Dr Oliver Mathematics
Mathematics: Advanced Higher
2011 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. Express (5)

$$\frac{13 - x}{x^2 + 4x - 5}$$

in partial fractions and hence obtain

$$\int \left(\frac{13 - x}{x^2 + 4x - 5} \right) \mathrm{d}x.$$

2. Use the binomial theorem to expand (3)

$$\left(\tfrac{1}{2}x - 3\right)^4$$

and simplify your answer.

3. (a) Obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ when y is defined as a function of x by the equation (3)

$$y + \mathrm{e}^y = x^2.$$

(b) Given (3)

$$\mathrm{f}(x) = \sin x \cos^3 x,$$

obtain $\mathrm{f}'(x)$.

4. (a) For what value of λ is (3)

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ -1 & \lambda & 6 \end{pmatrix}$$

singular?

(b) For (3)

$$\mathbf{A} = \begin{pmatrix} 2 & 2\alpha - \beta & -1 \\ 3\alpha + 2\beta & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix},$$

obtain values of α and β such that

$$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 2 & -5 & -1 \\ -1 & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}.$$

5. (a) Obtain the first four terms in the Maclaurin series of (4)

$$\sqrt{1+x},$$

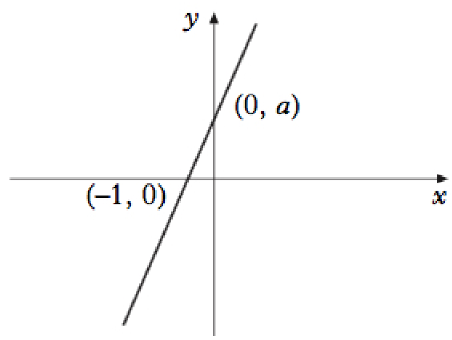
and hence write down the first four terms in the Maclaurin series of

$$\sqrt{1+x^2}.$$

(b) Hence obtain the first four terms in the Maclaurin series of (2)

$$\sqrt{(1+x)(1+x^2)}.$$

6. The diagram shows part of the graph of a function $\mathrm{f}(x)$. (4)



Sketch the graph of $|\,\mathrm{f}^{-1}(x)|$, showing the points of intersection with the axes.

7. A curve is defined by the equation (4)

$$y = \frac{\mathrm{e}^{\sin x}(2+x)^3}{\sqrt{1-x}} \text{ for } x < 1.$$

Calculate the gradient of the curve when $x = 0$.

8. (a) Write down an expression for (1)

$$\sum_{r=1}^{n} r^3 - \left(\sum_{r=1}^{n} r\right)^2.$$

(b) Write down an expression for (3)

$$\sum_{r=1}^{n} r^3 + \left(\sum_{r=1}^{n} r\right)^2.$$

9. Given that $y > -1$ and $x > -1$, obtain the general solution of the differential equation (5)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3(1+y)\sqrt{1+x},$$

expressing your answer in the form $y = \mathrm{f}(x)$.

10. Identify the locus in the complex plane given by (5)

$$|z-1| = 3.$$

Show in a diagram the region given by

$$|z-1| = 3.$$

11. (a) Obtain the exact value of (3)

$$\int_0^{\frac{1}{4}\pi} (\sec x - x)(\sec x + x)\,\mathrm{d}x.$$

(b) Find (4)

$$\int \frac{x}{\sqrt{1-49x^4}}\,\mathrm{d}x.$$

12. Prove by induction that (5)

$$8^n + 3^{n-2}$$

is divisible by 5 **for all integers** $n \geqslant 2$.

13. (a) The first three terms of an arithmetic sequence are (5)

$$a, \frac{1}{a}, 1,$$

where $a < 0$.
Obtain the value of a and the common difference.

(b) Obtain the smallest value of n for which the sum of the first n terms is greater than 1 000. (4)

14. (a) Find the general solution of the differential equation (7)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = \mathrm{e}^x + 12.$$

(b) Find the particular solution for which $y = -\frac{3}{2}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$ when $x = 0$. (3)

15. The lines L_1 and L_2 are given by the equations

$$\frac{x-1}{k} = \frac{y}{-1} = \frac{z+3}{1} \text{ and } \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2},$$

respectively.

Find

(a) the value of k for which L_1 and L_2 intersect and the point of intersection, (6)

(b) the acute angle between L_1 and L_2. (4)

16. Define

$$I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,\mathrm{d}x$$

for $n \geqslant 1$.

(a) Use integration by parts to show that (3)

$$I_n = \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,\mathrm{d}x.$$

(b) Find the values of A and B for which (5)

$$\frac{A}{(1+x^2)^n} + \frac{B}{(1+x^2)^{n+1}} \equiv \frac{x^2}{(1+x^2)^{n+1}},$$

and hence show that

$$I_{n+1} = \frac{1}{n \cdot 2^{n+1}} + \left(\frac{2n-1}{2n}\right) I_n.$$

(c) Hence obtain the exact value of (3)

$$\int_0^1 \frac{1}{(1+x^2)^3}\,\mathrm{d}x.$$