Dr Oliver Mathematics
Pigeonhole Principle

In this note, we will examine the pigeonhole principle. It is due to Johann Peter Gustav Lejeune Dirichlet: it is also known as Dirichlet's Box (or Drawer) Principle, or, as Dirichlet named it, *Schubfachprinzip* (drawer principle or shelf principle).

Theroem 1
If $(n+1)$ objects are put into n boxes, then at least one box contains two or more objects.

Solution
Suppose it is not true, that is, the ith box contains at most 1 object, for $i = 1, 2, \ldots, n$. Then the total number of objects contained in the n boxes can be at most

$$1 + 1 + \ldots + 1 = n,$$

which is one less than the number of objects distributed – contradiction. ■

Theroem 2
Let q_1, q_2, $\ldots$, q_n be positive integers. If

$$q_1 + q_2 + \ldots + q_n - n + 1$$

objects are put into n boxes, then either the 1st box contains at least q_1 objects, or the 2nd box contains at least q_2 objects, $\ldots$, the nth box contains at least q_n objects.

Solution
Suppose it is not true, that is, the ith box contains at most $q_i - 1$ objects, for $i = 1, 2, \ldots, n$. Then the total number of objects contained in the n boxes can be at most

$$(q_1 - 1) + (q_2 - 1) + \ldots + (q_n - 1) = q_1 + q_2 + \ldots + q_n - n,$$

which is one less than the number of objects distributed – contradiction. ■

Example 1
If you pick five cards from a standard deck of 52 cards, then at least two will be of the same suit.

Solution
Each of the five cards can belong to one of four suits. By the pigeonhole principle, two or more must belong to the same suit. ■

Example 2
A bag contains 7 red marbles, 7 white marbles, and 7 blue marbles. What is the minimum numbers of marbles you have to choose randomly from the bag to ensure that we get 5 marbles of same colour?

Solution
We need 4 red marbles, 4 white marbles, 4 blue marbles, and one other marble; so

$$(3 \times 4) + 1 = 13 \text{ marbles. } \blacksquare$$

Example 3
Consider a chess board with two of the diagonally opposite corners removed. Is it possible to cover the board with pieces of domino whose size is exactly two board squares?

Solution
No, it is not possible. Two diagonally opposite squares on a chess board are of the same colour. Therefore, when these are removed, the number of squares of one colour exceeds by 2 the number of squares of another colour. However, every piece of domino covers exactly two squares and these are of different colours. ■

Here are some examples for you to try.

1. If you pick five numbers from the integers 1 to 8, then two of them must add up to nine.

   Solution

   Every number can be paired with another to sum to nine. In all, there are four such pairs: the numbers 1 and 8, 2 and 7, 3 and 6, and lastly 4 and 5. Each of the five numbers belongs to one of those four pairs. By the pigeonhole principle, two of the numbers must be from the same pair.

2. A basket of fruit is being arranged out of apples, bananas, and oranges. What is the smallest number of pieces of fruit that should be put in the basket in order to guarantee that either there are at least 7 apples or at least 8 bananas or at least 5 oranges?

   Solution

   $$7 + 8 + 5 - 3 + 1 = \underline{\underline{18}}.$$

3. A box contains three pairs of socks coloured red, blue, and green, respectively. If the socks are chosen without looking, how many socks must be drawn to guarantee at least one matching pair?

Solution

It is possible for any selection of three or fewer socks to consist of only distinct socks. Given a selection of three socks, it is less likely though to have one red, one blue, and one green sock. However, if four socks are chosen, the pigeonhole principle ensures that two socks must be the same.

4. Show that given a set of n positive integers, there exists a non-empty subset whose sum is divisible by n.

Solution

Let the n integers be denoted by a_1, a_2, $\ldots$, a_n. Form the n sums

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ &\vdots \\ S_n &= a_1 + a_2 + \ldots + a_n. \end{aligned}$$

If one of these sums is divisible by n, then we are done.
Otherwise, by the pigeonhole principle, at least two of the sums must have the same remainder when divided by n (since only $(n-1)$ distinct remainders are possible). Pick two such sums S_i and S_j, with $i < j$. Then it follows that

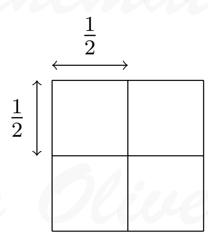
$$S_j - S_i = a_{i+1} + \ldots + a_n$$

must be divisible by n.

5. Consider any five points P_1, P_2, $\ldots$, P_5 in the interior of a square S of side length 1. Show that one can find two of the points at distance at most $\frac{\sqrt{2}}{2}$ apart.

Solution

Consider partitioning the square into 4 sub-squares as shown:



By the pigeonhole principle, 2 (or more) of the points must be in one sub-square. The distance between those two points must be less or equal to $\frac{\sqrt{2}}{2}$:

$$\begin{aligned} \text{distance} &= \sqrt{(\tfrac{1}{2})^2 + (\tfrac{1}{2})^2} \\ &= \sqrt{\tfrac{1}{4} + \tfrac{1}{4}} \\ &= \sqrt{\tfrac{1}{2}} \\ &= \tfrac{\sqrt{2}}{2}. \end{aligned}$$