Dr Oliver Mathematics
Applied Mathematics: Differential Equations

The total number of marks available is 72.
You must write down all the stages in your working.

1. (a) Given the differential equation (4)

$$\sin x \frac{\mathrm{d}y}{\mathrm{d}x} - 2y\cos x = 0,$$

find the general solution, expressing y explicitly in terms of x.

(b) Find the general solution of (5)

$$\sin x \frac{\mathrm{d}y}{\mathrm{d}x} - 2y\cos x = 3\sin^3 x.$$

2. Solve the differential equation (5)

$$\cos^2 x \frac{\mathrm{d}y}{\mathrm{d}x} = y,$$

given that $y > 0$ and that $y = 2$ when $x = 0$.

3. Obtain the solution of the differential equation (5)

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x^2 \mathrm{e}^x,$$

for which $y = 2$ when $x = 1$.

4. An industrial scientist finds that the differential equation

$$t\frac{\mathrm{d}x}{\mathrm{d}t} - 2x = 3t^2$$

models a production process.

(a) Find the general solution of the differential equation. (5)

(b) Hence find the particular solution given $x = 1$ when $t = 1$. (1)

5. An industrial process is modelled by the differential equation (7)

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{9t\mathrm{e}^{3t}}{y}$$

where $y > 0$ and $t \geqslant 0$.

Given that $y = 2$ when $t = 0$, find y explicitly in terms of t.

6. At any point (x, y) on a curve C, where $x \neq 0$, the gradient of the tangent is (9)

$$4 - \frac{3y}{x}.$$

Given that the point $(1, 3)$ lies on C, obtain an equation for C in the form $y = \mathrm{f}(x)$.

7. A turkey is taken from a refrigerator to be cooked. Its temperature is 4°C when it is placed in an oven preheated to 180°C.

Its temperature, T°C, after a time of x hours in the oven satisfies the equation

$$\frac{\mathrm{d}T}{\mathrm{d}x} = k(180 - T).$$

(a) Show that (4)

$$T = 180 - 176\mathrm{e}^{-kx}.$$

After an hour in the oven the temperature of the turkey is 30°C.

(b) Calculate the value of k correct to 2 decimal places. (2)

The turkey will be cooked when it reaches a temperature of 80°C.

(c) After how long (to the nearest minute) will the turkey be cooked? (3)

8. Find the general solution of the differential equation (6)

$$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 6,\ x \neq 0.$$

9. A flu-like virus starts to spread through the 20 000 inhabitants of Dumbarton. The situation can be modelled by the differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{N(20\,000 - N)}{10\,000},$$

where N is the number of people infected after t days and $0 < N < 20\,000$.

(a) How many people are infected when the infection is spreading most rapidly? (1)

(b) Express (5)

$$\frac{10\,000}{N(20\,000 - N)}$$

in partial fractions and show that

$$\ln\left(\frac{N}{20\,000 - N}\right) = 2t + c,$$

for some constant c.

Initially there were 100 people infected.

(c) Show that (4)

$$N = \frac{20\,000\,\mathrm{e}^{2t}}{199 + \mathrm{e}^{2t}}.$$

10. Find the general solution, in the form $y = \mathrm{f}(x)$, of the differential equation (6)

$$\frac{1}{\cos x}\frac{\mathrm{d}y}{\mathrm{d}x} + y\tan x = \tan x,\ 0 < x < \pi.$$