Dr Oliver Mathematics
Advance Level Further Mathematics
Further Mathematics 2: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.

1. (a) Express (1)

$$\frac{1}{(r+3)(r+4)}$$

in partial fractions.

Solution

$$\frac{1}{(r+3)(r+4)} \equiv \frac{(r+4)-(r+3)}{(r+3)(r+4)}$$
$$\equiv \underline{\underline{\frac{1}{r+3} - \frac{1}{r+4}}}.$$

(b) Hence, using the method of differences, show that (5)

$$\sum_{r=1}^{n} \frac{1}{(r+3)(r+4)} = \frac{n}{a(n+a)},$$

where a is a constant to be found.

Solution

$$\sum_{r=1}^{n} \frac{1}{(r+3)(r+4)} = \sum_{r=1}^{n} \left(\frac{1}{r+3} - \frac{1}{r+4} \right)$$
$$= (\tfrac{1}{4} - \tfrac{1}{5}) + (\tfrac{1}{5} - \tfrac{1}{6}) + \ldots + (\tfrac{1}{n+3} - \tfrac{1}{n+4})$$
$$= \tfrac{1}{4} - \tfrac{1}{n+4}$$
$$= \frac{(n+4)-4}{4(n+4)}$$
$$= \underline{\underline{\frac{n}{4(n+4)}}};$$

hence, $\underline{\underline{a = 4}}$.

(c) Find the exact value of (2)

$$\sum_{r=15}^{30} \frac{1}{(r+3)(r+4)}.$$

Solution

$$\begin{aligned}\sum_{r=15}^{30} \frac{1}{(r+3)(r+4)} &= \sum_{r=1}^{30} \frac{1}{(r+3)(r+4)} - \sum_{r=1}^{14} \frac{1}{(r+3)(r+4)} \\ &= \frac{30}{4 \times 34} - \frac{14}{4 \times 18} \\ &= \tfrac{15}{68} - \tfrac{7}{36} \\ &= \underline{\underline{\tfrac{4}{153}}}.\end{aligned}$$

2. A transformation from the z-plane to the w-plane is given by (4)

$$w = \frac{1 - \mathrm{i}z}{z},\ z \neq 0.$$

The transformation maps points on the real axis in the z-plane onto the line l in the w-plane.
Find an equation of the line l.

Solution

$z = x + 0\mathrm{i}$ and $w = u + \mathrm{i}v$:

$$u + \mathrm{i}v = \frac{1 - \mathrm{i}x}{x} = \frac{1}{x} - \mathrm{i};$$

hence, $\underline{\underline{u \neq 0, v = -1}}$.

3. (a) By writing (4)

$$\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4},$$

show that

(i) $\sin \frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$,

Solution

$$\begin{aligned}\sin\frac{\pi}{12} &= \sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\\ &= \sin\frac{\pi}{3}\cos\frac{\pi}{4}-\sin\frac{\pi}{4}\cos\frac{\pi}{3}\\ &= \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\times\frac{1}{2}\\ &= \underline{\underline{\frac{\sqrt{6}-\sqrt{2}}{4}}}.\end{aligned}$$

(ii) $\cos\frac{\pi}{12}=\frac{\sqrt{6}+\sqrt{2}}{4}$

Solution

$$\begin{aligned}\cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\\ &= \cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \frac{1}{2}\times\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}\\ &= \underline{\underline{\frac{\sqrt{6}+\sqrt{2}}{4}}}.\end{aligned}$$

(b) Hence find the exact values of z for which (5)

$$z^4=4\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right).$$

Give your answers in the form $z=a+\mathrm{i}b$ where $a,b\in\mathbb{Z}$.

Solution

$$\begin{aligned} & z^4=4\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)\\ \Rightarrow\ & z^4=4\left[\cos\left(\frac{\pi}{3}+2n\pi\right)+\mathrm{i}\sin\left(\frac{\pi}{3}+2n\pi\right)\right]\ \text{(for some } n\text{)}\\ \Rightarrow\ & z^4=4\left[\cos\frac{(6n+1)\pi}{3}+\mathrm{i}\sin\frac{(6n+1)\pi}{3}\right]\\ \Rightarrow\ & z=\sqrt{2}\left[\cos\frac{(6n+1)\pi}{12}+\mathrm{i}\sin\frac{(6n+1)\pi}{12}\right].\end{aligned}$$

$\underline{n=0}$:

$$\begin{aligned} z &= \sqrt{2}\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right) \\ &= \sqrt{2}\left[\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right) + \mathrm{i}\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\right] \\ &= \underline{\underline{\left(\frac{1+\sqrt{3}}{2}\right) + \mathrm{i}\left(\frac{-1+\sqrt{3}}{2}\right)}}. \end{aligned}$$

$\underline{n=1}$:

$$z = \underline{\underline{\sqrt{2}\left(\cos\frac{7\pi}{12} + \mathrm{i}\sin\frac{7\pi}{12}\right)}} \text{ or } \underline{\underline{\left(\frac{1-\sqrt{3}}{2}\right) + \mathrm{i}\left(\frac{1+\sqrt{3}}{2}\right)}}.$$

$\underline{n=2}$:

$$z = \underline{\underline{\sqrt{2}\left(\cos\frac{13\pi}{12} + \mathrm{i}\sin\frac{13\pi}{12}\right)}} \text{ or } \underline{\underline{\left(\frac{-1-\sqrt{3}}{2}\right) + \mathrm{i}\left(\frac{1-\sqrt{3}}{2}\right)}}.$$

$\underline{n=3}$:

$$z = \underline{\underline{\sqrt{2}\left(\cos\frac{19\pi}{12} + \mathrm{i}\sin\frac{19\pi}{12}\right)}} \text{ or } \underline{\underline{\left(\frac{-1+\sqrt{3}}{2}\right) + \mathrm{i}\left(\frac{-1-\sqrt{3}}{2}\right)}}.$$

4. Use algebra to find the set of values of x for which (7)

$$|x^2 - 2| > 4x.$$

Solution

Let's draw a sketch, shall we?

$\underline{-\sqrt{2} \leqslant x \leqslant \sqrt{2}}$:

$$\begin{aligned} x^2 - 2 > 4x &\Rightarrow x^2 - 4x - 2 > 0 \\ &\Rightarrow x^2 - 4x + 4 > 6 \\ &\Rightarrow (x-2)^2 > 6 \\ &\Rightarrow -\sqrt{6} < x - 2 \text{ or } x - 2 > \sqrt{6} \\ &\Rightarrow 2 - \sqrt{6} < x \text{ or } x > 2 + \sqrt{6}. \end{aligned}$$

$\underline{|x| > \sqrt{2}}$:

$$\begin{aligned} -(x^2 - 2) > 4x &\Rightarrow -x^2 + 2 > 4x \\ &\Rightarrow 2 > x^2 + 4x \\ &\Rightarrow 6 > x^2 + 4x + 4 \\ &\Rightarrow 6 > (x+2)^2 \\ &\Rightarrow -\sqrt{6} < x + 2 < \sqrt{6} \\ &\Rightarrow -2 - \sqrt{6} < x < -2 + \sqrt{6}. \end{aligned}$$

Hence,

$$\underline{\underline{x < -2 + \sqrt{6} \text{ or } x > 2 + \sqrt{6}}}.$$

5.
$$y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3x\frac{\mathrm{d}y}{\mathrm{d}x} - 3y^2 = 0.$$

Given that at $x = 0$, $y = 2$, and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$,

(a) show that, at $x = 0$, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \frac{3}{2}$. (6)

Solution

Insert all of the initial conditions:

$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 0 - 12 = 0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6.$$

Differentiate with respect to x:

$$\begin{aligned} & \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 3x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \\ \Rightarrow\quad & (1 \times 6) + 2\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + (3 \times 1) + 0 - (6 \times 2 \times 1) = 0 \\ \Rightarrow\quad & 2\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 3 \\ \Rightarrow\quad & \underline{\underline{\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \tfrac{3}{2}}}, \end{aligned}$$

as required.

(b) Find a series solution for y up to and including the term in x^3. (3)

Solution

$$\begin{aligned} y &= 2 + (1)x + \tfrac{1}{2}(6)x^2 + \tfrac{1}{6}(\tfrac{3}{2})x^3 + \ldots \\ &= \underline{\underline{2 + x + 3x^2 + \tfrac{1}{4}x^3 + \ldots}}. \end{aligned}$$

6. (a) Find the general solution of the differential equation (8)

$$6\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} - 6y = x - 6x^2.$$

Solution

Complementary function:

$$\begin{array}{ll} \text{add to:} & +5 \\ \text{multiply to:} & (+6)\times(-6) = -36 \end{array} \Bigg\} -4,\ +9$$

$$\begin{aligned} 6m^2 + 5m - 6 = 0 &\Rightarrow 6m^2 + 9m - 4m - 6 = 0 \\ &\Rightarrow 3m(2m+3) - 2(2m+3) = 0 \\ &\Rightarrow (3m-2)(2m+3) = 0 \\ &\Rightarrow m = -\tfrac{3}{2} \text{ or } m = \tfrac{2}{3}. \end{aligned}$$

and hence the complementary function is

$$y = A\mathrm{e}^{-\frac{3}{2}x} + B\mathrm{e}^{\frac{2}{3}x}.$$

Particular integral: try

$$y = Cx^2 + Dx + E \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2Cx + D \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2C.$$

Substitute into the differential equation:

$$\begin{aligned} -6C = -6 &\Rightarrow C = 1 \\ 5(2C) - 6D = 1 &\Rightarrow D = \tfrac{3}{2} \\ 6(2C) + 5D - 6E = 0 &\Rightarrow E = \tfrac{13}{4}. \end{aligned}$$

Hence the particular integral is

$$y = x^2 + \tfrac{3}{2}x + \tfrac{13}{4}.$$

General solution: hence the general solution is

$$\underline{\underline{y = A\mathrm{e}^{-\frac{3}{2}x} + B\mathrm{e}^{\frac{2}{3}x} + x^2 + \tfrac{3}{2}x + \tfrac{13}{4}}}.$$

(b) Find the particular solution for which $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}$ when $x = 0$. (5)

Solution

$$x = y = 0:\ 0 = A + B + 0 + 0 + \tfrac{13}{4} \Rightarrow A + B = -\tfrac{13}{4} \quad (1).$$

Differentiate with respect to x:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\tfrac{3}{2}A\mathrm{e}^{-\frac{3}{2}x} + \tfrac{2}{3}B\mathrm{e}^{\frac{2}{3}x} + 2x + \tfrac{3}{2}$$

and

$$\tfrac{3}{2} = -\tfrac{3}{2}A + \tfrac{2}{3}B + 0 + \tfrac{3}{2} \Rightarrow -9A + 4B = 0 \quad (2).$$

Now, $4 \times (1) - (2)$:

$$\begin{aligned} 13A = -13 &\Rightarrow A = -1 \\ &\Rightarrow B = -\tfrac{9}{4}. \end{aligned}$$

Hence, the particular solution is

$$\underline{\underline{y = -\mathrm{e}^{-\frac{3}{2}x} - \tfrac{9}{4}\mathrm{e}^{\frac{2}{3}x} + x^2 + \tfrac{3}{2}x + \tfrac{13}{4}}}.$$

7. The curve C shown in Figure 1 has polar equation

$$r = 2 + \sqrt{3}\cos\theta,\ 0 \leqslant \theta < 2\pi.$$

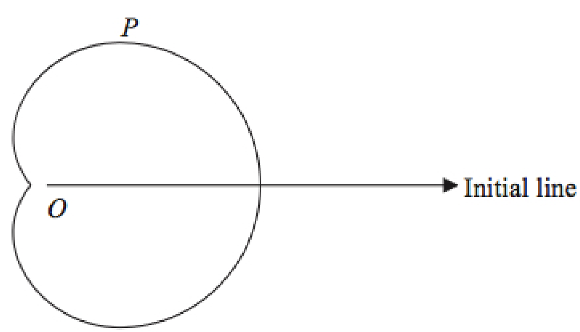

Figure 1: $r = 2 + \sqrt{3}\cos\theta$

The tangent to C at the point P is parallel to the initial line.

(a) Show that (6)

$$OP = \tfrac{1}{2}(3 + \sqrt{7}).$$

Solution

$$\begin{aligned} y = r\sin\theta \Rightarrow y &= \sin\theta(2+\sqrt{3}\cos\theta) \\ \Rightarrow y &= 2\sin\theta + \sqrt{3}\sin\theta\cos\theta \\ \Rightarrow y &= 2\sin\theta + \sqrt{3}\sin 2\theta \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}\theta} &= 2\cos\theta + \sqrt{3}\cos 2\theta. \end{aligned}$$

Now,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}\theta} = 0 &\Rightarrow 2\cos\theta + \sqrt{3}\cos 2\theta = 0 \\ &\Rightarrow 2\cos\theta + \sqrt{3}(2\cos^2\theta - 1) = 0 \\ &\Rightarrow 2\sqrt{3}\cos^2\theta + 2\cos\theta - \sqrt{3} = 0 \\ &\Rightarrow \cos\theta = \frac{-2\pm\sqrt{28}}{4\sqrt{3}} \\ &\Rightarrow \cos\theta = \frac{\sqrt{21}-\sqrt{3}}{6} \text{ or } \cos\theta = -\frac{\sqrt{21}\pm\sqrt{3}}{6}. \end{aligned}$$

We'll take the first one (why?):

$$\begin{aligned} OP &= 2 + \sqrt{3}\left(\frac{\sqrt{21}-\sqrt{3}}{6}\right) \\ &= 2 + \left(\frac{-1+\sqrt{7}}{2}\right) \\ &= \underline{\underline{\tfrac{1}{2}(3+\sqrt{7})}}, \end{aligned}$$

as required.

(b) Find the exact area enclosed by the curve C. (6)

Solution

$$\begin{aligned}
\text{Area} &= \tfrac{1}{2}\int_0^{2\pi} (2+\sqrt{3}\cos\theta)^2 \,\mathrm{d}\theta \\
&= \int_0^{\pi} (4 + 4\sqrt{3}\cos\theta + 3\cos^2\theta)\,\mathrm{d}\theta \\
&= \int_0^{\pi} \left[4 + 4\sqrt{3}\cos\theta + 3(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta)\right] \mathrm{d}\theta \\
&= \int_0^{\pi} \left(\tfrac{11}{2} + 4\sqrt{3}\cos\theta + \tfrac{3}{2}\cos 2\theta\right) \mathrm{d}\theta \\
&= \left[\tfrac{11}{2}\theta + 4\sqrt{3}\sin\theta + \tfrac{3}{4}\sin 2\theta\right]_{\theta=0}^{\pi} \\
&= (\tfrac{11}{2}\pi + 0 + 0) - (0 + 0 + 0) \\
&= \underline{\underline{\tfrac{11}{2}\pi}}.
\end{aligned}$$

8. (a) Using the substitution $t = x^2$, or otherwise, find (6)

$$\int 2x^5 \mathrm{e}^{-x^2}\,\mathrm{d}x.$$

Solution

$$t = x^2 \Rightarrow 1 = 2x\frac{\mathrm{d}x}{\mathrm{d}t} \Rightarrow \mathrm{d}t = 2x\,\mathrm{d}x$$

and

$$\begin{aligned}
\int 2x^5 \mathrm{e}^{-x^2}\,\mathrm{d}x &= \int (2x)(x^2)^2 \mathrm{e}^{-x^2}\,\mathrm{d}x \\
&= \int t^2 \mathrm{e}^{-t}\,\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
u = t^2 &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}t} = 2u \\
\frac{\mathrm{d}v}{\mathrm{d}t} = \mathrm{e}^{-t} &\Rightarrow v = -\mathrm{e}^{-t}.
\end{aligned}$$

$$= -t^2\mathrm{e}^{-t} + \int 2u\mathrm{e}^{-t}\,\mathrm{d}t$$

$$u = 2t \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}t} = 2$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = \mathrm{e}^{-t} \Rightarrow v = -\mathrm{e}^{-t}.$$

$$\begin{aligned} &= -t^2\mathrm{e}^{-t} + \left(-2t\mathrm{e}^{-t} + \int 2\mathrm{e}^{-t}\,\mathrm{d}t\right) \\ &= -t^2\mathrm{e}^{-t} - 2t\mathrm{e}^{-t} + \int 2\mathrm{e}^{-t}\,\mathrm{d}t \\ &= -t^2\mathrm{e}^{-t} - 2t\mathrm{e}^{-t} - 2\mathrm{e}^{-t} + c \\ &= \underline{\underline{-x^4\mathrm{e}^{-x^2} - 2x^2\mathrm{e}^{-x^2} - 2\mathrm{e}^{-x^2} + c}}. \end{aligned}$$

(b) Hence find the general solution of the differential equation (4)

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 2x^2\mathrm{e}^{-x^2},$$

giving your answer in the form $y = \mathrm{f}(x)$.

Solution

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 2x^2\mathrm{e}^{-x^2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{4}{x}\right)y = 2x\mathrm{e}^{-x^2}$$

$$\text{IF} = \int \mathrm{e}^{\frac{4}{x}}\,\mathrm{d}x = \int \mathrm{e}^{4\ln x}\,\mathrm{d}x = \int \mathrm{e}^{\ln x^4}\,\mathrm{d}x = x^4$$

$$\begin{aligned} &\Rightarrow x^4\frac{\mathrm{d}y}{\mathrm{d}x} + 4x^3y = 2x^5\mathrm{e}^{-x^2} \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(yx^4) = 2x^5\mathrm{e}^{-x^2} \\ &\Rightarrow yx^4 = -x^4\mathrm{e}^{-x^2} - 2x^2\mathrm{e}^{-x^2} - 2\mathrm{e}^{-x^2} + c \\ &\Rightarrow \underline{\underline{y = -\mathrm{e}^{-x^2} - \frac{2\mathrm{e}^{-x^2}}{x^2} - \frac{2\mathrm{e}^{-x^2}}{x^4} + \frac{c}{x^4}}}. \end{aligned}$$

Given that $y = 0$ when $x = 1$,

(c) find the particular solution of this differential equation, giving your solution in the form $y = \mathrm{f}(x)$. (3)

Solution

$$0 = -\mathrm{e}^{-1} - 2\mathrm{e}^{-1} - 2\mathrm{e}^{-1} + c \Rightarrow c = 5\mathrm{e}^{-1}$$

and so

$$\underline{\underline{y = -\mathrm{e}^{-x^2} - \frac{2\mathrm{e}^{-x^2}}{x^2} - \frac{2\mathrm{e}^{-x^2}}{x^4} + \frac{5\mathrm{e}^{-1}}{x^4}}}.$$