frac{1}{2}k}\mathrm{e}^x(\cos(x + \tfrac{1}{4}k\pi) - \sin(x + \tfrac{1}{4}k\pi)) \\ &= 2^{\frac{1}{2}k}\mathrm{e}^x \times \sqrt{2}\cos(x + \tfrac{1}{4}k\pi + \tfrac{1}{4}\pi) \\ &= 2^{\frac{1}{2}k+\frac{1}{2}}\mathrm{e}^x \cos(x + \tfrac{1}{4}(k+1)\pi) \\ &= 2^{\frac{1}{2}(k+1)}\mathrm{e}^x \cos(x + \tfrac{1}{4}(k+1)\pi),\end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{N}$, as required.

(b) Hence find the Maclaurin series expansion of $\mathrm{e}^x \cos x$, in ascending powers of $x$, up to and including the term in $x^4$. (3)

**Solution**

$$\begin{aligned} y\big|_{x=0} &= 1, \\ \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} &= 2^{\frac{1}{2}}\,\mathrm{e}^x \cos(x + \tfrac{1}{4}\pi)\Big|_{x=0} = 1, \\ \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} &= 2\,\mathrm{e}^x \cos(x + \tfrac{1}{2}\pi)\big|_{x=0} = 0, \\ \left.\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right|_{x=0} &= 2^{\frac{3}{2}}\,\mathrm{e}^x \cos(x + \tfrac{3}{4}\pi)\Big|_{x=0} = -2, \\ \left.\frac{\mathrm{d}^4y}{\mathrm{d}x^4}\right|_{x=0} &= 4\,\mathrm{e}^x \cos(x + \pi)\big|_{x=0} = -4, \end{aligned}$$

and hence

$$\begin{aligned} \mathrm{e}^x \cos x &= 1 + x + 0 + \frac{1}{3!}(-2)x^3 + 0 + \frac{1}{4!}(-4)x^4 + \dots \\ &= \underline{\underline{1 + x - \tfrac{1}{3}x^3 - \tfrac{1}{6}x^4 + \dots}} \end{aligned}$$

7. The variable $y$ satisfies the differential equation

$$4(1 + x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4x\frac{\mathrm{d}y}{\mathrm{d}x} = y.$$

At $x = 0$, $y = 1$, and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$.

(a) Find the value of $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at $x = 0$. (1)

**Solution**

$$4(1+0^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\times 0\times \tfrac{1}{2} = 1 \Rightarrow 4\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \tfrac{1}{4}}}.$$

(b) Find the value of $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ at $x = 0$. (4)

**Solution**

Take the derivative:

$$4(1+x^2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 8x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}y}{\mathrm{d}x}$$

and let $x = 0$:

$$\begin{aligned} & 4\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 8\times 0\times \tfrac{1}{4} + 4\times \tfrac{1}{2} + 4\times 0\times \tfrac{1}{4} = \tfrac{1}{2} \\ \Rightarrow\quad & 4\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2 = \tfrac{1}{2} \\ \Rightarrow\quad & 4\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -\tfrac{3}{2} \\ \Rightarrow\quad & \underline{\underline{\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -\tfrac{3}{8}}}. \end{aligned}$$

(c) Express $y$ as a series, in ascending powers of $x$, up to and including the term in $x^3$. (2)

**Solution**

$$\begin{aligned} y &= 1 + \tfrac{1}{2}x + \frac{1}{2!}(\tfrac{1}{4})x^2 + \frac{1}{3!}(-\tfrac{3}{8})x^3 + \ldots \\ &= \underline{\underline{1 + \tfrac{1}{2}x + \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 + \ldots}} \end{aligned}$$

(d) Find the value that the series gives for $y$ at $x = 0.1$, giving your answer to 5 decimal places. (1)

**Solution**

$$\begin{aligned} x = 0.1 &\Rightarrow y = 1 + \tfrac{1}{2}(0.1) + \tfrac{1}{8}(0.1)^2 - \tfrac{1}{16}(0.1)^3 \\ &\Rightarrow y = 1.051\,187\,5 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{y = 1.051\,19 \text{ (5 sf)}}}. \end{aligned}$$

8.

$$(1+2x)\frac{\mathrm{d}y}{\mathrm{d}x} = x + 4y^2.$$

(a) Show that (2)

$$(1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1 + 2(4y-1)\frac{\mathrm{d}y}{\mathrm{d}x}. \quad (1)$$

**Solution**

Take the derivative with respect to $x$:

$$\begin{aligned} (1+2x)\frac{\mathrm{d}y}{\mathrm{d}x} = x + 4y^2 &\Rightarrow (1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + 8y\frac{\mathrm{d}y}{\mathrm{d}x} \\ &\Rightarrow (1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1 + 8y\frac{\mathrm{d}y}{\mathrm{d}x} - 2\frac{\mathrm{d}y}{\mathrm{d}x} \\ &\Rightarrow \underline{\underline{(1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 1 + 2(4y-1)\frac{\mathrm{d}y}{\mathrm{d}x}}}. \end{aligned}$$

(b) Differentiate equation (1) with respect to $x$ to obtain an equation involving $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, $\frac{\mathrm{d}y}{\mathrm{d}x}$, $x$, $y$. (3)

**Solution**

Take the derivative with respect to $x$: e.g.,

$$\underline{\underline{(1+2x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2(4y-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 8\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}}.$$

Given that $y = \frac{1}{2}$ at $x = 0$,

(c) find a series solution for $y$, in ascending powers of $x$, up to and including the term in $x^3$. (6)

**Solution**

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = 0 + 4(\tfrac{1}{2})^2 = 1,$$
$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} = 1 + 2 \times (4 \times \tfrac{1}{2} - 1) \times 1 = 3,$$
$$\left.\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right|_{x=0} = 2 \times (4 \times \tfrac{1}{2} - 1) \times 3 + 8 \times 1^2 - 2 \times 3 = 8,$$

and hence

$$\begin{aligned} y &= \tfrac{1}{2} + x + \frac{1}{2!}(3)x^2 + \frac{1}{3!}(8)x^3 + \ldots \\ &= \underline{\underline{\tfrac{1}{2} + x + \tfrac{3}{2}x^2 + \tfrac{4}{3}x^3 + \ldots}} \end{aligned}$$

9. (a) Find the Taylor expansion of $\cos 2x$ in ascending powers of $(x - \frac{\pi}{4})$ up to and including the term in $(x - \frac{\pi}{4})^5$. (5)

**Solution**

$$y = \cos 2x \Rightarrow y|_{x=\frac{\pi}{4}} = 0,$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2\sin 2x \Rightarrow \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=\frac{\pi}{4}} = -2,$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4\cos 2x \Rightarrow \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=\frac{\pi}{4}} = 0,$$
$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 8\sin 2x \Rightarrow \left.\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right|_{x=\frac{\pi}{4}} = 8,$$
$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = 16\cos 2x \Rightarrow \left.\frac{\mathrm{d}^4y}{\mathrm{d}x^4}\right|_{x=\frac{\pi}{4}} = 0,$$
$$\frac{\mathrm{d}^5y}{\mathrm{d}x^5} = -32\sin 2x \Rightarrow \left.\frac{\mathrm{d}^5y}{\mathrm{d}x^5}\right|_{x=\frac{\pi}{4}} = -32,$$

and hence

$$\cos 2x = 0 - 2(x - \tfrac{\pi}{4}) + 0 + \frac{1}{3!}(8)(x - \tfrac{\pi}{4})^3 + 0 + \frac{1}{5!}(-32)(x - \tfrac{\pi}{4})^5 + \dots$$
$$= \underline{\underline{-2(x - \tfrac{\pi}{4}) + \tfrac{4}{3}(x - \tfrac{\pi}{4})^3 - \tfrac{4}{15}(x - \tfrac{\pi}{4})^5 + \dots}}.$$

(b) Use your answer to (a) to obtain an estimate of $\cos 2$, giving your answer to 6 decimal places. (3)

**Solution**

$$\cos 2x \approx 2(1 - \tfrac{\pi}{4}) + \tfrac{4}{3}(1 - \tfrac{\pi}{4})^3 - \tfrac{4}{15}(1 - \tfrac{\pi}{4})^5$$
$$= -0.416\,147\,367\,6 \text{ (FCD)}$$
$$= \underline{\underline{-0.416\,147 \text{ (6 dp)}}}.$$

10.

$$(1 - x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0.$$

At $x = 0$, $y = 2$, and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -1$.

(a) Find the value of $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$ at $x = 0$. (3)

**Solution**

$$x = 0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 0 + 4 = 0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4.$$

Differentiate with respect to $x$:

$$(1 - x^2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} + 2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

and set $x = 0$:

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 0 - 0 + 1 - 2 = 0 \Rightarrow \underline{\underline{\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 1}}.$$

(b) Express $y$ as a series in ascending powers of $x$, up to and including the term in $x^3$. (4)

**Solution**

$$\begin{aligned} y &= 2 - x + \frac{1}{2!}(-4)x^2 + \frac{1}{3!}(1)x^3 + \ldots \\ &= \underline{\underline{2 - x - 2x^2 + \tfrac{1}{6}x^3 + \ldots}} \end{aligned}$$

11.

$$(x^2+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2y^2 + (1-2x)\frac{\mathrm{d}y}{\mathrm{d}x}. \quad \text{(I)}$$

(a) By differentiating equation (I) with respect to $x$, show that (3)

$$(x^2+1)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = (1-4x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (4y-2)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

**Solution**

$$\begin{aligned} & (x^2+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2y^2 + (1-2x)\frac{\mathrm{d}y}{\mathrm{d}x} \\ \Rightarrow\quad & 2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (x^2+1)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 4y\frac{\mathrm{d}y}{\mathrm{d}x} + (1-2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} \\ \Rightarrow\quad & \underline{\underline{(x^2+1)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = (1-4x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (4y-2)\frac{\mathrm{d}y}{\mathrm{d}x}}}. \end{aligned}$$

Given that $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ at $x = 0$,

(b) find the series solution for $y$, in ascending powers of $x$, up to and including the term in $x^3$. (4)

**Solution**

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 2 + 1 = 3, \\ \frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= 3 + 2 = 5, \end{aligned}$$

and hence

$$\begin{aligned} y &= 1 + x + \frac{1}{2!}(3)x^2 + \frac{1}{3!}(5)x^3 + \dots \\ &= \underline{\underline{1 + x + \tfrac{3}{2}x^2 + \tfrac{5}{6}x^3 + \dots}} \end{aligned}$$

(c) Use your series to estimate the value of $y$ at $x = -0.5$, giving your answer to two decimal places. (1)

**Solution**

$$\begin{aligned} y &\approx 1 + (-0.5) + \tfrac{3}{2}(-0.5)^2 + \tfrac{5}{6}(-0.5)^3 \\ &= 0.770\,8\dot{3} \text{ (FCD)} \\ &= \underline{\underline{0.77 \text{ (2 dp)}}}. \end{aligned}$$

12. Given that $y = x^3 \ln x$,

(a) find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$, and $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$. (5)

**Solution**

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 3x^2 \ln x + x^3 \times \frac{1}{x} = \underline{\underline{3x^2 \ln x + x^2}}. \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 6x \ln x + 3x^2 \times \frac{1}{x} + 2x = \underline{\underline{6x \ln x + 5x}}. \\ \frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= 6 \ln x + 6x \times \frac{1}{x} + 5 = \underline{\underline{6 \ln x + 11}}. \end{aligned}$$

(b) Find the Taylor series expansion of $x^3 \ln x$ in ascending powers of $(x-1)$ up to and including the term in $(x-1)^3$. (3)

**Solution**

$$
\begin{aligned}
y\big|_{x=1} &= 1 \times 0 = 0, \\
\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1} &= 0 + 1 = 1, \\
\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=1} &= 0 + 5 = 5, \\
\left.\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right|_{x=1} &= 0 + 11 = 11,
\end{aligned}
$$

and hence

$$
\begin{aligned}
y &= 0 + (x-1) + \frac{1}{2!}(5)(x-1)^2 + \frac{1}{3!}(11)(x-1)^3 + \ldots \\
&= \underline{\underline{(x-1) + \tfrac{5}{2}(x-1)^2 + \tfrac{11}{6}(x-1)^3 + \ldots}}
\end{aligned}
$$

13.

$$y = \sec^2 x.$$

(a) Show that (4)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6\sec^4 x - 4\sec^2 x.$$

**Solution**

$$
\begin{aligned}
y = \sec^2 x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 2\sec x(\sec x \tan x) \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 2\sec^2 x \tan x \\
\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 4\sec x \tan x(\sec x \tan x) + 2\sec^2 x \sec^2 x \\
\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 4\sec^2 x \tan^2 x + 2\sec^4 x \\
\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 4\sec^2 x(\sec^2 x - 1) + 2\sec^4 x \\
\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 4\sec^4 x - 4\sec^2 x + 2\sec^4 x \\
\Rightarrow \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6\sec^4 x - 4\sec^2 x.}}
\end{aligned}
$$

(b) Find a Taylor series expansion of $\sec^2 x$ in ascending powers of $(x - \frac{\pi}{4})$ up to and including the term in $(x - \frac{\pi}{4})^3$. (6)

**Solution**

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = 24 \sec^3 x(\sec x \tan x) - 8 \sec x(\sec x \tan x)$$

and

$$y\big|_{x=\frac{\pi}{4}} = 2,$$
$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0\frac{\pi}{4}} = 4,$$
$$\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=\frac{\pi}{4}} = 16,$$
$$\left.\frac{\mathrm{d}^3 y}{\mathrm{d}x^3}\right|_{x=\frac{\pi}{4}} = 80,$$

and hence

$$\begin{aligned}\tan x &= 2 + 4(x - \tfrac{\pi}{4}) + \frac{1}{2!}(16)(x - \tfrac{\pi}{4})^2 + \frac{1}{3!}(80)(x - \tfrac{\pi}{4})^3 + \dots \\ &= \underline{\underline{2 + 4(x - \tfrac{\pi}{4}) + 8(x - \tfrac{\pi}{4})^2 + \tfrac{40}{3}(x - \tfrac{\pi}{4})^3 + \dots}}.\end{aligned}$$

14. The displacement $x$ metres of a particle at time $t$ seconds is given by the differential equation (5)

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x + \cos x = 0.$$

When $t = 0$, $x = 0$, and $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2}$.

Find a Taylor series solution for $x$ in ascending powers of $t$, up to and including the term in $t^3$.

**Solution**

Differentiate with respect to $x$:

$$\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} + \frac{\mathrm{d}x}{\mathrm{d}t} - \sin x \frac{\mathrm{d}x}{\mathrm{d}t} = 0 \Rightarrow \frac{\mathrm{d}^3 x}{\mathrm{d}t^3} = \frac{\mathrm{d}x}{\mathrm{d}t}(\sin x - 1).$$

$$\left.\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right|_{x=0} = -1,$$
$$\left.\frac{\mathrm{d}^3x}{\mathrm{d}t^3}\right|_{x=0} = -\tfrac{1}{2},$$

and hence

$$\begin{aligned} x &= 0 + \tfrac{1}{2}t + \frac{1}{2!}(-1)t^2 + \frac{1}{3!}(-\tfrac{1}{2})t^3 + \ldots \\ &= \underline{\underline{\tfrac{1}{2}t - \tfrac{1}{2}t^2 - \tfrac{1}{12}t^3 + \ldots}} \end{aligned}$$

15.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \mathrm{e}^x\left(2y\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + 1\right).$$

(a) Show that (3)

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \mathrm{e}^x\left[2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + ky\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + 1\right],$$

where $k$ is a constant to be found.

**Solution**

Differentiate with respect to $x$:

$$\begin{aligned} \frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= \mathrm{e}^x\left(2y\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + 1\right) + \mathrm{e}^x\left[2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\right] \\ &= \underline{\underline{\mathrm{e}^x\left[2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 4y\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + 1\right]}}; \end{aligned}$$

hence, $\underline{\underline{k = 4}}$.

Given that, at $x = 0$, $y = 1$, and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$,

(b) find a series solution for $y$ in ascending powers of $x$, up to and including the term in $x^3$. (4)

**Solution**

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} = 1(2\times 1\times 2 + 1^2 + 1) = 6,$$
$$\left.\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right|_{x=0} = 1(2\times 1\times 6 + 2\times 2^2 + 4\times 1\times 2 + 1^2 + 1) = 30,$$

and hence

$$\begin{aligned} y &= 1 + 2x + \frac{1}{2!}(6)x^2 + \frac{1}{3!}(30)x^3 + \ldots \\ &= \underline{\underline{1 + 2x + 3x^2 + 5x^3 + \ldots}} \end{aligned}$$

16.

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = 3x + y^2.$$

(a) Show that (2)

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (1-2y)\frac{\mathrm{d}y}{\mathrm{d}x} = 3.$$

**Solution**

Differentiate with respect to $x$:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 3 + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &\Rightarrow x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 3 \\ &\Rightarrow \underline{\underline{x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (1-2y)\frac{\mathrm{d}y}{\mathrm{d}x} = 3.}} \end{aligned}$$

Given that $y = 1$ at $x = 1$,

(b) find a series solution for $y$ in ascending powers of $(x-1)$, up to and including the term in $(x-1)^3$. (8)

**Solution**

Differentiate with respect to $x$:

$$\begin{aligned} & x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \\ \Rightarrow\quad & \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}(2y - x - 1) + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2. \end{aligned}$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1} = 3 + 1^2 = 4,$$
$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=1} = 3 + 4 = 7,$$
$$\left.\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right|_{x=1} = 0 + 2 \times 4^2 = 32,$$

and hence

$$\begin{aligned}\tan x &= 1 + 4(x-1) + \frac{1}{2!}(7)(x-1)^2 + \frac{1}{3!}(32)(x-1)^3 + \ldots \\ &= \underline{\underline{1 + 4(x-1) + \tfrac{7}{2}(x-1)^2 + \tfrac{16}{3}(x-1)^3 + \ldots}}.\end{aligned}$$

17. (5)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y - \sin x = 0$$

Given that $y = \frac{1}{2}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{8}$ when $x = 0$, find a series expansion for $y$ in terms of $x$, up to and including the term in $x^3$.

**Solution**

Using the differential equation,

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} + 2 - 0 = 0 \Rightarrow \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} = -2.$$

Now differentiate the differential equation:

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 4\frac{\mathrm{d}y}{\mathrm{d}x} - \cos x = 0$$

and substitute the values in:

$$\left.\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right|_{x=0} + 4 \times \tfrac{1}{8} - 1 = 0 \Rightarrow \left.\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right|_{x=0} = \tfrac{1}{2}.$$

So

$$\begin{aligned}y &= \tfrac{1}{2} + \tfrac{1}{8}x + \frac{-2}{2!}x^2 + \frac{\frac{1}{2}}{3!}x^3 + \ldots \\ &= \underline{\underline{\tfrac{1}{2} + \tfrac{1}{8}x - x^2 + \tfrac{1}{12}x^3 + \ldots}}\end{aligned}$$

18. Given that

$$y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 5y = 0,$$

(a) Find $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ in terms of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, $\frac{\mathrm{d}y}{\mathrm{d}x}$, and $x$. (4)

**Solution**

Differentiate the differential equation:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \Rightarrow \quad \frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= \underline{\underline{-\frac{1}{y}\left(3\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x}\right)}}, \end{aligned}$$

or equivalent.

Given that $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ at $x = 0$,

(b) find a series expansion for $y$ in ascending powers of $x$, up to and including the term in $x^3$. (5)

**Solution**

Using the original differential equation,

$$2\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} + 2^2 + 10 = 0 \Rightarrow \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} = -7.$$

Using the result of part (a),

$$\left.\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right|_{x=0} = -\tfrac{1}{2}\left(3 \times 2 \times (-7) + 5 \times 2\right) = 16.$$

Hence

$$\begin{aligned} y &= 2 + 2x + \frac{-7}{2!}x^2 + \frac{16}{3!}x^3 + \dots \\ &= \underline{\underline{2 + 2x - \tfrac{7}{2}x^2 + \tfrac{8}{3}x^3 + \dots}} \end{aligned}$$

19. (8)

$$y = \sqrt{8 + \mathrm{e}^x},\ x \in \mathbb{R}.$$

Find the series expansion for $y$ in ascending powers of $x$, up to and including the term in $x^2$, giving each coefficient in its simplest form.

**Solution**

$$\begin{aligned} y &= (8+\mathrm{e}^x)^{\frac{1}{2}} \Rightarrow y = 3 \text{ when } x = 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \tfrac{1}{2}\mathrm{e}^x(8+\mathrm{e}^x)^{-\frac{1}{2}} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{6} \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \tfrac{1}{2}\mathrm{e}^x(8+\mathrm{e}^x)^{-\frac{1}{2}} - \tfrac{1}{4}\mathrm{e}^{2x}(8+\mathrm{e}^x)^{-\frac{3}{2}} \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \tfrac{17}{108} \end{aligned}$$

Hence

$$\underline{\underline{y = 3 + \tfrac{1}{6}x + \tfrac{17}{216}x^2 + \ldots}}$$

20.

$$y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y = 0.$$

(a) Find an expression for $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ in terms of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, $\frac{\mathrm{d}y}{\mathrm{d}x}$, and $y$. (4)

**Solution**

Differentiate with respect to $x$:

$$\begin{aligned} &\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 4\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \\ \Rightarrow\ & y\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = -2\frac{\mathrm{d}y}{\mathrm{d}x} - 5\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \\ \Rightarrow\ & \underline{\underline{\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \frac{-2\frac{\mathrm{d}y}{\mathrm{d}x} - 5\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}}{y}}}. \end{aligned}$$

Given that $y = 2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0.5$ at $x = 0$,

(b) find a series expansion for $y$ in ascending powers of $x$, up to and including the term in $x^3$. (5)

**Solution**

$$y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y = 0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2y}{y}.$$

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} = \frac{-2\times 0.5^2 - 2\times 2}{2} = -\tfrac{9}{4},$$
$$\left.\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right|_{x=0} = \frac{-2\times 0.5 - 5\times 0.5\times(-\tfrac{9}{4})}{2} = \tfrac{37}{16},$$

and hence

$$\begin{aligned}\tan x &= 2 + \tfrac{1}{2}x + \frac{1}{2!}(-\tfrac{9}{4})x^2 + \frac{1}{3!}(\tfrac{37}{16})x^3 + \ldots\\ &= \underline{\underline{2 + \tfrac{1}{2}x - \tfrac{9}{4}x^2 + \tfrac{37}{96}x^3 + \ldots}}.\end{aligned}$$

21.

$$y = \tan^2 x,\ -\tfrac{\pi}{2} < x < \tfrac{\pi}{2}.$$

(a) Show that (4)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6\sec^4 x - 4\sec^2 x.$$

**Solution**

$$\begin{aligned}y = \tan^2 x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 2\tan x\sec^2 x\\ \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= (2\sec^2 x)\sec^2 x + 2\tan x(2\sec x\cdot\sec x\tan x)\\ \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 2\sec^4 x + 4\sec^2 x\tan^2 x\\ \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 2\sec^4 x + 4\sec^2 x(\sec^2 x - 1)\\ \Rightarrow \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6\sec^4 x - 4\sec^2 x.}}\end{aligned}$$

(b) Hence show that (3)

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 8\sec^2 x\tan x(A\sec^2 x + B),$$

where $A$ and $B$ are constants to be found.

**Solution**

$$\begin{aligned}\frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= 24\sec^3 x(\sec x\tan x) - 8\sec x(\sec x\tan x)\\ &= 24\sec^4 x\tan x - 8\sec^2 x\tan x\\ &= 8\sec^2 x\tan x(3\sec^2 x - 1);\end{aligned}$$

so $\underline{\underline{A = 3}}$ and $\underline{\underline{B = -1}}$.

(c) Find the Taylor series expansion of $\tan^2 x$, in ascending powers of $(x - \frac{\pi}{3})$, up to and including the term in $(x - \frac{\pi}{3})^3$. (4)

**Solution**

$$\begin{aligned} y\big|_{x=\frac{\pi}{3}} &= 3,\\ \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0\frac{\pi}{3}} &= 8\sqrt{3},\\ \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=\frac{\pi}{3}} &= 80,\\ \left.\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right|_{x=\frac{\pi}{3}} &= 352\sqrt{3},\end{aligned}$$

and hence

$$\begin{aligned}\tan^2 x &= 3 + 8\sqrt{3}(x - \tfrac{\pi}{3}) + \frac{1}{2!}(80)(x - \tfrac{\pi}{3})^2 + \frac{1}{3!}(352\sqrt{3})(x - \tfrac{\pi}{3})^3 + \dots\\ &= \underline{\underline{3 + 8\sqrt{3}(x - \tfrac{\pi}{3}) + 40(x - \tfrac{\pi}{3})^2 + \tfrac{176}{3}\sqrt{3}(x - \tfrac{\pi}{3})^3 + \dots}}.\end{aligned}$$

22. (a) Find the Taylor series expansion about $\frac{\pi}{4}$ of $\tan x$, in ascending powers of $(x - \frac{\pi}{4})$, up to and including the term in $(x - \frac{\pi}{4})^3$. (7)

**Solution**

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \sec^2 x \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 2\sec x(\sec x \tan x) \\
&= 2\sec^2 x \tan x \\
\frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= 2\sec^2 x(\sec^2 x) + 4\sec x(\sec x \tan x)\tan x \\
&= 2\sec^4 x + 4\sec^2 x \tan^2 x.
\end{aligned}$$

Now,

$$\begin{aligned}
y\big|_{x=\frac{\pi}{4}} &= 1, \\
\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0\frac{\pi}{4}} &= 2, \\
\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=\frac{\pi}{4}} &= 4, \\
\left.\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right|_{x=\frac{\pi}{4}} &= 16,
\end{aligned}$$

and hence

$$\begin{aligned}
\tan x &= 1 + 2(x - \tfrac{\pi}{4}) + \frac{1}{2!}(4)(x - \tfrac{\pi}{4})^2 + \frac{1}{3!}(16)(x - \tfrac{\pi}{4})^3 + \dots \\
&= \underline{\underline{1 + 2(x - \tfrac{\pi}{4}) + 2(x - \tfrac{\pi}{4})^2 + \tfrac{8}{3}(x - \tfrac{\pi}{4})^3 + \dots}}
\end{aligned}$$

(b) Deduce that an approximation for $\tan \frac{5\pi}{12}$ is (2)

$$1 + \tfrac{1}{3}\pi + \tfrac{1}{18}\pi^2 + \tfrac{1}{81}\pi^3.$$

**Solution**

$$\begin{aligned}
\tan \tfrac{5\pi}{12} &\approx 1 + 2(\tfrac{5\pi}{12} - \tfrac{\pi}{4}) + 2(\tfrac{5\pi}{12} - \tfrac{\pi}{4})^2 + \tfrac{8}{3}(\tfrac{5\pi}{12} - \tfrac{\pi}{4})^3 \\
&= 1 + 2(\tfrac{\pi}{6}) + 2(\tfrac{\pi}{6})^2 + \tfrac{8}{3}(\tfrac{\pi}{6})^3 \\
&= \underline{\underline{1 + \tfrac{1}{3}\pi + \tfrac{1}{18}\pi^2 + \tfrac{1}{81}\pi^3}}.
\end{aligned}$$

23.
$$y = \ln\left(\frac{1}{1-2x}\right), \ |x| < \tfrac{1}{2}.$$

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$, and $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$. (4)

**Solution**

$$\begin{aligned} y = \ln\left(\frac{1}{1-2x}\right) &\Rightarrow y = \ln 1 - \ln(1-2x) \\ &\Rightarrow y = -\ln(1-2x) \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{1-2x} \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 2(1-2x)^{-1}}} \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4(1-2x)^{-2}}} \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 16(1-2x)^{-3}}}. \end{aligned}$$

(b) Hence, or otherwise, find the series expansion of $\ln\left(\dfrac{1}{1-2x}\right)$ about $x = 0$, in ascending powers of $x$, up to and including the term in $x^3$. Give each coefficient in its simplest form. (3)

**Solution**

$$\begin{aligned} y\big|_{x=0} &= 0, \\ \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} &= 2, \\ \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} &= 4, \\ \left.\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right|_{x=0} &= 16, \end{aligned}$$

and hence

$$\begin{aligned} y &= 0 + 2x + \frac{1}{2!}(4)x^2 + \frac{1}{3!}(16)x^3 + \dots \\ &= \underline{\underline{2x + 2x^2 + \tfrac{8}{3}x^3 + \dots}} \end{aligned}$$

(c) Use your expansion to find an approximate value for $\ln\left(\frac{3}{2}\right)$, giving your answer to 3 decimal places. (3)

**Solution**

$$\begin{aligned}\frac{1}{1-2x} = \frac{3}{2} &\Rightarrow 1 - 2x = \tfrac{2}{3} \\ &\Rightarrow -2x = -\tfrac{1}{3} \\ &\Rightarrow x = \tfrac{1}{6}.\end{aligned}$$

Now,

$$\begin{aligned}\ln\left(\tfrac{3}{2}\right) &\approx 2(\tfrac{1}{6}) + 2(\tfrac{1}{6})^2 + \tfrac{8}{3}(\tfrac{1}{6})^3 \\ &= 0.401\,234\,567\,9 \text{ (FCD)} \\ &= \underline{\underline{0.401 \text{ (3 dp)}}}.\end{aligned}$$