Dr Oliver Mathematics
Mathematics: Advanced Higher
2023 Paper 2: Calculator
2 hours

The total number of marks available is 65.
You must write down all the stages in your working.

1. The function f is defined by (2)

$$\mathrm{f}(x) = 2\sin^{-1} 3x.$$

Find $\mathrm{f}'(x)$.

2. Find (2)

$$\int \left(\frac{x^2}{x^3 + 10} \right) \mathrm{d}x.$$

3. Matrix $\mathbf{A}$ is defined by

$$\mathbf{A} = \begin{pmatrix} 2 & 2x & 4 \\ x & -1 & 0 \\ 1 & 0 & -2 \end{pmatrix}, \text{ where } x \in \mathbb{R}.$$

(a) Find a simplified expression for the determinant of $\mathbf{A}$. (2)

(b) Hence, determine whether $\mathbf{A}^{-1}$ exists for all values of x. (1)

4. Calculate the gradient of the tangent to the curve with equation (3)

$$x^2y^2 - 2y = \sin 3x$$

at the point $(0, 0)$.

5. (a) Write down and simplify the general term in the binomial expansion of (3)

$$\left(3x - \frac{2}{x^2}\right)^8.$$

(b) Hence, or otherwise, determine the coefficient of x^{-1}. (2)

6. (a) Use the Euclidean algorithm to find d, the greatest common divisor of 703 and 399. (1)

(b) Find integers a and b such that (2)

$$d = 703a + 399b.$$

(c) Hence find integers p and q such that (1)

$$76 = 703p + 399q.$$

7. (a) Solve the differential equation (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 6\mathrm{e}^{5x},$$

given that when $x = 0$, $y = -1$.

Express y in terms of x.

The solution of the differential equation in (a) is also a solution of

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 5\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = k\mathrm{e}^{2x},\ k \in \mathbb{R}.$$

(b) Find the value of k. (2)

8. The fourth and seventh terms of a geometric sequence are 9 and 243 respectively.

(a) Find the

(i) common ratio, (1)

(ii) first term. (1)

(b) Show that (2)

$$\frac{S_{2n}}{S_n} = 1 + 3^n,$$

where S_n represents the sum of the first n terms of this geometric sequence.

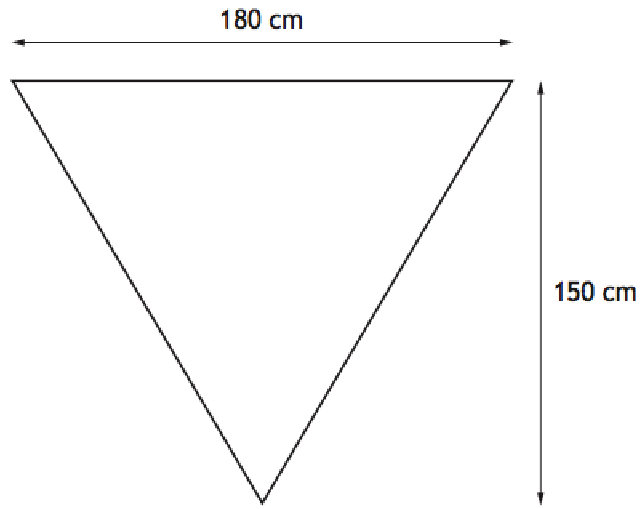
9. Express 572_{10} in base 9. (2)

10. A curve is defined by (5)

$$y = x^{5x^2},\ \text{where } x > 0.$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x.

11. On a building site, water is stored in a container.



The container is a cone with diameter 180 cm at its widest point and height of 150 cm. A cross section of the cone is shown below.

(a) Show that when the water level is at a height of h cm, $0 \leqslant h \leqslant 150$, the volume of water in the container can be written as (1)

$$V = \frac{3\pi h^3}{25}.$$

Water is pumped into the container at a constant rate of 10 litres per second.

(b) Find the rate at which the height is increasing when $h = 125$. (5)

12. Prove by induction that, for all positive integers n, (5)

$$\sum_{r=1}^{n} 2^{r-1} r = 2^n(n-1) + 1.$$

13. Points scored in the long jump element of the decathlon can be calculated using a solution of the differential equation (6)

$$(m - 220)\frac{\mathrm{d}P}{\mathrm{d}m} = 1.4P,\ m > 220,$$

where m is the distance jumped in centimetres and P the points scored.

Given that a jump of 807 centimetres scores 1 079 points, find an expression for P in terms of m.

14. A complex number is defined by (4)

$$w = a + b\text{i},$$

where a and b are positive real numbers.

Given

$$w^2 = 8 + 6\text{i},$$

determine the values of a and b.

15. A function $\mathrm{f}(x)$ has the following properties:

- $f'(x) = \dfrac{x+1}{1+(x+1)^4}$ and
- the first term in the Maclaurin expansion of $\mathrm{f}(x)$ is 1.

(a) Find the Maclaurin expansion of $\mathrm{f}(x)$ up to and including the term in x^2. (3)

(b) Use the substitution (3)

$$u = (x+1)^2$$

to find

$$\int \frac{x+1}{1+(x+1)^4}\,\mathrm{d}x.$$

(c) Determine an expression for $\mathrm{f}(x)$. (2)