Dr Oliver Mathematics
Proof by Contradiction

In this note, we will examine proof by contradiction.

To prove that a statement, P, is true, we begin by assuming that P is false and show that it leads to a contradiction.

Example 1
$\sqrt{2}$ is irrational.

Solution Suppose $\sqrt{2}$ *is* rational. Then

$$\sqrt{2} = \frac{a}{b}$$

for some $a \in \mathbb{N}$ and $b \in \mathbb{N}$ in simplest form. Now,

$$\begin{aligned} \sqrt{2} = \frac{a}{b} &\Rightarrow a = \sqrt{2}b \\ &\Rightarrow a^2 = 2b^2. \end{aligned}$$

Now, $2b^2$ is even so a^2 is even.
And, if a^2 is even, then a is even.
Next, $a = 2c$ for some integer c.
Now,

$$\begin{aligned} a^2 = 2b^2 &\Rightarrow (2c)^2 = 2b^2 \\ &\Rightarrow 4c^2 = 2b^2 \\ &\Rightarrow 2c^2 = b^2. \end{aligned}$$

Now, $2c^2$ is even so b^2 is even.
And, if b^2 is even, then b is even – contradiction.

Therefore, $\sqrt{2}$ is irrational. ■

Example 2
There are infinitely many prime numbers.

Solution Let us assume there *is* a greatest prime number.
Let p_1, p_2, p_3, . . . , p_n be the list of prime numbers.
Now, consider

$$N = p_1 p_2 p_3 \ldots p_n + 1.$$

Is this a prime?
p_1? No, because we are left with remainder 1.

p_2? No, because we are left with remainder 1.
$\vdots$
p_n? No, because we are left with remainder 1.
So, if you divide N by one of the primes on our list, you get a remainder of 1. So N is not divisible by any of the primes p_1, p_2, p_3, ..., p_n.

However, by the Fundamental Theorem of Arithmetic, N must have a prime factor (which might be either itself or some smaller number). This contradicts our assumption that p_1, p_2, p_3, ..., p_n was a list of all the prime numbers.

Therefore, there are infinitely many prime numbers. ■

Example 3
There is no greatest even integer.

Solution Suppose there *is* a greatest even integer, N. For every even integer n, $N \geqslant n$. Now, consider

$$M = N + 2.$$

Then, M is an even integer because it is a sum of even integers. Also, $M > N$ since $M = N + 2$. Therefore, M is an integer that is greater than the greatest integer – contradiction.

Therefore, there is no greatest even integer. ■

Here are some examples for you to try.

1. $\sqrt{3}$ is irrational.
2. $\sqrt{5}$ is irrational.
3. $\sqrt{6}$ is irrational.
4. $\sqrt[3]{2}$ is irrational.
5. Prove that $\log_5 10$ is irrational.
6. There exist no integers a and b for which $18a + 6b = 1$.
7. If $|x| < \epsilon$ for every real number $\epsilon > 0$, then $x = 0$.
8. $\sqrt{2} + \sqrt{6} < \sqrt{15}$.
9. If n is an integer and $n^3 + 5$ is odd, then n is even.
10. For all integers n, if n^2 is odd, then n is odd.
11. The sum of any rational number and any irrational number is irrational.
12. There do not exist two prime numbers p and q for which $p - q = 97$.