Dr Oliver Mathematics

$A = \frac{1}{6}|a|(\beta - \alpha)^3$

In this note, we will investigate a neat trick using only a parabola, a line which cuts it, and the resulting area:

$$\boxed{A = \tfrac{1}{6}|a|(\beta - \alpha)^3.}$$

1 Introduction

We will do an example.

Example 1
We will take the Core Mathematics C2 2005 January as an example.

The line with equation $y = 3x + 20$ cuts the curve with equation $y = x^2 + 6x + 10$ and the points A and B, as shown in Figure 1.

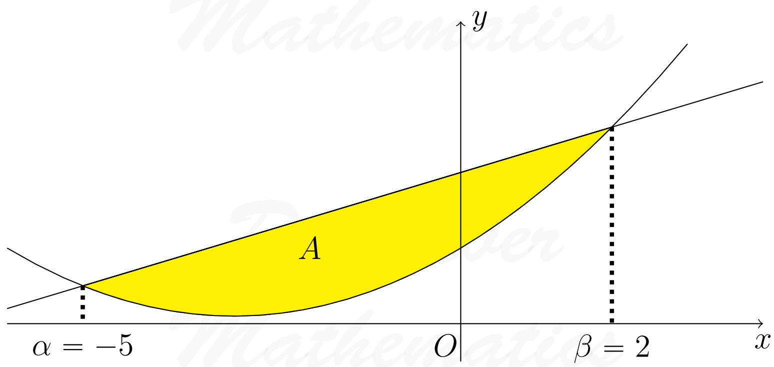

Figure 1: $y = 3x + 20$ and $y = x^2 + 6x + 10$

Use calculus to find exact area of A.

Solution 1a
The area may be found by

$$\text{area} = \text{area of the trapezium} - \text{area of the curve}.$$

Well,

$$\begin{aligned} x = -5 &\Rightarrow y = 3(-5) + 20 = 5, \\ x = 2 &\Rightarrow y = 3(2) + 20 = 26, \end{aligned}$$

so, $A(-5, 5)$ and $B(2, 26)$.

Now,

$$\begin{aligned}\text{area of the trapezium} &= \tfrac{1}{2} \times [2 - (-5)] \times (5 + 26) \\ &= \tfrac{1}{2} \times 7 \times 31 \\ &= 108\tfrac{1}{2}\end{aligned}$$

and

$$\begin{aligned}\text{area of the curve} &= \int_{-5}^{2} (x^2 + 6x + 10)\,\mathrm{d}x \\ &= \left[\tfrac{1}{3}x^3 + 3x^2 + 10x\right]_{x=-5}^{2} \\ &= (\tfrac{8}{3} + 12 + 20) - (-\tfrac{125}{3} + 75 - 50) \\ &= 51\tfrac{1}{3}.\end{aligned}$$

Thus

$$\begin{aligned}\text{area} &= 108\tfrac{1}{2} - 51\tfrac{1}{3} \\ &= \underline{\underline{57\tfrac{1}{6}\ \text{units}^2}}.\end{aligned}$$

So far, so what?

Solution 1b
There is another method.

Compare: $y = ax^2 + bx + c$ with $y = x^2 + 6x + 10$:

$$A = \tfrac{1}{6}|a|(\beta - \alpha)^3.$$

Well, $a = 1$, $\alpha = -5$, and $\beta = 2$:

$$\begin{aligned}A &= \tfrac{1}{6}|1|[2 - (-5)]^3 \\ &= \tfrac{1}{6}(7^3) \\ &= \tfrac{1}{6}(343) \\ &= \underline{\underline{57\tfrac{1}{6}}}.\end{aligned}$$

2 The Theory

Consider the parabola

$$y = ax^2 + bx + c$$

which crosses the x-axis at $x = \alpha$ and $x = \beta$, where $\alpha < \beta$, and which has a shaded area of A:

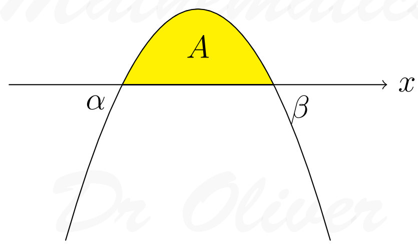

Figure 2: $y = ax^2 + bx + c$

Now,

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a(x - \alpha)(x - \beta) \end{aligned}$$

$\times$	x	$-\alpha$
x	x^2	$-\alpha x$
$-\beta$	$-\beta x$	$+\alpha\beta$

$$= a\left[x^2 - (\alpha + \beta)x + \alpha\beta\right].$$

Now,

$$\begin{aligned} A &= \int_{\alpha}^{\beta} (ax^2 + bx + c)\,\mathrm{d}x \\ &= \int_{\alpha}^{\beta} a\left[x^2 - (\alpha + \beta)x + \alpha\beta\right]\,\mathrm{d}x \\ &= a\left[\tfrac{1}{3}x^3 - \tfrac{1}{2}(\alpha + \beta)x^2 + \alpha\beta x\right]_{x=\alpha}^{\beta} \\ &= a\left\{\left(\tfrac{1}{3}\beta^3 - \tfrac{1}{2}(\alpha + \beta)\beta^2 + \alpha\beta^2\right) - \left(\tfrac{1}{3}\alpha^3 - \tfrac{1}{2}(\alpha + \beta)\alpha^2 + \alpha^2\beta\right)\right\} \\ &= a\left\{\tfrac{1}{3}\beta^3 - \tfrac{1}{2}(\alpha + \beta)\beta^2 + \alpha\beta^2 - \tfrac{1}{3}\alpha^3 + \tfrac{1}{2}(\alpha + \beta)\alpha^2 - \alpha^2\beta\right\} \end{aligned}$$

rearrange the terms:

$$\begin{aligned} &= a\left\{\tfrac{1}{3}\beta^3 - \tfrac{1}{3}\alpha^3 - \tfrac{1}{2}(\alpha + \beta)\beta^2 + \tfrac{1}{2}(\alpha + \beta)\alpha^2 + \alpha\beta^2 - \alpha^2\beta\right\} \\ &= a\left\{\tfrac{1}{3}(\beta^3 - \alpha^3) - \tfrac{1}{2}(\alpha + \beta)(\beta^2 - \alpha^2) + \alpha\beta(\beta - \alpha)\right\} \end{aligned}$$

recall $x^3 - y^3 \equiv (x - y)(x^2 + xy + y^2)$:

$$= a\left\{\tfrac{1}{3}(\beta - \alpha)(\beta^2 + \alpha\beta + \alpha^2) - \tfrac{1}{2}(\alpha + \beta)(\beta - \alpha)(\beta + \alpha) + \alpha\beta(\beta - \alpha)\right\}$$

extract $\frac{1}{6}(\beta - \alpha)$ as a common factor:

$$\begin{aligned}
&= \tfrac{1}{6}a(\beta-\alpha)\left\{2(\beta^2+\alpha\beta+\alpha^2) - 3(\alpha+\beta)(\beta+\alpha) + 6\alpha\beta\right\} \\
&= \tfrac{1}{6}a(\beta-\alpha)\left\{2\beta^2+2\alpha\beta+2\alpha^2 - 3\alpha^2 - 6\alpha\beta - 3\beta^2 + 6\alpha\beta\right\} \\
&= \tfrac{1}{6}a(\beta-\alpha)\left\{2\alpha\beta - \alpha^2 - \beta^2\right\} \\
&= -\tfrac{1}{6}a(\beta-\alpha)\left\{\beta^2 - 2\alpha\beta + \alpha^2\right\} \\
&= -\tfrac{1}{6}a(\beta-\alpha)(\beta-\alpha)^2 \\
&= -\tfrac{1}{6}a(\beta-\alpha)^3
\end{aligned}$$

Since the shape of the quadratic function is concave down, the leading coefficient a is *negative*.

Although the area A appears to be negative, the total value is positive since the coefficient a is negative here.

Even if the graph of a quadratic function is concave up, the area is the same. However, since the value of a here is positive, the area becomes negative:

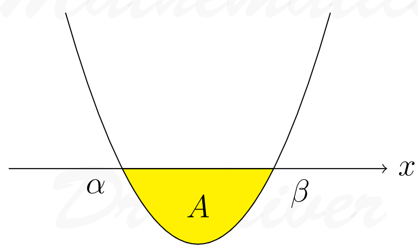

Figure 3: $y = ax^2 + bx + c$

So, since the area is *positive* in any case, we can take the absolute value of the total value.

Since $(\beta - \alpha)$ is always positive here, you can think of the absolute value as being only in a.

So,

$$A = \tfrac{1}{6}|a|(\beta - \alpha)^3,$$

as required.

What about

$$y = ax^2 + bx + c$$

intersecting with

$$y = mx + n?$$

Well, at the points of intersection,

$$
\begin{aligned}
ax^2 + bx + c = mx + n &\Rightarrow ax^2 + bx - mx + c - n = 0 \\
&\Rightarrow ax^2 + (b - m)x - (c - n) = 0 \\
&\Rightarrow a(x - \alpha)(x - \beta) = 0
\end{aligned}
$$

Since the x-coordinates of the parabola and the straight line intersect at two different point, α and β, we can see that it is the same as the first two explanations.

Hence,

$$\boxed{A = \frac{1}{6}|a|(\beta - \alpha)^3,}$$

as required.