Dr Oliver Mathematics
Further Mathematics
Further Vectors
Past Examination Questions

This booklet consists of 26 questions across a variety of examination topics.
The total number of marks available is 250.

1. The points A, B, and C lie on the plane Π and, relative to a fixed origin O, they have position vectors

$$\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k},\ \mathbf{b} = -\mathbf{i} + 2\mathbf{j},\ \text{and } \mathbf{c} = 5\mathbf{i} - 3\mathbf{j} + 7\mathbf{k},$$

respectively.

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. (4)

(b) Find an equation of Π in the form $\mathbf{r}.\mathbf{n} = p$. (2)

The point D has position vector $5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

(c) Calculate the volume of the tetrahedron $ABCD$. (4)

2. (a) Explain why, for any two vectors $\mathbf{a}$ and $\mathbf{a}$, $\mathbf{a}.\mathbf{b} \times \mathbf{a} = 0$. (2)

(b) Given vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ such that $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, where $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{c}$, show that (2)

$$\mathbf{b} - \mathbf{c} = \lambda \mathbf{a},$$

where λ is a scalar.

3. The line l_1 has equation

$$\mathbf{r} = \mathbf{i} + 6\mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{k})$$

and the line l_2 has equation

$$\mathbf{r} = 3\mathbf{i} + p\mathbf{j} + \mu(\mathbf{i} - 2\mathbf{j} + \mathbf{k}),$$

where p is a constant. The plane Π_1 contains l_1 and l_2.

(a) Find a vector which is normal to Π_1. (2)

(b) Show that an equation for Π_1 is $6x + y - 4z = 16$. (2)

(c) Find the value of p. (1)

The plane Π_2 has equation $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 2$.

(d) Find an equation for the line of intersection of Π_1 and Π_2, giving your answer in the form (5)

$$(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}.$$

4. The plane Π passes through the points

$$P(-1, 3, -2),\ Q(4, -1, -1),\ \text{and}\ R(3, 0, c),$$

where c is a constant.

(a) Find, in terms of c, $\overrightarrow{RP} \times \overrightarrow{RQ}$. (3)

Given that $\overrightarrow{RP} \times \overrightarrow{RQ} = 3\mathbf{i} + d\mathbf{j} + \mathbf{k}$, where d is a constant,

(b) find the value of c and show that $d = 4$, (2)

(c) find an equation of Π in the form $\mathbf{r}.\mathbf{n} = p$, where p is a constant. (3)

The point S has position vector $\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$. The point S' is the image of S under reflection in Π.

(d) Find the position vector of S'. (5)

5. The points A, B, and C lie on the plane Π_1 and, relative to a fixed origin O, they have position vectors

$$\mathbf{a} = \mathbf{i} + 3\mathbf{j} - \mathbf{k},\ \mathbf{b} = 3\mathbf{i} + 3\mathbf{j} - 4\mathbf{k},\ \text{and}\ \mathbf{c} = 5\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

(a) Find $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$. (4)

(b) Find an equation for Π_1, giving your answer in the form $\mathbf{r}.\mathbf{n} = p$. (2)

The plane Π_2 has cartesian equation $x + z = 3$ and Π_1 and Π_2 intersect in the line l.

(c) Find an equation for l, giving your answer in the form $(\mathbf{r} - \mathbf{p}) \times \mathbf{q} = \mathbf{0}$. (4)

The point P is the point on l that is the nearest to the origin O.

(d) Find the coordinates of P. (4)

6. The points A, B, and C have position vectors, relative to a fixed origin O,

$$\begin{aligned}\mathbf{a} &= 2\mathbf{i} - \mathbf{j},\\ \mathbf{b} &= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k},\ \text{and}\\ \mathbf{c} &= 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k},\end{aligned}$$

respectively. The plane Π passes through A, B, and C.

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. (4)

(b) Show that a cartesian equation of Π is $3x - y + 2z = 7$. (2)

The line l has equation

$$(\mathbf{r} - 5\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = \mathbf{0}.$$

The line l and the plane Π intersect at the point T.

(c) Find the coordinates of T. (5)

(d) Show that A, B, and T lie on the same straight line. (3)

7. Figure 1 shows a pyramid $PQRST$ with base $PQRS$.

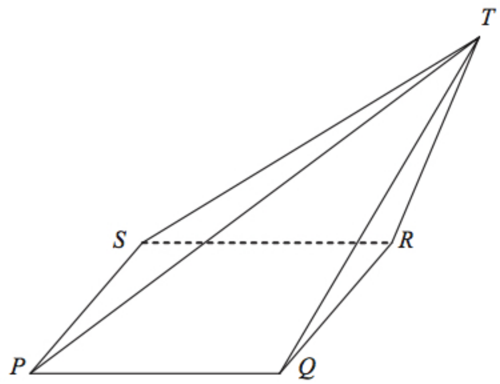

Figure 1: a pyramid $PQRST$

The coordinates of P, Q, and R are $P(1, 0, -1)$, $Q(2, -1, 1)$, and $R(3, -3, 2)$. Find

(a) Find $\overrightarrow{PQ} \times \overrightarrow{PR}$. (3)

(b) a vector equation for the plane containing the face $PQRS$, giving your answer in the form $\mathbf{r}.\mathbf{n} = d$. (2)

The plane Π contains the face PST. The vector equation of Π is $\mathbf{r}.(\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}) = 6$.

(c) Find cartesian equations of the line through P and S. (5)

(d) Hence show that PS is parallel to QR. (2)

Given that $PQRS$ is a parallelogram and that T has coordinates $(5, 2, -1)$,

(e) find the volume of the pyramid $PQRST$. (3)

8. The points A, B, and C have position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ respectively, relative to a fixed origin O, as shown in Figure 2.

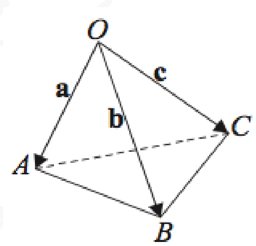

Figure 2: the points A, B, and C

It is given that

$$\mathbf{a} = \mathbf{i} + \mathbf{j},\ \mathbf{b} = 3\mathbf{i} - \mathbf{j} + \mathbf{k},\ \text{and } \mathbf{c} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}.$$

Calculate

(a) $\mathbf{b} \times \mathbf{c}$, (3)

(b) $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$, (2)

(c) the area of triangle OBC, (2)

(d) the volume of the tetrahedron $OABC$. (1)

9. The lines l_1 and l_2 have equations

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} \alpha \\ -4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}.$$

If the lines l_1 and l_2 interest,

(a) the value of α, (4)

(b) an equation for the plane containing the lines l_1 and l_2, giving your answer in the form $ax + by + cz + d = 0$, where a, b, c, and d are constants. (4)

For other values of α, the lines l_1 and l_2 do not intersect and are skew lines. Given that $\alpha = 2$,

(c) find the shortest distance between the lines l_1 and l_2. (3)

10. Given that

$$\mathbf{a} = \mathbf{i} + 7\mathbf{j} + 9\mathbf{k} \text{ and } \mathbf{b} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k},$$

(a) show that $\mathbf{a} \times \mathbf{b} = c(2\mathbf{i} + \mathbf{j} - k\mathbf{k})$, and state the value of the constant c. (2)

The plane Π_1 passes through the point $(3, 1, 3)$ and the vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to Π_1.

(b) Find a cartesian equation for the plane Π_1. (2)

The line l_1 has equation $\mathbf{r} = \mathbf{i} - 2\mathbf{k} + \lambda\mathbf{a}$.

(c) Show that the line l_1 lies in the plane Π_1. (2)

The line l_2 has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \mu\mathbf{b}$. The line l_2 lies in a plane Π_2, which is parallel to the plane Π_1.

(d) Find a cartesian equation of the plane Π_2. (2)

(e) Find the distance between the planes Π_1 and Π_2. (3)

11. The plane Π has vector equation

$$\mathbf{r} = 3\mathbf{i} + \mathbf{k} + \lambda(-4\mathbf{i} + \mathbf{j}) + \mu(6\mathbf{i} - 2\mathbf{j} + \mathbf{k}).$$

(a) Find an equation of Π in the form $\mathbf{r}.\mathbf{n} = p$, where $\mathbf{n}$ is a vector perpendicular to Π and p is a constant. (5)

The point P has coordinates $(6, 13, 5)$. The line l passes through P and is perpendicular to Π. The line l intersects Π at the point N.

(b) Show that the coordinates of N are $(3, 1, -1)$. (4)

The point R lies on Π and has coordinates $(1, 0, 2)$.

(c) Find the perpendicular distance from N to the line PR. Give your answer to 3 significant figures. (5)

12. The plane P has equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}.$$

(a) Find a vector perpendicular to the plane P. (2)

The line l passes through the point $A(1, 3, 3)$ and meets P at $(3, 1, 2)$. The acute angle between the plane P and the line l is α.

(b) Find α to the nearest degree. (4)

(c) Find the perpendicular distance from A to the plane P. (4)

13. The straight line l_1 is mapped onto the straight line l_2 by the transformation represented (5)

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \\ 3 & -2 & 1 \end{pmatrix}.$$

The equation of l_2 is $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$, where $\mathbf{a} = 4\mathbf{i} + \mathbf{j} + 7\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

Find a vector equation for the line l_1.

14. The position vectors of the points A, B, and C relative to an origin O are $\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$, $7\mathbf{i} - 3\mathbf{k}$, and $4\mathbf{i} + 4\mathbf{j}$. Find

(a) $\overrightarrow{AC} \times \overrightarrow{BC}$, (4)

(b) the area of triangle ABC, (2)

(c) an equation of the plane ABC in the form $\mathbf{r}.\mathbf{n} = p$. (2)

15. The straight line l_1 is mapped onto the straight line l_2 by the transformation represented by the matrix $\mathbf{M}$, where (5)

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}.$$

The equation of l_1 is $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=\mathbf{0}$, where $\mathbf{a}=3\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

Find a vector equation for the line l_2.

16. The plane Π_1 has vector equation

$$\mathbf{r}.(3\mathbf{i}-4\mathbf{j}+2\mathbf{k})=5.$$

(a) Find the perpendicular distance from the point $(6,2,12)$ to the plane Π_1. (3)

The plane Π_2 has vector equation

$$\mathbf{r}=\lambda(2\mathbf{i}+\mathbf{j}+5\mathbf{k})+\mu(\mathbf{i}-\mathbf{j}-2\mathbf{k}),$$

where λ and μ are scalar parameters.

(b) find the acute angle between Π_1 and Π_2 giving your answer to the nearest degree. (5)

(c) Find an equation of the line of intersection of the two planes in the form $\mathbf{r}\times\mathbf{a}=\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors. (6)

17. Two skew lines l_1 and l_2 have equations

$$\begin{aligned} l_1: &\quad \mathbf{r}=(\mathbf{i}-\mathbf{j}+\mathbf{k})+\lambda(4\mathbf{i}+3\mathbf{j}+2\mathbf{k}),\\ l_2: &\quad \mathbf{r}=(3\mathbf{i}+7\mathbf{j}+2\mathbf{k})+\mu(-4\mathbf{i}+6\mathbf{j}+\mathbf{k}),\end{aligned}$$

respectively, where λ and μ are real parameters.

(a) Find a vector in the direction of the common perpendicular to l_1 and l_2. (2)

(b) Find the shortest distance between these two lines. (5)

18. The plane Π_1 has vector equation (9)

$$\mathbf{r}=\begin{pmatrix}1\\-1\\2\end{pmatrix}+s\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}1\\2\\-2\end{pmatrix},$$

where s and t are real parameters. The plane Π_1 is transformed to the plane Π_2 by the transformation represented by the matrix $\mathbf{T}$, where

$$\mathbf{T}=\begin{pmatrix}2&0&3\\0&2&-1\\0&1&2\end{pmatrix}.$$

Find an equation of the plane Π_2 in the form $\mathbf{r}.\mathbf{n}=p$.

19. The line l passes through the point $P(2,1,3)$ and is perpendicular to the plane Π whose vector equation is

$$\mathbf{r}.(\mathbf{i}-2\mathbf{j}-\mathbf{k})=3.$$

Find

(a) a vector equation of the line l, (2)

(b) the position vector of the point where l meets Π. (4)

(c) Hence find the perpendicular distance of P from Π. (2)

20. (4)

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 5 & 0 \end{pmatrix}.$$

The transformation $M : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $\mathbf{M}$. Find a cartesian equation of the image, under this transformation, of the line

$$x = \frac{y}{2} = \frac{z}{-1}.$$

21. The position vectors of the points A, B, and C from a fixed origin O are

$$\mathbf{a} = \mathbf{i} - \mathbf{j},\ \mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k},\ \text{and}\ \mathbf{c} = 2\mathbf{j} + \mathbf{k},$$

respectively.

(a) Using vector products, find the area of the triangle ABC. (4)

(b) Show that (3)

$$\tfrac{1}{6}\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = 0.$$

(c) Hence or otherwise, state what can be deduced about the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. (1)

22. The plane Π_1 has vector equation

$$\mathbf{r}.(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = 5.$$

The plane Π_2 has vector equation

$$\mathbf{r}.(-\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) = 7.$$

(a) Find a vector equation for the line of intersection of Π_1 and Π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and λ is a scalar parameter. (6)

The plane Π_3 has cartesian equation $x - y + 2z = 31$.

(b) Using your answer to part (a), or otherwise, find the coordinates of the point of intersection of the planes Π_1, Π_2, and Π_3. (3)

23. The points A, B, and C have position vectors

$$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix},$$

respectively.

(a) Find a vector equation of the straight line AB. (2)

(b) Find a cartesian form of the equation of the straight line AB. (2)

The plane Π contains the points A, B, and C.

(c) Find a vector equation of Π in the form $\mathbf{r}.\mathbf{n} = p$. (4)

(d) Find the perpendicular distance from the origin to Π. (2)

24. The plane Π_1 has equation

$$x - 5y - 2z = 3.$$

The plane Π_2 has equation

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k}),$$

where λ and μ are scalar parameters.

(a) Show that Π_1 is perpendicular to Π_2. (4)

(b) Find a cartesian equation for Π_2. (2)

(c) Find an equation for the line of intersection of Π_1 and Π_2 giving your answer in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors to be found. (6)

25. The plane Π_1 has equation

$$x - 2y - 3z = 5$$

and the plane Π_2 has equation

$$6x + y - 4z = 7.$$

(a) Find, to the nearest degree, the acute angle between Π_1 and Π_2. (3)

The point P has coordinates $(2, 3, -1)$. The line l is perpendicular to Π_1 and passes through the point P. The line l intersects Π_2 at the point Q.

(b) Find the coordinates of Q. (4)

The plane Π_3 passes through the point Q and is perpendicular to Π_1 and Π_2.

(c) Find an equation of the plane Π_3 in the form $\mathbf{r}.\mathbf{n} = p$. (4)

26. The straight line l_2 is mapped onto the straight line l_1 by the transformation represented by the matrix (6)

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & -1 & -1 \\ -6 & -1 & -2 \end{pmatrix}.$$

Given that l_2 has cartesian equation

$$\frac{x-1}{5} = \frac{y+2}{2} = \frac{z-3}{1},$$

find a cartesian equation of the line l_1.