Dr Oliver Mathematics
Induction

In this note, we explore induction.

There are three steps to doing induction:

(a) Show that a propositional form $P(x)$ is true for some basis case.

(b) Assume that $P(n)$ is true for some n, and show that this implies that $P(n+1)$ is true.

(c) Then, by the principle of induction, the propositional form $P(x)$ is true for all n greater or equal to the basis case.

In the first example, we start with a famous case.

Example 1
Prove that

$$1 + 2 + \ldots + n = \tfrac{1}{2}n(n+1).$$

Solution
Base case: Let us see what the case looks like taking the LHS and RHS separately.

$$\begin{aligned} 1 &= 1, \\ \tfrac{1}{2} \times 1 \times 2 &= 1, \end{aligned}$$

and we agree. So $n = 1$ is true.

Induction: Suppose the solution is true for $n = k$, i.e.,

$$1 + 2 + \ldots + k = \tfrac{1}{2}k(k+1).$$

Then

$$\begin{aligned} 1 + 2 + \ldots + k + (k+1) &= \tfrac{1}{2}k(k+1) + (k+1) \text{ (by the inductive hypothesis)} \\ &= \tfrac{1}{2}(k+1)[k+2], \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{N}$, as required. ■.

Example 2
Prove that

$$1^2 + 2^2 + \ldots + n^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

Solution

Base case: Let us see what the case looks like taking the LHS and RHS separately.

$$1^2 = 1,$$
$$\tfrac{1}{6} \times 1 \times 2 \times 3 = 1,$$

and we agree. So $n = 1$ is true.

Induction: Suppose the solution is true for $n = k$, i.e.,

$$1^2 + 2^2 + \ldots + k^2 = \tfrac{1}{6}k(k+1)(2k+1).$$

Then

$$\begin{aligned} 1^2 + 2^2 + \ldots + k^2 + (k+1)^2 &= \tfrac{1}{6}k(k+1)(2k+1) + (k+1)^2 \text{ (by the inductive hypothesis)} \\ &= \tfrac{1}{6}(k+1)[k(2k+1) + 6(k+1)] \\ &= \tfrac{1}{3}(k+1)(2k^2 + 7k + 6) \\ &= \tfrac{1}{3}(k+1)(k+2)(2k+3) \\ &= \tfrac{1}{3}(k+1)(k+2)[2(k+1)+1], \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{N}$, as required. ■.

Here are two examples for you to try: if you like that, go the the "A Level Further Mathematics" page and click on "Induction Questions" link.

1. Prove that
$$1^3 + 2^3 + \ldots + n^3 = \tfrac{1}{4}n^2(n+1)^2.$$

Solution

Base case: Let us see what the case looks like taking the LHS and RHS separately.

$$1^3 = 1,$$
$$\tfrac{1}{4} \times 1^2 \times 2^2 = 1,$$

and we agree. So $n = 1$ is true.

Induction: Suppose the solution is true for $n = k$, i.e.,

$$1^3 + 2^3 + \ldots + k^3 = \tfrac{1}{4}k^2(k+1)^2.$$

Then

$$\begin{aligned} & 1^3 + 2^3 + \ldots + k^3 + (k+1)^3 \\ = \; & \tfrac{1}{4}k^2(k+1)^2 + (k+1)^3 \text{ (by the inductive hypothesis)} \\ = \; & \tfrac{1}{4}(k+1)^2[k^2 + 4(k+1)] \\ = \; & \tfrac{1}{3}(k+1)^2(k^2 + 4k + 4) \\ = \; & \tfrac{1}{3}(k+1)^2(k+2)^2, \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{N}$, as required.

2. Prove by induction that $11^n - 6$ is divisible by 5 for every positive integer n.

Solution

Base case: if $n = 1$, we have

$$11^1 - 6 = 11 - 6 = 5,$$

so $n = 1$ is true.

Induction: Suppose the solution is true for $n = k$, i.e.,

$$11^k - 6$$

is divisible by 5. Then

$$11^k - 6 = 5m \Rightarrow 11^k = 5m + 6$$

for some positive integer m. Then

$$\begin{aligned} 11^{k+1} - 6 &= (11 \times 11^k) - 6 \\ &= 11(5m + 6) - 6 \text{ (by the inductive hypothesis)} \\ &= 55m + 60 \\ &= 5(11m + 12), \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{N}$, as required.