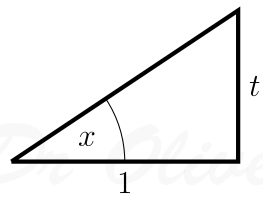
Dr Oliver Mathematics
The t-Substitutions

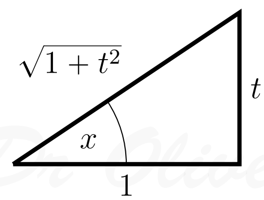
The t-substitutions are often referred to as "the world's sneakiest substitutions" and you need to recognise when they are needed.

1 $t = \tan x$

We start off with a right-angled triangle, with opposite side t, adjacent side 1, with the angle x:



We mark in the hypotenuse:



Now,

$$\sin x = \frac{t}{\sqrt{1+t^2}}$$
$$\cos x = \frac{1}{\sqrt{1+t^2}}$$
$$\tan x = t;$$

the expressions for $\operatorname{cosec} x$, $\sec x$, and $\cot x$ follow.

Finally, what about $\mathrm{d}x$? Now,

$$\begin{aligned} t = \tan x &\Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x} = \sec^2 x \\ &\Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x} = 1 + \tan^2 x \\ &\Rightarrow \mathrm{d}x = \frac{1}{1+t^2}\,\mathrm{d}t. \end{aligned}$$

We can summarise the formulas:

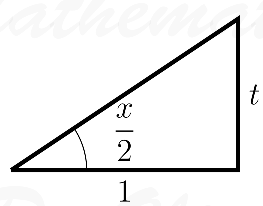
Expression	Replacement
$\sin x$	$\dfrac{t}{\sqrt{1+t^2}}$
$\cos x$	$\dfrac{1}{\sqrt{1+t^2}}$
$\tan x$	t
$\operatorname{cosec} x$	$\dfrac{\sqrt{1+t^2}}{t}$
$\sec x$	$\sqrt{1+t^2}$
$\cot x$	$\dfrac{1}{t}$
$\mathrm{d}x$	$\dfrac{1}{1+t^2}\,\mathrm{d}t$

Example 1
Find

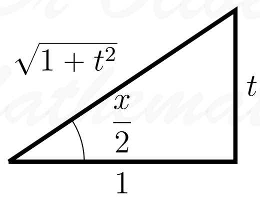
$$\int_0^{\frac{\pi}{4}} \frac{1}{3\cos^2 x + \sin^2 x}\,\mathrm{d}x.$$

2 $t = \tan\left(\dfrac{x}{2}\right)$

This is also known as the *Weierstrass substitution*. We start off with a right-angled triangle, with opposite side t, adjacent side 1, with the angle $\dfrac{x}{2}$:



We mark in the hypotenuse:



Now,

$$\begin{aligned}
\sin \frac{x}{2} &= \frac{t}{\sqrt{1+t^2}} \\
\cos \frac{x}{2} &= \frac{1}{\sqrt{1+t^2}} \\
\tan \frac{x}{2} &= t.
\end{aligned}$$

Next, we use the double-angle formulas:

$$\begin{aligned}
\sin x &\equiv \sin 2\left(\frac{x}{2}\right) \\
&\equiv 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) \\
&\equiv 2 \times \frac{t}{\sqrt{1+t^2}} \times \frac{1}{\sqrt{1+t^2}} \\
&\equiv \frac{2t}{1+t^2},
\end{aligned}$$

$$\begin{aligned}
\cos x &\equiv \cos 2\left(\frac{x}{2}\right) \\
&\equiv \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) \\
&\equiv \frac{1}{1+t^2} - \frac{t^2}{1+t^2} \\
&\equiv \frac{1-t^2}{1+t^2}
\end{aligned}$$

and

$$\begin{aligned}
\tan x &\equiv \frac{\sin x}{\cos x} \\
&\equiv \frac{2t}{1+t^2} \times \frac{1+t^2}{1-t^2} \\
&\equiv \frac{2t}{1-t^2};
\end{aligned}$$

the expressions for $\operatorname{cosec} x$, $\sec x$, and $\cot x$ follow.

Finally, what about $\mathrm{d}x$? Now,

$$
\begin{aligned}
t = \tan\left(\frac{x}{2}\right) &\Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x} = \tfrac{1}{2}\sec^2\left(\frac{x}{2}\right) \\
&\Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1+\tan^2\left(\frac{x}{2}\right)}{2} \\
&\Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1+t^2}{2} \\
&\Rightarrow \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t.
\end{aligned}
$$

We can summarise the formulas:

Expression	Replacement
$\sin x$	$\dfrac{2t}{1+t^2}$
$\cos x$	$\dfrac{1-t^2}{1+t^2}$
$\tan x$	$\dfrac{2t}{1-t^2}$
$\operatorname{cosec} x$	$\dfrac{1+t^2}{2t}$
$\sec x$	$\dfrac{1+t^2}{1-t^2}$
$\cot x$	$\dfrac{1-t^2}{2t}$
$\mathrm{d}x$	$\dfrac{2}{1+t^2}\,\mathrm{d}t$

Example 2
Find

$$\int \frac{1}{2\sin x - \cos x + 3}\,\mathrm{d}x.$$

3 Examples

Here are some examples for you to try.

1. (a) Using $t = \tan\left(\frac{x}{2}\right)$, find
$$\int \operatorname{cosec} x \, \mathrm{d}x.$$
   (b) Using $t = \tan x$, find
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \operatorname{cosec} 2x \, \mathrm{d}x.$$

2. Using $t = \tan\left(\frac{x}{2}\right)$, find
$$\int \frac{1}{1 + \sin x} \, \mathrm{d}x.$$

3. Using $t = \tan\left(\frac{x}{2}\right)$, find
$$\int \frac{1}{1 + \sin x + \cos x} \, \mathrm{d}x.$$

4. Using $t = \tan\left(\frac{x}{2}\right)$, find
$$\int \frac{1}{\sin x + \tan x} \, \mathrm{d}x.$$

5. Using $t = \tan\left(\frac{x}{2}\right)$, find
$$\int \sec x \, \mathrm{d}x.$$

6. Using $t = \tan\left(\frac{x}{2}\right)$, find
$$\int \frac{1}{1 + \sec x} \, \mathrm{d}x.$$

7. (a) Given that
$$\frac{1}{(1+t)(1+t^2)} \equiv \frac{A}{1+t} + \frac{Bt + C}{1+t^2},$$
   find the values of the constants A, B, and C.

   (b) Hence, using $t = \tan x$, find
$$\int \frac{1}{1 + \tan x} \, \mathrm{d}x.$$

8. (a) Given that

$$\frac{2}{3t^2 - 10t + 3} \equiv \frac{A}{3t-1} + \frac{B}{t-3},$$

find the values of the constants A and B.

(b) Hence, using $t = \tan\left(\frac{x}{2}\right)$, find

$$\int \frac{1}{3 - 5\sin x}\, \mathrm{d}x.$$

9. Find

$$\int \frac{1}{4 + 12\cos^2 x}\, \mathrm{d}x.$$

10. Find

$$\int \frac{1}{5 + 3\cos x}\, \mathrm{d}x.$$

11. Find

$$\int \frac{1}{5 - 3\cos x}\, \mathrm{d}x.$$

12. Find

$$\int \frac{1}{2 + \cos x}\, \mathrm{d}x.$$

13. Find, to 3 decimal places,

$$\int_0^{\frac{\pi}{4}} \frac{1}{5\cos^2 x + 3\cos^2 x}\, \mathrm{d}x.$$

14. (a) Given that

$$\frac{2}{(5-t)(1+5t)} \equiv \frac{A}{5-t} + \frac{B}{1+5t},$$

find the values of the constants A and B.

(b) Find

$$\int \frac{1}{12\sin x + 5\cos x}\, \mathrm{d}x.$$

(c) Find, to 3 decimal places,

$$\int_3^{3.1} \frac{1}{12\sin x + 5\cos x}\, \mathrm{d}x.$$