Dr Oliver Mathematics
Mathematics: Higher
2009 Paper 1: Non-Calculator
1 hour 30 minutes

The total number of marks available is 70.
You must write down all the stages in your working.

Section A

1. A sequence is defined by (2)

$$u_{n+1} = 3u_n + 4$$

with $u_1 = 2$.

What is the value of u_3?

A. 34

B. 21

C. 18

D. 13

Solution

A

$$u_2 = 3u_1 + 4 = 3 \times 2 + 4 = 10$$
$$u_3 = 3u_2 + 4 = 3 \times 10 + 4 = 34.$$

2. A circle has equation (2)

$$x^2 + y^2 + 8x + 6y - 75 = 0.$$

What is the radius of this circle?

A. 5

B. 10

C. $\sqrt{75}$

D. $\sqrt{175}$

Solution

B

$$\begin{aligned} x^2+y^2+8x+6y-75=0 &\Rightarrow x^2+8x+y^2+6y=75 \\ &\Rightarrow (x^2+8x+16)+(y^2+6y+9)=75+16+9 \\ &\Rightarrow (x+4)^2+(y+3)^2=100 \\ &\Rightarrow (x+4)^2+(y+3)^2=10^2. \end{aligned}$$

3. Triangle PQR has vertices at $P(-3,-2)$, $Q(-1,4)$, and $R(3,6)$. (2)
PS is a median.
What is the gradient of PS?
   - A. -2
   - B. $-\frac{7}{4}$
   - C. 1
   - D. $\frac{7}{4}$

Solution

D

$S(1,5)$ and the gradient of PS is

$$\frac{5-(-2)}{1-(-3)}=\tfrac{7}{4}.$$

4. A curve has equation (2)

$$y=5x^3-12x.$$

What is the gradient of the tangent at the point $(1,-7)$?
   - A. -7
   - B. -5
   - C. 3
   - D. 5

Solution

C

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 15x^2 - 12$$

and

$$x = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3.$$

5. Here are two statements about the points $S(2, 3)$ and $T(5, -1)$: (2)
(1) The length of $ST = 5$ units;
(2) The gradient of $ST = \frac{4}{3}$.

Which of the following is true?

A. Neither statement is correct.

B. Only statement (1) is correct.

C. Only statement (2) is correct.

D. Both statements are correct.

Solution

B

$$ST = \sqrt{3^2 + 4^2} = 5$$

and

$$\text{gradient} = \frac{3 - (-1)}{2 - 5} = -\tfrac{4}{3}.$$

6. A sequence is generated by the recurrence relation (2)

$$u_{n+1} = 0.7u_n + 10.$$

What is the limit of this sequence as $n \to \infty$?

A. $\dfrac{100}{3}$

B. $\dfrac{100}{7}$

C. $\dfrac{17}{100}$

D. $\dfrac{3}{10}$

Solution

A

Let u be the number that they are converging to. Now,

$$u = 0.7u + 10 \Rightarrow 0.3u = 10$$
$$\Rightarrow u = \frac{100}{3}.$$

7. If the exact value of $\cos x$ is $\frac{1}{\sqrt{5}}$, find the exact value of $\cos 2x$. (2)

   A. $-\frac{3}{5}$

   B. $-\frac{2}{\sqrt{5}}$

   C. $-\frac{2}{\sqrt{5}}$

   D. $\frac{3}{5}$

Solution

A

$$\begin{aligned}\cos 2x &= 2\cos^2 x - 1\\ &= 2\left(\frac{1}{\sqrt{5}}\right)^2 x - 1\\ &= \tfrac{2}{5} - 1\\ &= -\tfrac{3}{5}.\end{aligned}$$

8. What is the derivative of (2)

$$\frac{1}{4x^3},\ x \neq 0?$$

   A. $\dfrac{1}{12x^2}$

   B. $-\dfrac{1}{12x^2}$

   C. $\dfrac{4}{x^4}$

   D. $-\dfrac{3}{4x^4}$

Solution

D

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{4x^3}\right) &= \tfrac{1}{4}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-3}\right)\\ &= \tfrac{1}{4}\times\left(-3x^{-4}\right)\\ &= -\frac{3}{4x^4}.\end{aligned}$$

9. The line with equation (2)

$$y = 2x$$

intersects the circle with equation

$$x^2 + y^2 = 5$$

at the points J and K.

What are the x-coordinates of J and K?

A. $x_J = 1, x_K = -1$

B. $x_J = 2, x_K = -2$

C. $x_J = 1, x_K = -2$

D. $x_J = -1, x_K = 2$

Solution

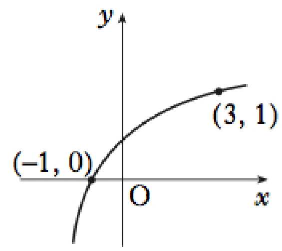
A

$$\begin{aligned}x^2 + y^2 = 5 &\Rightarrow x^2 + (2x)^2 = 5\\ &\Rightarrow 5x^2 = 5\\ &\Rightarrow x^2 = 1\\ &\Rightarrow x = \pm 1\end{aligned}$$

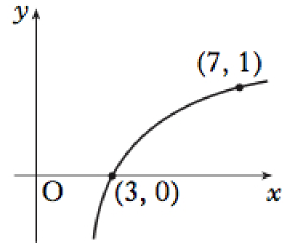
10. Which of the following graphs has equation (2)

$$y = \log_5(x-2)?$$

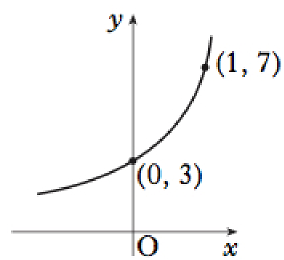
A



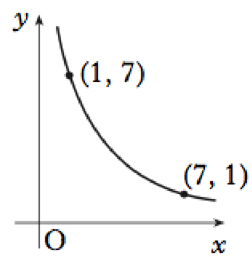
B



C



D



Solution

B

11. How many solutions does the equation (2)

$$(4\sin x - \sqrt{5})(\sin x + 1) = 0$$

have in the interval $0 \leqslant x < 2\pi$?

A. 4

B. 3

C. 2

D. 1

Solution

B

$$(4\sin x - \sqrt{5})(\sin x + 1) = 0 \Rightarrow \sin x = \frac{\sqrt{5}}{4} \text{ or } \sin x = -1$$

and the are 3 solutions (2 solutions for $\sin x = \frac{\sqrt{5}}{4}$ and one for $\sin x = -1$).

12. A function f is given by (2)

$$\mathrm{f}(x) = 2x^2 - x - 9.$$

Which of the following describes the nature of the roots of $\mathrm{f}(x) = 0$?

A. No real roots

B. Equal roots

C. Real distinct roots

D. Rational distinct roots

Solution

C

$$\begin{aligned} b^2 - 4ac &= (-1)^2 - 4 \times 2 \times (-9) \\ &= 73. \end{aligned}$$

13. k and a are given by (2)

$$\begin{aligned} k\sin a^\circ &= 1, \\ k\cos a^\circ &= \sqrt{3}, \end{aligned}$$

where $k > 0$ and $0 \leqslant a < 90$.

What are the values of k and a?

A. $k = 2$ and $a = 60$

B. $k = 2$ and $a = 30$

C. $k = \sqrt{10}$ and $a = 60$

D. $k = \sqrt{10}$ and $a = 30$

Solution

B

$$k = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

and

$$a = \tan^{-1} \frac{1}{\sqrt{3}} = 30.$$

14. If (2)

$$f(x) = 2\sin\left(3x - \frac{\pi}{2}\right) + 5,$$

what is the range of values of $f(x)$?

A. $-1 \leqslant f(x) \leqslant 11$

B. $2 \leqslant f(x) \leqslant 8$

C. $3 \leqslant f(x) \leqslant 7$

D. $-3 \leqslant f(x) \leqslant 7$

Solution

C

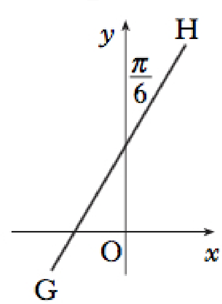
The minimum is

$$2 \times (-1) + 5 = 3$$

and the maximum is

$$2 \times 1 + 5 = 7.$$

15. The line GH makes an angle of $\frac{\pi}{6}$ radians with the y-axis, as shown in the diagram. (2)



What is the gradient of GH?

A. $\sqrt{3}$

B. $\frac{1}{2}$

C. $\frac{1}{\sqrt{2}}$

D. $\frac{\sqrt{3}}{2}$

Solution

A

$$\pi - \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

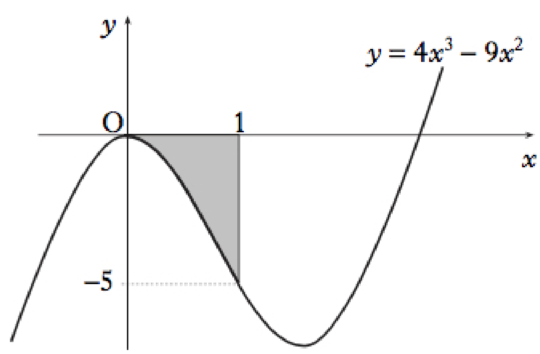
and

$$\tan^{-1}\frac{\pi}{3} = \sqrt{3}.$$

16. The graph of (2)

$$y = 4x^3 - 9x^2$$

is shown in the diagram.



Which of the following gives the area of the shaded section?

A. $\left[x^4 - 3x^3\right]_{x=-5}^{0}$

B. $-\left[x^4 - 3x^3\right]_{x=0}^{1}$

C. $\left[12x^2 - 18x\right]_{x=-5}^{0}$

D. $-\left[12x^2 - 18x\right]_{x=0}^{1}$

Solution

B

$$\int_0^1 (4x^3 - 9x^2)\,\mathrm{d}x = \left[x^4 - 3x^3\right]_{x=0}^1$$

and we want to take it away (why?)

17. The vector $\mathbf{u}$ has components (2)

$$\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}.$$

Which of the following is a unit vector parallel to $\mathbf{u}$?

A. $-\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}$

B. $-3\mathbf{i} + 4\mathbf{k}$

C. $-\frac{3}{\sqrt{7}}\mathbf{i} + \frac{4}{\sqrt{7}}\mathbf{k}$

D. $-\frac{1}{3}\mathbf{i} + \frac{1}{4}\mathbf{k}$

Solution

A

$$|\mathbf{u}| = \sqrt{3^2 + 0 + 4^2} = 5.$$

18. Given that (2)

$$\mathrm{f}(x) = (4 - 3x^2)^{-\frac{1}{2}}$$

on a suitable domain, find $\mathrm{f}'(x)$.

A. $-3x(4 - 3x^2)^{-\frac{1}{2}}$

B. $-\frac{1}{2}(4 - 3x^2)^{-\frac{3}{2}}$

C. $2(4 - 3x^3)^{\frac{1}{2}}$

D. $3x(4 - 3x^2)^{-\frac{3}{2}}$

Solution

D

$$\begin{aligned} f'(x) &= -\tfrac{1}{2}(4-3x^2)^{-\frac{3}{2}} \times (-6x) \\ &= 3x(4-3x^2)^{-\frac{3}{2}}. \end{aligned}$$

19. For what values of x is (2)

$$6 + x - x^2 < 0?$$

A. $x > 3$ only

B. $x < -2$ only

C. $x < -2$ or $x > 3$

D. $-3 < x < 2$

Solution

C

$$\begin{aligned} 6 + x - x^2 = 0 &\Rightarrow (2-x)(3+x) = 0 \\ &\Rightarrow x = -3 \text{ or } x = 2 \end{aligned}$$

and we want the x-values when it is negative.

20. (2)

$$A = 2\pi r^2 + 6\pi r.$$

What is the rate of change of A with respect to r when $r = 2$?

A. 10π

B. 12π

C. 14π

D. 20π

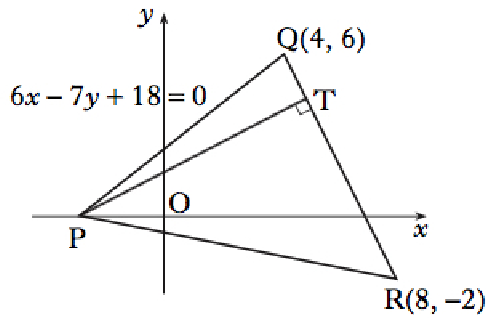
Solution

C

$$\left.\frac{dA}{dr}\right|_{r=2} = \left. 4\pi r + 6\pi \right|_{r=2} = 14\pi.$$

Section B

21. Triangle PQR has vertex P on the x-axis, as shown in the diagram.



Q and R are the points $(4, 6)$ and $(8, -2)$ respectively.
The equation of PQ is

$$6x - 7y + 18 = 0.$$

(a) State the coordinates of P. (1)

Solution

$$\begin{aligned} y = 0 &\Rightarrow 6x + 18 = 0 \\ &\Rightarrow 6x = -18 \\ &\Rightarrow x = -3; \end{aligned}$$

hence, $\underline{\underline{P(-3, 0)}}$.

(b) Find the equation of the altitude of the triangle from P. (3)

Solution

$$\begin{aligned} m_{QR} &= \frac{6 - (-2)}{4 - 8} \\ &= \frac{8}{-4} \\ &= -2 \end{aligned}$$

and the gradient of PT is $\frac{1}{2}$. Finally, the equation is

$$y - 0 = \tfrac{1}{2}(x + 3) \Rightarrow \underline{\underline{y = \tfrac{1}{2}x + \tfrac{3}{2}}}.$$

The altitude from P meets the line QR at T.

(c) Find the coordinates of T. (4)

Solution

The equation of QR is

$$y - 6 = -2(x - 4) \Rightarrow y = -2x + 14$$

and now intersect:

$$\begin{aligned} -2x + 14 = \tfrac{1}{2}x + \tfrac{3}{2} &\Rightarrow \tfrac{5}{2}x = \tfrac{25}{2} \\ &\Rightarrow x = 5 \\ &\Rightarrow y = 4; \end{aligned}$$

hence, $\underline{\underline{T(5, 4)}}$.

22. D, E, and F have coordinates $(10, -8, -15)$, $(1, -2, -3)$, and $(-2, 0, 1)$. respectively.

(a) (i) Show that D, E, and F are collinear. (4)

Solution

$$\overrightarrow{DE} = \begin{pmatrix} 9 \\ -6 \\ -12 \end{pmatrix}$$

and

$$\begin{aligned} \overrightarrow{DF} &= \begin{pmatrix} 12 \\ -8 \\ -16 \end{pmatrix} \\ &= \tfrac{4}{3}\begin{pmatrix} 9 \\ -6 \\ -12 \end{pmatrix} \\ &= \tfrac{4}{3}\overrightarrow{DE}; \end{aligned}$$

hence, $\underline{\underline{D, E, \text{ and } F \text{ are collinear}}}$.

(ii) Find the ratio in which E divides DF.

Solution

$\underline{\underline{3 : 1}}$.

G has coordinates $(k, 1, 0)$.

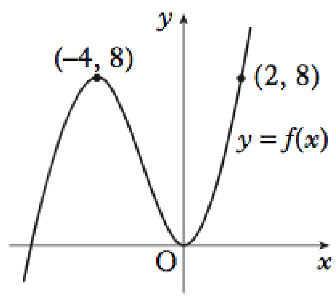
(b) Given that DE is perpendicular to GE, find the value of k. (4)

Solution

Given that DE is perpendicular to GE,

$$\begin{pmatrix} 9 \\ -6 \\ -12 \end{pmatrix} \cdot \begin{pmatrix} k-1 \\ 3 \\ 3 \end{pmatrix} = 0 \Rightarrow 9(k-1) - 18 - 36 = 0$$
$$\Rightarrow 9(k-1) = 54$$
$$\Rightarrow k - 1 = 6$$
$$\Rightarrow \underline{\underline{k = 7}}.$$

23. The diagram shows a sketch of the function $y = \mathrm{f}(x)$.



(a) Copy the diagram and on it sketch the graph of $y = \mathrm{f}(2x)$. (2)

Solution

E.g., it goes through $\underline{\underline{(-2, 8), (0, 0), \text{ and } (1, 8)}}$.

(b) On a separate diagram, sketch the graph of $y = 1 - \mathrm{f}(2x)$. (3)

Solution

E.g., it goes through $\underline{\underline{(-2, -7), (0, 1), \text{ and } (1, -7)}}$.

24. (a) Using the fact that (3)

$$\frac{7\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4},$$

find the exact value of $\sin\left(\frac{7\pi}{12}\right)$.

Solution

$$\begin{aligned}\sin\left(\frac{7\pi}{12}\right) &= \sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\\ &= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)\\ &= \left(\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}\right)+\left(\frac{\sqrt{2}}{2}\times\frac{1}{2}\right)\\ &= \underline{\underline{\frac{\sqrt{6}+\sqrt{2}}{4}}}.\end{aligned}$$

(b) Show that (2)

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B.$$

Solution

$$\begin{aligned}\sin(A+B)+\sin(A-B) &= (\sin A\cos B+\sin B\cos A)\\ &\quad+(\sin A\cos B-\sin B\cos A)\\ &= \underline{\underline{2\sin A\cos B}},\end{aligned}$$

as required.

(c) (i) Express $\frac{\pi}{12}$ in terms of $\frac{\pi}{3}$ and $\frac{\pi}{4}$. (4)

Solution

$$\frac{\pi}{12}=\underline{\underline{\frac{\pi}{3}-\frac{\pi}{4}}}.$$

(ii) Hence or otherwise find the exact value of

$$\sin\left(\frac{7\pi}{12}\right)+\sin\left(\frac{\pi}{12}\right).$$

Solution

$$\begin{aligned}
\sin\left(\frac{7\pi}{12}\right) + \sin\left(\frac{\pi}{12}\right) &= 2\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) \\
&= 2 \times \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} \\
&= \underline{\underline{\frac{\sqrt{6}}{2}}}.
\end{aligned}$$