Dr Oliver Mathematics
Further Mathematics
Second Order Differential Equations
Past Examination Questions

This booklet consists of 37 questions across a variety of examination topics.
The total number of marks available is 435.

1. Find the general solution of the differential equation (6)

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = e^{3x}.$$

Solution

Complementary function:

$$m^2 - 6m + 8 = 0 \Rightarrow (m-2)(m-4) = 0 \Rightarrow m = 2 \text{ or } m = 4$$

and hence the complementary function is

$$y = Ae^{2x} + Be^{4x}.$$

Particular integral: try

$$y = Ce^{3x} \Rightarrow \frac{dy}{dx} = 3Ce^{3x} \Rightarrow \frac{d^2y}{dx^2} = 9Ce^{3x}.$$

Substitute into the differential equation:

$$9C - 18C + 8C = 1 \Rightarrow C = -1.$$

Hence the particular integral is $y = -e^{3x}$.

General solution: hence the general solution is

$$\underline{\underline{y = Ae^{2x} + Be^{4x} - e^{3x}}}.$$

2. Find the general solution of the differential equation (7)

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 4x.$$

Solution

Complementary function:

$$m^2 - 8m + 16 = 0 \Rightarrow (m-4)^2 = 0 \Rightarrow m = 4 \text{ (repeated root)}$$

and hence the complementary function is

$$y = (A + Bx)\mathrm{e}^{4x}.$$

Particular integral: try

$$y = Cx + D \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = C \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0.$$

Substitute into the differential equation:

$$0 - 8C + 16(Cx + D) \equiv 4x \Rightarrow C = \tfrac{1}{4}, D = \tfrac{1}{8}.$$

Hence the particular integral is $y = \frac{1}{4}x + \frac{1}{8}$.

General solution: hence the general solution is

$$\underline{\underline{y = (A + Bx)\mathrm{e}^{4x} + \tfrac{1}{4}x + \tfrac{1}{8}.}}$$

3. (a) Find the general solution of the differential equation (7)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 17y = 17x + 36.$$

Solution

Complementary function:

$$m^2 + 2m + 17 = 0 \Rightarrow (m+1)^2 + 16 = 0 \Rightarrow m = -1 \pm 4\mathrm{i}$$

and hence the complementary function is

$$y = \mathrm{e}^{-x}(A\sin 4x + B\cos 4x).$$

Particular integral: try

$$y = Cx + D \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = C \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0.$$

Substitute into the differential equation:

$$0 + 2C + 17(Cx + D) \equiv 17x + 36 \Rightarrow C = 1, D = 2.$$

So the particular integral is $y = x + 2$.

General solution: hence the general solution is

$$\underline{\underline{y = \mathrm{e}^{-x}(A\sin 4x + B\cos 4x) + x + 2}}.$$

(b) Show that, when x is large and positive, the solution approximates to a linear function and state the equation of the linear function. (2)

Solution

For all x,

$$-\sqrt{A^2 + B^2} \leqslant A\sin 4x + B\cos 4x \leqslant \sqrt{A^2 + B^2}.$$

Since $\mathrm{e}^{-x} \to 0$ as $x \to \infty$,

$$\mathrm{e}^{-x}(A\sin 4x + B\cos 4x) \to 0$$

as $x \to \infty$. Hence

$$y = \mathrm{e}^{-x}(A\sin 4x + B\cos 4x) + x + 2 \to \underline{\underline{x + 2}} \text{ as } x \to \infty.$$

4. Find the general solution of the differential equation (9)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 65\sin 2x.$$

Solution

Complementary function:

$$m^2 + 4m + 5 = 0 \Rightarrow (m + 2)^2 + 1 = 0 \Rightarrow m = -2 \pm \mathrm{i}$$

and hence the complementary function is

$$y = \mathrm{e}^{-2x}(A\sin x + B\cos x).$$

Particular integral: try

$$y = C\sin 2x + D\cos 2x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2C\cos 2x - 2D\sin 2x$$
$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4C\sin 2x - 4D\cos 2x.$$

Substitute into the differential equation and equate like terms:

$$\sin 2x: \quad -4C - 8D + 5C = 65 \Rightarrow C - 8D = 65$$
$$\cos 2x: \quad -4D + 8C + 5D = 0 \Rightarrow 8C + D = 0.$$

Solving gives $C = 1$ and $D = -8$ and hence the particular integral is

$$y = \sin 2x - 8\cos 2x.$$

General solution: the general solution is

$$\underline{\underline{y = \mathrm{e}^{-2x}(A\sin x + B\cos x) + \sin 2x - 8\cos 2x}}.$$

5. The variables x and y satisfy the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = \mathrm{e}^{3x}.$$

(a) Find the complementary function. (3)

Solution

The auxiliary equation is

$$m^2 - 6m + 9 = 0 \Rightarrow (m-3)^2 = 0 \Rightarrow m = 3 \text{ (repeated root)}$$

and hence the complementary function is

$$\underline{\underline{y = (A + Bx)\mathrm{e}^{3x}}}.$$

(b) Explain briefly why there is no particular integral if either of the forms $y = k\mathrm{e}^{3x}$ or $y = kx\mathrm{e}^{3x}$. (1)

Solution

Both of these functions are part of the complementary function and so they each satisfy the equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0.$$

(c) Given that there is a particular integral of the form $y = kx^2e^{3x}$, find the value of k. (5)

Solution

$$\frac{dy}{dx} = (2kx + 3kx^2)e^{3x} \text{ and } \frac{d^2y}{dx^2} = (2k + 12kx + 9kx^2)e^{3x}$$

Substitute into the differential equation and equate like terms:

$$e^{3x}: \quad 2k + 0 + 0 = 1 \Rightarrow \underline{\underline{k = \tfrac{1}{2}}}$$

$$xe^{3x}: \quad 12k - 12k + 0 = 0 \text{ (giving no information)}$$

$$x^2e^{3x}: \quad 9k - 18k + 9k = 0 \text{ (giving no information).}$$

6. Solve the differential equation (10)

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{-x}$$

given that $y \to 0$ as $x \to \infty$ and that $\frac{dy}{dx} = -3$ when $x = 0$.

Solution

The auxiliary equation is

$$m^2 - 2m - 3 = 0 \Rightarrow (m - 3)(m + 1) = 0 \Rightarrow m = -1 \text{ or } m = 3$$

and hence the complementary function is

$$y = Ae^{-x} + Be^{3x}.$$

Since e^{-x} is part of the complementary function, for the particular integral try

$$y = Cxe^{-x} \Rightarrow \frac{dy}{dx} = (C - Cx)e^{-x} \Rightarrow \frac{d^2y}{dx^2} = (Cx - 2C)e^{-x}.$$

Substitute into the differential equation and equate like terms:

$$\begin{aligned} \mathrm{e}^{-x}: &\quad -2C - 2C = 2 \\ x\mathrm{e}^{-x}: &\quad C + 2C - 3C = 0 \text{ (and this gives us no information).} \end{aligned}$$

and hence $C = -\frac{1}{2}$. So the general solution is

$$y = A\mathrm{e}^{-x} + B\mathrm{e}^{3x} - \tfrac{1}{2}x\mathrm{e}^{-x}.$$

Since $y \to 0$ as $x \to \infty$ we need $B = 0$ or else the function will become unbounded as $\mathrm{e}^{3x} \to \infty$ as $x \to \infty$. Now

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -A\mathrm{e}^{-x} + (-\tfrac{1}{2} + \tfrac{1}{2}x)\mathrm{e}^{-x}.$$

So

$$x = 0, \frac{\mathrm{d}y}{\mathrm{d}x} = -3 \Rightarrow -3 = -A - \tfrac{1}{2} \Rightarrow A = \tfrac{5}{2}.$$

Hence

$$\underline{\underline{y = \tfrac{5}{2}\mathrm{e}^{-x} - \tfrac{1}{2}x\mathrm{e}^{-x}}}.$$

7. The variables x and y satisfy the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} = 12\mathrm{e}^{2x}.$$

(a) Find the general solution of the differential equation. (6)

Solution

The auxiliary equation is

$$m^2 + 4m = 0 \Rightarrow m(m+4) = 0 \Rightarrow m = 0 \text{ or } m = -4$$

and hence the complementary function is

$$y = A + B\mathrm{e}^{-4x}.$$

For the particular integral try

$$y = Cxe^{2x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2C\mathrm{e}^{2x} \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4C\mathrm{e}^{2x}.$$

Substitute into the differential equation:

$$4C + 8C = 12 \Rightarrow C = 1.$$

So the particular integral is $y = \mathrm{e}^{2x}$ and hence the general solution is

$$\underline{\underline{y = A + B\mathrm{e}^{-4x} + \mathrm{e}^{2x}}}.$$

(b) It is given that the curve which represents a particular solution of the differential equation has gradient 6 when $x = 0$ and approximates to $y = \mathrm{e}^{2x}$ when x is large and positive. Find the equation of the curve. (4)

Solution

$\frac{\mathrm{d}y}{\mathrm{d}x} = -4B\mathrm{e}^{-4x} + 2\mathrm{e}^{2x}$ and hence $x = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -4B + 2 \Rightarrow B = -1$.

Since y is approximated by e^{2x} we also need $A = 0$. Hence

$$\underline{\underline{y = -\mathrm{e}^{-4x} + \mathrm{e}^{2x}}}.$$

8. A differential equation is given by

$$\sin^2 x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2\sin x \cos x \frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 2\sin^4 x \cos x,\ 0 < x < \pi.$$

(a) Show that the substitution $y = u \sin x$, where u is a function of x, transforms this differential equation into (5)

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + u = \sin 2x.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\cos x + \sin x \frac{\mathrm{d}u}{\mathrm{d}x}$$

and

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(u\cos x + \sin x \frac{\mathrm{d}u}{\mathrm{d}x}\right) \\ &= \frac{\mathrm{d}u}{\mathrm{d}x}\cos x - u\sin x + \cos x \frac{\mathrm{d}u}{\mathrm{d}x} + \sin \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \\ &= 2\cos x \frac{\mathrm{d}u}{\mathrm{d}x} - u\sin x + \sin \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}. \end{aligned}$$

Hence

$$\begin{aligned}
&\sin^2 x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\sin x\cos x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 2\sin^4 x\cos x \\
\Rightarrow\quad &\sin^2 x\left(2\cos x\frac{\mathrm{d}u}{\mathrm{d}x} - u\sin x + \sin\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right) \\
&-2\sin x\cos x\left(u\cos x + \sin x\frac{\mathrm{d}u}{\mathrm{d}x}\right) + 2u\sin x = 2\sin^4 x\cos x \\
\Rightarrow\quad &-u\sin^3 x + \sin^3 x\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - 2u\sin x\cos^2 x + 2u\sin x = 2\sin^4 x\cos x \\
\Rightarrow\quad &-u\sin^3 x + \sin^3 x\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 2u\sin x(1-\cos^2 x) = 2\sin^4 x\cos x \\
\Rightarrow\quad &\sin^3 x\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + u\sin^3 x = 2\sin^4 x\cos x \\
\Rightarrow\quad &\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + u = 2\sin x\cos x \\
\Rightarrow\quad &\underline{\underline{\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + u = \sin 2x,}}
\end{aligned}$$

as required.

(b) Hence find the general solution to the differential equation (6)

$$\sin^2 x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\sin x\cos x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 2\sin^4 x\cos x$$

giving your answer in the form $y = \mathrm{f}(x)$.

Solution

The auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm\mathrm{i}$$

and hence the complementary function is

$$u = A\sin x + B\cos x.$$

Since the differential equation involves only the function and the second derivative for the particular integral we can try

$$u = C\sin 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2C\cos 2x \Rightarrow \frac{\mathrm{d}^2u}{\mathrm{d}x^2} = -4C\sin 2x.$$

Substitute into the differential equation:

$$-4C + C = 1 \Rightarrow C = -\tfrac{1}{3}.$$

So the particular integral is $u = -\frac{1}{3}\sin 2x$ and hence the general solution is

$$u = A\sin x + B\cos x - \tfrac{1}{3}\sin 2x.$$

Since $y = u\sin x$,

$$\underline{\underline{y = \sin x\left(A\sin x + B\cos x - \tfrac{1}{3}\sin 2x\right)}}.$$

9. The differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = \sin kx$$

is to be solved, where k is a constant.

(a) In the case $k = 2$, by using a particular integral of the form $ax\cos 2x + bx\sin 2x$, find the general solution. (7)

Solution

The auxiliary equation is

$$m^2 + 4 = 0 \Rightarrow m = \pm 2\mathrm{i}$$

and hence the complementary function is

$$y = A\sin 2x + B\cos 2x.$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a\cos 2x - 2ax\sin 2x + b\sin 2x + 2bx\cos 2x$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4a\sin 2x - 4ax\cos 2x + 4b\cos 2x - 4bx\sin 2x.$$

Substitute into the differential equation and equate like terms:

$$\begin{aligned}
\sin 2x: &\quad -4a + 0 = 1 \Rightarrow a = -\tfrac{1}{2}\\
\cos 2x: &\quad 4b + 0 = 0 \Rightarrow b = 0\\
x\sin 2x: &\quad -4b + 4b = 0 \text{ (giving us no information)}\\
x\cos 2x: &\quad -4a + 4a = 0 \text{ (giving us no information)}.
\end{aligned}$$

So the particular integral is $y = -\frac{1}{2}x\sin 2x$ and hence the general solution is

$$\underline{\underline{y = A\sin 2x + B\cos 2x - \tfrac{1}{2}x\sin 2x}}.$$

(b) Describe briefly the behaviour of your solution for y when $x \to \infty$. (2)

Solution
$|y| \to \infty$ as $x \to \infty$ as the the terms in $\sin 2x$ and $\cos 2x$ are bounded but the final term is unbounded.

(c) In the case $k \neq 2$, explain briefly whether y would exhibit the same behaviour as in part (b) when $x \to \infty$. (2)

Solution
No: if $k \neq 2$ then the particular integral would have the form $y = C \sin kx + D \cos kx$ and this, like the terms in the complementary function, is bounded.

10. The variables x and y satisfy the differential equation

$$2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 5\mathrm{e}^{-2x}.$$

(a) Find the complementary function of the differential equation. (2)

Solution
The auxiliary equation is

$$2m^2 + 3m - 2 = 0 \Rightarrow (2m-1)(m+2) = 0 \Rightarrow m = \tfrac{1}{2} \text{ or } m = -2$$

and hence the complementary function is

$$\underline{\underline{y = A\mathrm{e}^{\frac{1}{2}x} + B\mathrm{e}^{-2x}}}.$$

(b) Given that there is a particular integral of the form $y = px\mathrm{e}^{-2x}$, find the constant p. (4)

Solution

$$y = px\mathrm{e}^{-2x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = (p - 2px)\mathrm{e}^{-2x} \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (-4p + 4px)\mathrm{e}^{-2x}.$$

Substitute into the differential equation and equate like terms:

$$\mathrm{e}^{-2x}: \quad -8p + 3p + 0 = 5 \Rightarrow \underline{\underline{p = -1}}$$

$$x\mathrm{e}^{-2x}: \quad 8p - 6p - 2p = 0 \text{ (giving no information)}.$$

(c) Find the solution of the differential equation for which $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4$ when $x = 0$. (5)

Solution

The general solution is

$$y = A\mathrm{e}^{\frac{1}{2}x} + B\mathrm{e}^{-2x} - x\mathrm{e}^{-2x}.$$

Now $y = 0$ when $x = 0$ and hence $A + B = 0$. Next,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{2}A\mathrm{e}^{\frac{1}{2}x} - 2B\mathrm{e}^{-2x} - \mathrm{e}^{-2x} + 2x\mathrm{e}^{-2x}$$

and, since $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4$ when $x = 0$, we have $4 = \tfrac{1}{2}A - 2B - 1$. Solving these simultaneous equations gives $A = 2$ and $B = -2$. So the solution is

$$\underline{\underline{y = 2\mathrm{e}^{\frac{1}{2}x} - 2\mathrm{e}^{-2x} - x\mathrm{e}^{-2x}}}.$$

11. Find the solution of the differential equation (11)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = \mathrm{e}^{-x}$$

for which $y = \dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$.

Solution

The auxiliary equation is

$$m^2 + 2m + 5 = 0 \Rightarrow (m+1)^2 + 4 = 0 \Rightarrow m = -1 \pm 2\mathrm{i}$$

and so the complementary function is

$$y = \mathrm{e}^{-x}(A\sin 2x + B\cos 2x).$$

For the particular integral, try $y = C\mathrm{e}^{-x}$. (Note that this is not part of the complementary function: both $\mathrm{e}^{-x}\sin 2x$ and $\mathrm{e}^{-x}\cos 2x$ are but not e^{-x} on its own.) Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -C\mathrm{e}^{-x} \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = C\mathrm{e}^{-x}.$$

Substitute into the differential equation:

$$C - 3C + 5C = 1 \Rightarrow C = \tfrac{1}{3}.$$

So the particular integral is $y = \frac{1}{3}\mathrm{e}^{-x}$ and hence the general solution is

$$y = \mathrm{e}^{-x}(\tfrac{1}{3} + A\sin 2x + B\cos 2x).$$

Now, since $y = 0$ when $x = 0$, we have

$$0 = \tfrac{1}{3} + B \Rightarrow B = -\tfrac{1}{3}.$$

Next,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{e}^{-x}(\tfrac{1}{3} + A\sin 2x - \tfrac{1}{3}\cos 2x) + \mathrm{e}^{-x}(2A\cos 2x + \tfrac{2}{3}\sin 2x).$$

So,

$$x = 0, \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 0 = \tfrac{1}{3} - \tfrac{1}{3} + 2A \Rightarrow A = 0.$$

So the solution is

$$\underline{\underline{y = \mathrm{e}^{-x}(\tfrac{1}{3} - \tfrac{1}{3}\cos 2x)}}.$$

12. The variables x and y satisfy the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 16y = 8\cos 4x.$$

(a) Find the complementary function of the differential equation. (2)

Solution

The auxiliary equation is

$$m^2 + 16 = 0 \Rightarrow m = \pm 4\mathrm{i}$$

and so the complementary function is

$$\underline{\underline{y = A\sin 4x + B\cos 4x}}.$$

(b) Given that there is a particular integral of the form $y = px\sin 4x$, where p is a constant, find the general solution of the differential equation. (6)

Solution

$$\frac{dy}{dx} = p\sin 4x + 4px\cos 4x \text{ and } \frac{d^2y}{dx^2} = 8p\cos 4x - 16px\sin 4x.$$

Substitute into the differential equation and equate like terms:

$$\begin{aligned} x\sin 4x: &\quad -16p + 16p = 0 \text{ (and this tells us nothing)} \\ \cos 4x: &\quad 8p + 0 = 8 \Rightarrow p = 1. \end{aligned}$$

So the particular integral is $y = x\sin 4x$ and hence the general solution is

$$\underline{\underline{y = A\sin 4x + B\cos 4x + x\sin 4x}}.$$

(c) Find the solution of the equation for which $y = 2$ and $\frac{dy}{dx} = 0$ when $x = 0$. (4)

Solution

$x = 0$ and $y = 2$ tells us that $B = 2$.

$$\frac{dy}{dx} = 4A\cos 4x - 8\sin 4x + \sin 4x + 4x\cos 2x$$

and so $x = \frac{dy}{dx} = 0$ tells us that $A = 0$. Hence

$$\underline{\underline{y = 2\cos 4x + x\sin 4x}}.$$

13. (a) Find the general solution of the differential equation (7)

$$3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = -2x + 13.$$

Solution

The auxiliary equation is

$$3m^2 + 5m - 2 = 0 \Rightarrow (3m-1)(m+2) = 0 \Rightarrow m = \tfrac{1}{3} \text{ or } m = -2$$

and hence the complementary function is

$$y = Ae^{\frac{1}{3}x} + Be^{-2x}.$$

For the particular integral, try

$$y = Cx + D \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = C \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0.$$

Substitute into the differential equation and equate like terms:

$$5C - 2(Cx + D) = -2x + 13 \Rightarrow C = -1 \text{ and } D = -4.$$

So the particular integral is $y = -x - 4$ and the general solution is

$$\underline{\underline{y = A\mathrm{e}^{\frac{1}{3}x} + B\mathrm{e}^{-2x} - x - 4.}}$$

(b) Find the particular solution for which $y = -\frac{7}{2}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$. (5)

Solution

$$x = 0, y = -\tfrac{7}{2} \Rightarrow -\tfrac{7}{2} = A + B - 4.$$

Differentiate:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{3}A\mathrm{e}^{\frac{1}{3}x} - 2B\mathrm{e}^{-2x} - 1 \Rightarrow 0 = \tfrac{1}{3}A - 2B.$$

Solve these simultaneous equations to get $A = \frac{3}{7}$ and $B = \frac{1}{14}$. So the solution is

$$\underline{\underline{y = \tfrac{3}{7}\mathrm{e}^{\frac{1}{3}x} + \tfrac{1}{14}\mathrm{e}^{-2x} - x - 4.}}$$

(c) Write down the function to which y approximates when x is large and positive. (1)

Solution

Since $\mathrm{e}^{-2x} \to 0$ as $x \to \infty$, we have

$$\underline{\underline{y = \tfrac{3}{7}\mathrm{e}^{\frac{1}{3}x} - x - 4.}}$$

14. (a) Find the complementary function of the differential equation (2)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = \operatorname{cosec} x.$$

Solution
The auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm\mathrm{i}$$

and so the complementary function is

$$\underline{\underline{y = A\sin x + B\cos x}}.$$

(b) It is given that

$$y = p(\ln \sin x)\sin x + qx\cos x,$$

where p and q are constants, is a particular integral of the differential equation.

(i) Show that (6)

$$p - 2(p+q)\sin^2 x \equiv 1.$$

Solution

$$\begin{aligned} & y = p(\ln \sin x)\sin x + qx\cos x \\ \Rightarrow\quad & \frac{\mathrm{d}y}{\mathrm{d}x} = p\cos x + p(\ln \sin x)\cos x + q\cos x - qx\sin x \\ \Rightarrow\quad & \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -q\sin x - p(\ln \sin x)\sin x + p\cos^2 x\operatorname{cosec} x - 2q\sin x - qx\cos x. \end{aligned}$$

Since this satisfies the differential equation,

$$\begin{aligned} & -p\sin x + p\cos^2 x\operatorname{cosec} x - 2q\sin x = \operatorname{cosec} x \\ \Rightarrow\quad & -p\sin x - 2q\sin x + p(1-\sin^2 x)\operatorname{cosec} x = \operatorname{cosec} x \\ \Rightarrow\quad & -p\sin x - 2q\sin x + p\operatorname{cosec} x - p\sin x = \operatorname{cosec} x \\ \Rightarrow\quad & -2p\sin^2 x - 2q\sin^2 x + p = 1 \\ \Rightarrow\quad & \underline{\underline{p - 2(p+q)\sin^2 x = 1}}, \end{aligned}$$

as required.

(ii) Deduce the values of p and q. (2)

Solution
Since this an identity,

$$x = 0 \Rightarrow \underline{\underline{p = 1}} \text{ and } x = \tfrac{\pi}{2} \Rightarrow \underline{\underline{q = -1}}.$$

(c) Write down the general solution of the differential equation. State the set of values of x, in the interval $0 \leqslant x \leqslant 2\pi$, for which the solution is valid, justifying your answer. (3)

Solution

The general solution is

$$\underline{\underline{y = A\sin x + B\cos x + (\ln\sin x)\sin x - x\cos x}}.$$

y does not exist when $\sin x = 0$ and hence the required range of values for x is

$$\underline{\underline{0 < x < \pi \text{ and } \pi < x < 2\pi}}.$$

15. (a) Find the general solution of the differential equation (6)

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 6\frac{\mathrm{d}y}{\mathrm{d}t} + 10y = \mathrm{e}^{2t},$$

giving your answer in the form $y = \mathrm{f}(t)$.

Solution

The auxiliary equation is

$$m^2 - 6m + 10 = 0 \Rightarrow (m-3)^2 + 1 = 0 \Rightarrow m = 3 \pm \mathrm{i}$$

and hence the complementary function is

$$y = \mathrm{e}^{3t}(A\sin t + B\cos t).$$

For the particular integral try

$$y = C\mathrm{e}^{2t} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 2C\mathrm{e}^{2t} \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 4C\mathrm{e}^{2t}.$$

Substitute into the differential equation:

$$4C - 12C + 10C = 1 \Rightarrow C = \tfrac{1}{2}$$

and hence the particular integral is $y = \frac{1}{2}\mathrm{e}^{2t}$. So the general solution is

$$\underline{\underline{y = \mathrm{e}^{3t}(A\sin t + B\cos t) + \tfrac{1}{2}\mathrm{e}^{2t}}}.$$

(b) Given that $x = t^{\frac{1}{2}}$, $x > 0$, $t > 0$, and y is a function of x, show that (5)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4t\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t}.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \left(\tfrac{1}{2}t^{-\frac{1}{2}}\right) = 2t^{\frac{1}{2}}\frac{\mathrm{d}y}{\mathrm{d}t}$$

and hence

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(2t^{\frac{1}{2}}\frac{\mathrm{d}y}{\mathrm{d}t}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(2t^{\frac{1}{2}}\frac{\mathrm{d}y}{\mathrm{d}t}\right) \div \frac{\mathrm{d}x}{\mathrm{d}t}\\ &= \left(t^{-\frac{1}{2}}\frac{\mathrm{d}y}{\mathrm{d}t} + 2t^{\frac{1}{2}}\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\right) \div \left(\tfrac{1}{2}t^{-\frac{1}{2}}\right)\\ &= \underline{\underline{2\frac{\mathrm{d}y}{\mathrm{d}t} + 4t\frac{\mathrm{d}^2y}{\mathrm{d}t^2}}},\end{aligned}$$

as required.

(c) Hence show that the substitution $x = t^{\frac{1}{2}}$ transforms the differential equation (2)

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (12x^2 + 1)\frac{\mathrm{d}y}{\mathrm{d}x} + 40x^3y = 4x^3\mathrm{e}^{2x^2}$$

into

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 6\frac{\mathrm{d}y}{\mathrm{d}t} + 10y = \mathrm{e}^{2t}.$$

Solution

$$
\begin{aligned}
& x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (12x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x} + 40x^3y = 4x^3\mathrm{e}^{2x^2} \\
\Rightarrow\ & t^{\frac{1}{2}}\left(4t\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t}\right) - (12t+1)\left(2t^{\frac{1}{2}}\frac{\mathrm{d}y}{\mathrm{d}t}\right) + 40t^{\frac{3}{2}}y = 4t^{\frac{3}{2}}\mathrm{e}^{2t} \\
\Rightarrow\ & 4t^{\frac{3}{2}}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2t^{\frac{1}{2}}\frac{\mathrm{d}y}{\mathrm{d}t} - 24t^{\frac{3}{2}}\frac{\mathrm{d}y}{\mathrm{d}t} - 2t^{\frac{1}{2}}\frac{\mathrm{d}y}{\mathrm{d}t} + 40t^{\frac{3}{2}}y = 4t^{\frac{3}{2}}\mathrm{e}^{2t} \\
\Rightarrow\ & 4t^{\frac{3}{2}}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 24t^{\frac{3}{2}}\frac{\mathrm{d}y}{\mathrm{d}t} + 40t^{\frac{3}{2}}y = 4t^{\frac{3}{2}}\mathrm{e}^{2t} \\
\Rightarrow\ & \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 6\frac{\mathrm{d}y}{\mathrm{d}t} + 10y = \mathrm{e}^{2t}}},
\end{aligned}
$$

as required.

(d) Hence write down the general solution of the differential equation (1)

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (12x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x} + 40x^3y = 4x^3\mathrm{e}^{2x^2}.$$

Solution

$$\underline{\underline{y = \mathrm{e}^{3x^2}(A\sin x^2 + B\cos x^2) + \tfrac{1}{2}\mathrm{e}^{2x^2}}}.$$

16. (a) Show that the substitution $x = \mathrm{e}^t$ transforms the differential equation (7)

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 30 + 20\sin(\ln x)$$

into

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 5\frac{\mathrm{d}y}{\mathrm{d}t} + 6y = 3 + 20\sin t.$$

Solution

This is a standard substitution and you need to know how to do this both correctly and quickly:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\times\mathrm{e}^t = x\frac{\mathrm{d}y}{\mathrm{d}x}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}\right)\times\frac{\mathrm{d}x}{\mathrm{d}t}\\ &= \left(\frac{\mathrm{d}y}{\mathrm{d}x} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\times x\\ &= x\frac{\mathrm{d}y}{\mathrm{d}x} + x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}.\end{aligned}$$

Finally, $x = \mathrm{e}^t$ and so $t = \ln x$. Hence

$$\begin{aligned}& x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 30 + 20\sin(\ln x)\\ \Rightarrow\quad & \left(x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) - 5x\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 3 + 20\sin(\ln x)\\ \Rightarrow\quad & \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 5\frac{\mathrm{d}y}{\mathrm{d}t} + 6y = 3 + 20\sin t,}}\end{aligned}$$

as required.

(b) Find the general solution of (11)

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 5\frac{\mathrm{d}y}{\mathrm{d}t} + 6y = 3 + 20\sin t.$$

Solution

The auxiliary equation is

$$m^2 - 5m + 6 = 0 \Rightarrow (m-2)(m-3) = 0 \Rightarrow m = 2 \text{ or } m = 3,$$

and hence the complementary function is

$$y = A\mathrm{e}^{2t} + B\mathrm{e}^{3t}.$$

For the particular integral, try

$$\begin{aligned} y = C + D\sin t + E\cos t &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = D\cos t - E\sin t\\ &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -D\sin t - E\cos t.\end{aligned}$$

If we substitute into the differential equation and compare like terms:

$$\begin{aligned}\text{constant}: &\quad 6C = 3 \Rightarrow C = \tfrac{1}{2}\\ \sin t: &\quad -D + 5E + 6D = 20 \Rightarrow 5D + 5E = 20\\ \cos t: &\quad -E - 5D + 6E = 0 \Rightarrow -5D + 5E = 0,\end{aligned}$$

and hence $D = 2$ and $E = 2$. Hence the particular integral is

$$y = \tfrac{1}{2} + 2\sin t + 2\cos t$$

and so the general solution is

$$\underline{\underline{y = Ae^{2t} + Be^{3t} + \tfrac{1}{2} + 2\sin t + 2\cos t.}}$$

(c) Write down the general solution of (1)

$$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 3 + 20\sin(\ln x).$$

Solution
By 'write down' they are not suggesting that you cannot have a line of working, merely that you are in a position to answer this from your previous work. We just need to replace e^t by x and note that, for example, $e^{2t} = (e^t)^2 = x^2$. Hence

$$\underline{\underline{y = Ax^2 + Bx^3 + \tfrac{1}{2} + 2\sin(\ln x) + 2\cos(\ln x).}}$$

17. (a) Show that the transformation $y = vx$ transforms the equation (5)

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (2 + 9x^2)y = x^5 \qquad (\dagger)$$

into the equation

$$\frac{d^2v}{dx^2} + 9v = x^2. \qquad (\ddagger)$$

Solution

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2\frac{\mathrm{d}v}{\mathrm{d}x} + x\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}.$$

Substitute into (†):

$$\begin{aligned} & x^2\left(2\frac{\mathrm{d}v}{\mathrm{d}x} + x\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right) - 2x\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (2+9x^2)(vx) = x^5 \\ \Rightarrow\quad & 2x^2\frac{\mathrm{d}v}{\mathrm{d}x} + x^3\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - 2vx - 2x^2\frac{\mathrm{d}v}{\mathrm{d}x} + 2vx + 9vx^3 = x^5 \\ \Rightarrow\quad & x^3\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + 9vx^3 = x^5 \\ \Rightarrow\quad & \underline{\underline{\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + 9v = x^2.}} \end{aligned}$$

(b) Solve the differential equation (‡) to find v as a function of x. (6)

Solution

<u>Complementary function</u>: The characteristic equation is

$$m^2 + 9 = 0 \Rightarrow m = \pm 3$$

and so the complementary function is $v = A\sin 3x + B\cos 3x$.

<u>Particular integral</u>:

$$v = \mu x^2 + \nu x + \xi \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2\mu x + \nu \Rightarrow \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = 2\mu$$

and now

$$2\mu + 9(\mu x^2 + \nu x + \xi) = x^2 \Rightarrow \mu = \tfrac{1}{9}, \nu = 0, \xi = -\tfrac{1}{11}$$

and we have

$$v = \tfrac{1}{9}x^2 - \tfrac{2}{81}.$$

<u>General solution</u>:

$$\underline{\underline{v = A\sin 3x + B\cos 3x + \tfrac{1}{9}x^2 - \tfrac{2}{81}}}.$$

(c) Hence state the general solution of the differential equation (†). (1)

Solution

Hence the general solution of the differential equation (†) is

$$\underline{\underline{y = x(A\sin 3x + B\cos 3x + \tfrac{1}{9}x^2 - \tfrac{1}{9}x - \tfrac{2}{81})}}.$$

18. (a) Find the general solution of the differential equation (6)

$$2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = 2t + 9.$$

Solution

Complementary function:

$$\begin{aligned} 2m^2 + 5m + 2 = 0 &\Rightarrow (2m+1)(m+2) = 0 \\ &\Rightarrow m = -2 \text{ or } m = -\tfrac{1}{2}, \end{aligned}$$

and the complementary function is

$$x = A\mathrm{e}^{-2t} + B\mathrm{e}^{-\frac{1}{2}t}.$$

Particular integral:

$$x = at + b \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = a \Rightarrow \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 0$$

and

$$0 + 5a + 2(at + b) = 2t + 9 \Rightarrow a = 1, b = 2$$

and the particular integral is

$$x = t + 2.$$

General solution: Hence, the general solution is

$$\underline{\underline{x = A\mathrm{e}^{-2t} + B\mathrm{e}^{-\frac{1}{2}t} + t + 2}}.$$

(b) Find the particular solution of this differential equation for which $x = 3$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = -1$ when $t = 0$. (4)

Solution

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -2A\mathrm{e}^{-2t} - \tfrac{1}{2}B\mathrm{e}^{-\frac{1}{2}t} + 1$$

and

$$3 = A + B + 2 \qquad (1)$$
$$-1 = -2A - \tfrac{1}{2}B + 1. \qquad (2)$$

Solve:

$$B = 1 - A \Rightarrow -1 = -2A - \tfrac{1}{2}(1 - A) + 1 \Rightarrow -\tfrac{3}{2} = -\tfrac{3}{2}A$$

and so

$$A = 1, B = 0.$$

Hence

$$\underline{\underline{x = \mathrm{e}^{-2t} + t + 2}}.$$

The particular solution in part (b) is used to model the motion of a particle P on the x-axis. At time t seconds ($t \geqslant 0$), P is x metres from the origin O.

(c) Show that the minimum distance between O and P is $\frac{1}{2}(5 + \ln 2)$ m and justify that the distance is a minimum. (4)

Solution

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} = 0 &\Rightarrow -2\mathrm{e}^{-2t} + 1 = 0\\ &\Rightarrow \mathrm{e}^{-2t} = \tfrac{1}{2}\\ &\Rightarrow -2t = \ln\tfrac{1}{2}\\ &\Rightarrow t = -\tfrac{1}{2}\ln\tfrac{1}{2}\\ &\Rightarrow t = \tfrac{1}{2}\ln 2.\end{aligned}$$

Now,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 4\mathrm{e}^{-2t} > 0$$

and this is a minimum and

$$\begin{aligned}x &= \mathrm{e}^{-\ln 2} + \tfrac{1}{2}\ln 2 + 2\\ &= \mathrm{e}^{\ln\frac{1}{2}} + \tfrac{1}{2}\ln 2 + 2\\ &= \tfrac{1}{2} + \tfrac{1}{2}\ln 2 + 2\\ &= \underline{\underline{\tfrac{1}{2}(5 + \ln 2)}}.\end{aligned}$$

19. Given that $3x\sin 2x$ is a particular integral of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = k\cos 2x,$$

where k is a constant,

(a) calculate the value of k, (4)

Solution

Particular integral: try $y = 3x\sin 2x$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\sin 2x + 6x\cos 2x \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12\cos 2x - 12x\sin 2x.$$

Now,

$$\begin{aligned} & (12\cos 2x - 12x\sin 2x) + 4(3x\sin 2x) = k\cos 2x \\ \Rightarrow\quad & 12\cos 2x = k\cos 2x \\ \Rightarrow\quad & \underline{\underline{k = 12}}. \end{aligned}$$

(b) find the particular solution of the differential equation for which at $x = 0$, $y = 2$, and for which $x = \frac{\pi}{4}$, $y = \frac{\pi}{2}$. (4)

Solution

Complementary function: The characteristic equation is

$$m^2 + 4 = 0 \Rightarrow m = \pm 2$$

and so the complementary function is $y = A\sin 2x + B\cos 2x$.

General solution: Hence the general solution is

$$y = A\sin 2x + B\cos 2x + 3x\sin 2x.$$

$x = 0, y = 2$: $2 = 0 + B + 0 \Rightarrow B = 2$.
$x = \frac{\pi}{4}, y = \frac{\pi}{2}$: $\frac{\pi}{2} = A + 0 + \frac{3\pi}{4} \Rightarrow A = -\frac{\pi}{4}$.
Hence, the particular solution of the differential equation is

$$\underline{\underline{y = -\tfrac{\pi}{4}\sin 2x + 2\cos 2x + 3x\sin 2x.}}$$

20. A scientist is modelling the amount of a chemical in the human bloodstream. The amount x of the chemical, measured in mg l^{-1}, at a time t hours satisfies the differential

equation

$$2x\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 6\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = x^2 - 3x^4,\ x > 0.$$

(a) Show that the substitution $y = \dfrac{1}{x^2}$ transforms this differential equation into (5)

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + y = 3. \quad (\dagger)$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{2}{x^3}\,\frac{\mathrm{d}x}{\mathrm{d}t}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{2}{x^3}\,\frac{\mathrm{d}x}{\mathrm{d}t}\right) \\ &= \frac{6}{x^4}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \frac{2}{x^3}\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2}.\end{aligned}$$

Now,

$$\begin{aligned}& 2x\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 6\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = x^2 - 3x^4 \\ \Rightarrow\quad & \frac{2}{x^3}\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{6}{x^4}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = \frac{1}{x^2} - 3 \\ \Rightarrow\quad & -\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = y - 3 \\ \Rightarrow\quad & \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + y = 3.}}\end{aligned}$$

(b) Find the general solution of the differential equation (†). (4)

Solution

Complementary function:

$$m^2 + 1 = 0 \Rightarrow m = \pm 1$$

and the complementary function is $y = A\cos t + B\sin t$.

Particular integral: We try $y = c$:

$$\frac{d^2y}{dt^2} = \frac{dy}{dt} = 0$$

and $y = 3$.

General solution: So, the general solution is

$$\underline{\underline{y = A\cos t + B\sin t + 3}}.$$

Given that at time $t = 0$, $x = \frac{1}{2}$ and $\frac{dx}{dt} = 0$,

(c) find an expression for x in terms of t, (4)

Solution

$$y = A\cos t + B\sin t + 3 \Rightarrow \frac{1}{x^2} = A\cos t + B\sin t + 3.$$

Now,

$$x = \tfrac{1}{2}, t = 0 \Rightarrow 4 = A + 0 + 3 \Rightarrow A = 1.$$

Next,

$$\frac{1}{x^2} = A\cos t + B\sin t + 3 \Rightarrow -\frac{2}{x^3}\frac{dx}{dt} = -A\sin t + B\cos t$$

and

$$x = \tfrac{1}{2}, \frac{dx}{dt} = 0 \Rightarrow 0 = 0 + B \Rightarrow B = 0.$$

Hence,

$$\begin{aligned}\frac{1}{x^2} = \cos t + 3 &\Rightarrow x^2 = \frac{1}{\cos t + 3}\\ &\Rightarrow \underline{\underline{x = \sqrt{\frac{1}{\cos t + 3}}}},\end{aligned}$$

because $x \geqslant 0$.

(d) write down the maximum values of x as t varies. (1)

Solution

$$t = \pi \Rightarrow \underline{\underline{x = \sqrt{\frac{1}{2}}}}.$$

21. For the differential equation (12)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 2x(x+3),$$

find the solution for which $x = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$, and $y = 1$.

Solution

Complementary function:

$$\begin{aligned} m^2 + 3m + 2 = 0 &\Rightarrow (m+1)(m+2) = 0 \\ &\Rightarrow m = -1 \text{ or } m = -2, \end{aligned}$$

and so we have

$$y = A\mathrm{e}^{-2x} + B\mathrm{e}^{-x}.$$

Particular integral: Try

$$y = Cx^2 + Dx + E \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2Cx + D \Rightarrow \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2C.$$

Now,

$$2C + 3(2Cx + D) + 2(Cx^2 + Dx + E) \equiv 2x^2 + 6x.$$

Solve:
$\underline{x^2}$: $2C = 2 \Rightarrow C = 1$.
$\underline{x}$: $6C + 2D = 6 \Rightarrow D = 0$.
constant: $2C + 3D + 2E = 2 \Rightarrow E = -1$.

Hence,

$$y = x^2 - 1.$$

General solution: The general solution is

$$y = A\mathrm{e}^{-2x} + B\mathrm{e}^{-x} + x^2 - 1.$$

Now,

$$x = 0, y = 1 \Rightarrow 1 = A + B + 0 - 1 \Rightarrow A + B = 2.$$

Differentiate:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2A\mathrm{e}^{-2x} - B\mathrm{e}^{-x} + 2x$$

and

$$1 = -2A - B + 0 \Rightarrow 2A + B = -1.$$

Solve:

$$2A + (2 - A) = -1 \Rightarrow A = -3, B = 5$$

and

$$\underline{\underline{y = -3\mathrm{e}^{-2x} + 5\mathrm{e}^{-x} + x^2 - 1}}.$$

22. (a) Find the general solution of the differential equation (8)

$$3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = x^2.$$

Solution

Complementary function:

$$3m^2 - m - 2 = 0 \Rightarrow (3m+2)(m-1) = 0$$
$$\Rightarrow m = -\tfrac{2}{3} \text{ or } m = 1,$$

and so we have

$$y = A\mathrm{e}^{-\frac{2}{3}x} + B\mathrm{e}^{x}.$$

Particular integral: try

$$y = Cx^2 + Dx + E \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2Cx + D \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2C.$$

Now,

$$3(2C) - (2Cx + D) - 2(Cx^2 + Dx + E) \equiv x^2.$$

Solve:
x^2: $-2C = 1 \Rightarrow C = -\frac{1}{2}$.
x: $-2C - 2D = 0 \Rightarrow D = \frac{1}{2}$.
constant: $6C - D - 2E = 0 \Rightarrow E = -\frac{7}{4}$.

Hence,

$$y = -\tfrac{1}{2}x^2 + \tfrac{1}{2}x - \tfrac{7}{4}.$$

General solution: The general solution is

$$\underline{\underline{y = Ae^{-\frac{2}{3}x} + Be^x - \tfrac{1}{2}x^2 + \tfrac{1}{2}x - \tfrac{7}{4}}}.$$

(b) Find the particular solution for which, at $x = 0$, $y = 2$ and $\dfrac{dy}{dx} = 3$. (6)

Solution

$$x = 0, y = 2 \Rightarrow 2 = A + B - \tfrac{7}{4} \Rightarrow A + B = \tfrac{15}{4}.$$

Now,

$$\frac{dy}{dx} = -\tfrac{2}{3}Ae^{-\frac{2}{3}x} + Be^x - x + \tfrac{1}{2}$$

and

$$3 = -\tfrac{2}{3}A + B + \tfrac{1}{2} \Rightarrow -\tfrac{2}{3}A + B = \tfrac{5}{2}.$$

Solve:

$$\begin{aligned} B = \tfrac{15}{4} - A &\Rightarrow -\tfrac{2}{3}A + (\tfrac{15}{4} - A) = \tfrac{5}{2} \\ &\Rightarrow -\tfrac{5}{3}A = -\tfrac{5}{4} \\ &\Rightarrow A = \tfrac{3}{4} \\ &\Rightarrow B = 3; \end{aligned}$$

Hence, the particular solution is

$$\underline{\underline{y = \tfrac{3}{4}e^{-\frac{2}{3}x} + 3e^x - \tfrac{1}{2}x^2 + \tfrac{1}{2}x - \tfrac{7}{4}}}.$$

23. (a) Find, in terms of k, the general solution of the differential equation (7)

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = kt + 5,$$

where k is a constant and $t > 0$.

Solution

Complementary function:

$$\begin{aligned} m^2 + 4m + 3 = 0 &\Rightarrow (m + 1)(m + 3) = 0 \\ &\Rightarrow m = -1 \text{ or } m = -3, \end{aligned}$$

and so we have

$$x = Ae^{-3t} + Be^{-t}.$$

<u>Particular integral</u>: try try

$$x = Ct + D \Rightarrow \frac{dx}{dt} = C \Rightarrow \frac{d^2x}{dt^2} = 0.$$

Now,

$$0 + 4C + 3(Ct + D) = kt + 5 \Rightarrow C = \tfrac{1}{3}k, D = \tfrac{5}{3} - \tfrac{4}{9}k.$$

<u>General solution</u>: The general solution is

$$\underline{\underline{x = Ae^{-3t} + Be^{-t} + \tfrac{1}{3}kt + \tfrac{5}{3} - \tfrac{4}{9}k}}.$$

For large values of t, this general solution may be approximated by a linear function.

(b) Given that $k = 6$, find the equation of this linear function. (2)

Solution

$$k = 6 \Rightarrow x = Ae^{-3t} + Be^{-t} + 2t - 1$$

and hence the linear function is

$$\underline{\underline{x \approx 2t - 1}}.$$

24.

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 5y = 4e^x.$$

(a) Show that $\lambda x e^x$ is a particular integral of the differential equation, where λ is a constant to be found. (4)

Solution

<u>Particular integral</u>: try $y = \lambda x e^x$:

$$\frac{dy}{dx} = \lambda x e^x + \lambda e^x \text{ and } \frac{d^2y}{dx^2} = \lambda x e^x + 2\lambda e^x.$$

Now,

$$\begin{aligned} & [\lambda x + 2\lambda + 4(\lambda x + \lambda) - 5(\lambda x)]\,\mathrm{e}^x = 4\mathrm{e}^x \\ \Rightarrow\ & \lambda x + 2\lambda + 4(\lambda x + \lambda) - 5(\lambda x) = 4 \\ \Rightarrow\ & 6\lambda = 4 \\ \Rightarrow\ & \underline{\underline{\lambda = \tfrac{2}{3}}}. \end{aligned}$$

(b) Find general solution of the differential equation. (4)

Solution

Complementary function:

$$\begin{aligned} m^2 + 4m - 5 = 0 &\Rightarrow (m+5)(m-1) = 0 \\ &\Rightarrow m = -5 \text{ or } m = 1, \end{aligned}$$

and so we have

$$y = A\mathrm{e}^{-5x} + B\mathrm{e}^x.$$

General solution: The general solution is

$$\underline{\underline{y = A\mathrm{e}^{-5x} + B\mathrm{e}^x + \tfrac{2}{3}x\mathrm{e}^x}}.$$

(c) Find the particular solution for which $y = -\frac{2}{3}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{3}$ at $x = 0$. (5)

Solution

$$x = 0, y = -\tfrac{2}{3} \Rightarrow -\tfrac{2}{3} = A + B + 0 \Rightarrow A + B = -\tfrac{2}{3}.$$

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -5A\mathrm{e}^{-5x} + B\mathrm{e}^x + \tfrac{2}{3}x\mathrm{e}^x + \tfrac{2}{3}\mathrm{e}^x$$

and

$$x = 0, \frac{\mathrm{d}y}{\mathrm{d}x} = -\tfrac{4}{3} \Rightarrow -\tfrac{4}{3} = -5A + B + \tfrac{2}{3} \Rightarrow -5A + B = -2.$$

Solve:

$$6A = \tfrac{4}{3} \Rightarrow A = \tfrac{2}{9}, B = -\tfrac{8}{9}$$

and the particular solution is

$$\underline{\underline{y = \tfrac{2}{9}\mathrm{e}^{-5x} - \tfrac{8}{9}\mathrm{e}^x + \tfrac{2}{3}x\mathrm{e}^x}}.$$

25. Find the general solution of the differential equation (8)

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 10x = e^{-4t}.$$

Solution

<u>Complementary function</u>:

$$m^2 + 6m + 10 = 0 \Rightarrow (m+3)^2 + 1 = 0$$
$$\Rightarrow m = -3 \pm i,$$

and so the complementary function is

$$x = e^{-3t}(A\cos t + B\sin t).$$

<u>Particular integral</u>: try $x = Ce^{-4t}$:

$$\frac{dx}{dt} = -4Ce^{-4t} \text{ and } \frac{d^2x}{dt^2} = 16Ce^{-4t}.$$

Now,

$$(16C - 24C + 10C)e^{-4t} = e^{-4t} \Rightarrow 2C = 1 \Rightarrow C = \tfrac{1}{2}$$

and so the particular integral is

$$x = \tfrac{1}{2}e^{-4t}.$$

<u>General solution</u>: The general solution is

$$\underline{\underline{x = e^{-3t}(A\cos t + B\sin t) + \tfrac{1}{2}e^{-4t}}}.$$

26. Find the general solution of the differential equation (10)

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 5\cos t.$$

Solution

<u>Complementary function</u>:

$$m^2 + 6m + 9 = 0 \Rightarrow (m+3)^2 = 0 \Rightarrow m = -3 \text{ (only)}$$

and the complementary function is

$$x = (A + Bt)e^{-3t}.$$

Particular integral: try $x = C\cos t + D\sin t$:

$$\frac{dx}{dt} = -C\sin t + D\cos t \text{ or } \frac{d^2x}{dt^2} = -C\cos t - D\sin t.$$

Now,

$$\begin{aligned}\underline{\sin t}: &\quad -D - 6C + 9D = 0 \Rightarrow -6C + 8D = 0\\ \underline{\cos t}: &\quad -C + 6D + 9C = 5 \Rightarrow 8C + 6D = 5.\end{aligned}$$

Solve:

$$C = \tfrac{4}{3}D \Rightarrow \tfrac{50}{3}D = 5 \Rightarrow D = \tfrac{3}{10}, C = \tfrac{2}{5}$$

and we have

$$x = \tfrac{2}{5}\cos t + \tfrac{3}{10}\sin t.$$

General solution: The general solution is

$$\underline{\underline{x = (A + Bt)e^{-3t} + \tfrac{2}{5}\cos t + \tfrac{3}{10}\sin t}}.$$

27.

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{-t}.$$

Given that $x = 0$ and $\frac{dx}{dt} = 2$ at $t = 0$,

(a) find x in terms of t. (8)

Solution

Complementary function:

$$\begin{aligned}m^2 + 5m + 6 = 0 &\Rightarrow (m + 2)(m + 3) = 0\\ &\Rightarrow m = -3 \text{ or } m = -2,\end{aligned}$$

and so we have

$$x = Ae^{-3t} + Be^{-2t}.$$

Particular integral: try $x = Ce^{-t}$:

$$\frac{dx}{dt} = -Ce^{-t} \text{ or } \frac{d^2x}{dt^2} = Ce^{-t}.$$

Now,

$$(C - 5C + 6C)\mathrm{e}^{-t} = 2\mathrm{e}^{-t} \Rightarrow C = 1$$

and we have

$$x = \mathrm{e}^{-t}.$$

General solution: The general solution is

$$x = A\mathrm{e}^{-3t} + B\mathrm{e}^{-2t} + \mathrm{e}^{-t}.$$

Particular solution:

$$x = 0, t = 0 \Rightarrow 0 = A + B + 1 \Rightarrow A + B = -1.$$

Now,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3A\mathrm{e}^{-3t} - 2B\mathrm{e}^{-2t} - \mathrm{e}^{-t}$$

which means

$$x = 0, \frac{\mathrm{d}x}{\mathrm{d}t} = 2 \Rightarrow 2 = -3A - 2B - 1 \Rightarrow -3A - 2B = 3.$$

Solve:

$$B = -A - 1 \Rightarrow -3A - 2(-A - 1) = 3 \Rightarrow -A = 1 \Rightarrow A = -1, B = 0,$$

and we have

$$\underline{\underline{x = -\mathrm{e}^{-3t} + \mathrm{e}^{-t}}}.$$

The particular solution in part (a) is used to model the motion of a particle P on the x-axis. At time t seconds, where $t \geqslant 0$, P is x metres from the origin O.

(b) Show that the maximum distance between O and P is $\frac{2\sqrt{3}}{9}$ m and justify that the distance is a maximum. (7)

Solution

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} = 0 &\Rightarrow 3\mathrm{e}^{-3t} - \mathrm{e}^{-t} = 0\\ &\Rightarrow \mathrm{e}^{-3t}(3 - \mathrm{e}^{2t}) = 0\\ &\Rightarrow \mathrm{e}^{2t} = 3\\ &\Rightarrow t = \tfrac{1}{2}\ln 3.\end{aligned}$$

Now,

$$\begin{aligned} x &= -\mathrm{e}^{-\frac{3}{2}\ln 3} + \mathrm{e}^{-\frac{1}{2}\ln 3} \\ &= \mathrm{e}^{-\ln 3^{\frac{1}{2}}} - \mathrm{e}^{\ln 3^{-\frac{3}{2}}} \\ &= 3^{-\frac{1}{2}} - 3^{-\frac{3}{2}} \\ &= \tfrac{1}{\sqrt{3}} - \tfrac{1}{3\sqrt{3}} \\ &= \underline{\underline{\tfrac{2\sqrt{3}}{9}}}. \end{aligned}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -9\mathrm{e}^{-3t} + \mathrm{e}^{-t}$$

and

$$\left.\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right|_{x=\frac{1}{2}\ln 3} = -9\mathrm{e}^{-\frac{3}{2}\ln 3} + \mathrm{e}^{-\frac{1}{2}\ln 3} = -\tfrac{2\sqrt{3}}{3} < 0,$$

and this is a maximum.

28. (a) Find the value of λ for which $y = \lambda x \sin 5x$ is a particular integral of the differential equation (4)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 25y = 3\cos 5x.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lambda \sin 5x + 5\lambda x \cos 5x$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 5\lambda\cos 5x + 5\lambda x \cos 5x - 25\lambda x \sin 5x = 10\lambda\cos 5x - 25\lambda x\sin 5x.$$

Then

$$\begin{aligned} & 10\lambda\cos 5x - 25\lambda x\sin 5x + 25(\lambda x \sin 5x) = 3\cos 5x \\ \Rightarrow\ & 10\lambda\cos 5x = 3\cos 5x \\ \Rightarrow\ & \underline{\underline{\lambda = \tfrac{3}{10}}}. \end{aligned}$$

(b) Using your answer to part (a), the general solution of the differential equation (3)

Solution

Complementary function:

$$m^2 + 25 = 0 \Rightarrow m = \pm 5\text{i}$$

and so we have the complementary function is

$$y = A\cos 5x + B\sin 5x.$$

General solution: The general solution is

$$\underline{\underline{y = A\cos 5x + B\sin 5x + \tfrac{3}{10}x\sin 5x.}}$$

Given that at $x = 0$, $y = 0$, and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 5$,

(c) find the particular solution of this differential equation, giving your solution in the form $y = \mathrm{f}(x)$. (5)

Solution

$$x = 0, y = 0 \Rightarrow 0 = A + 0 + 0 \Rightarrow A = 0.$$

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5B\cos 5x + \tfrac{3}{10}\sin 5x + \tfrac{3}{2}x\cos 5x$$

and

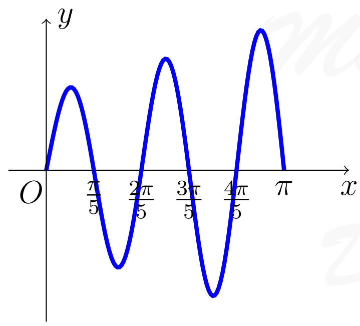
$$x = 0, \frac{\mathrm{d}y}{\mathrm{d}x} = 5 \Rightarrow 5 = 5B + 0 + 0 \Rightarrow B = 1.$$

Hence

$$\underline{\underline{y = \sin 5x + \tfrac{3}{10}x\sin 5x.}}$$

(d) Sketch the curve with equation $y = \mathrm{f}(x)$ for $0 \leqslant x \leqslant \pi$. (2)

Solution



29. The differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + 9x = \cos 3t,\ t \geqslant 0,$$

describes the motion of a particle along the x-axis.

(a) Find the general solution to this differential equation. (8)

Solution

Complementary function:

$$m^2 + 6m + 9 = 0 \Rightarrow (m+3)^2 = 0 \Rightarrow m = -3 \text{ (only)}$$

and the complementary function is

$$x = (A + Bt)\mathrm{e}^{-3t}.$$

Particular integral: try $x = C\cos 3t + D\sin 3t$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -3C\sin 3t + 3D\cos 3t \text{ and } \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -9C\cos 3t - 9D\sin 3t.$$

Now,

$$\begin{aligned} \underline{\sin t}: &\quad -9D - 18C + 9D = 0 \Rightarrow C = 0, \\ \underline{\cos t}: &\quad -9C + 18D + 9C = 1 \Rightarrow D = \tfrac{1}{18}, \end{aligned}$$

and this gives

$$x = \tfrac{1}{18}\sin 3t.$$

General solution: The general solution is

$$\underline{\underline{x = (A + Bt)\mathrm{e}^{-3t} + \tfrac{1}{18}\sin 3t.}}$$

(b) Find the particular solution of this differential equation for which, at $t = 0$, $x = \frac{1}{2}$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$. (5)

Solution

$$x = \tfrac{1}{2}, t = 0 \Rightarrow \tfrac{1}{2} = A + 0 + 0 \Rightarrow A = \tfrac{1}{2}.$$

Now,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (-3A + B - 3Bt)\mathrm{e}^{-3t} + \tfrac{1}{6}\cos 3t$$

and

$$t = 0, \frac{\mathrm{d}x}{\mathrm{d}t} = 0 \Rightarrow 0 = -3A + B + \tfrac{1}{6} \Rightarrow B = \tfrac{4}{3}.$$

So we have

$$\underline{\underline{x = (\tfrac{1}{2} + \tfrac{4}{3}t)\mathrm{e}^{-3t} + \tfrac{1}{18}\sin 3t.}}$$

On the graph of the particular solution defined in part (b), the first turning point for $t > 30$ is the point A.

(c) Find the approximate values for the coordinates of A. (2)

Solution

Now,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (-3A + B - 3Bt)\mathrm{e}^{-3t} + \tfrac{1}{6}\cos 3t$$

which means

$$\frac{\mathrm{d}x}{\mathrm{d}t} \approx \tfrac{1}{6}\cos 3t$$

and we want

$$\begin{aligned}\cos 3t = 0 &\Rightarrow 3t = \tfrac{1}{2}\pi, \tfrac{3}{2}\pi, \ldots, \tfrac{(2n-1)}{2}\pi, \ldots \\ &\Rightarrow t = \tfrac{1}{6}\pi, \tfrac{3}{6}\pi, \ldots, \tfrac{(2n-1)}{6}\pi, \ldots\end{aligned}$$

Now,

$$\begin{aligned}\tfrac{(2n-1)}{6}\pi > 30 &\Rightarrow 2n - 1 > \frac{180}{\pi} \\ &\Rightarrow 2n > \frac{180}{\pi} + 1 \\ &\Rightarrow n > \frac{1}{2}\left[\frac{180}{\pi} + 1\right] \\ &\Rightarrow n > 29.147\,889\,76 \text{ (FCD)},\end{aligned}$$

so we take $n = 30$ which gives $\underline{\underline{t = \frac{59}{6}\pi}}$ and

$$x = (\tfrac{1}{2} + \tfrac{4}{3}t)e^{-3(\frac{59\pi}{6})} + \tfrac{1}{18}\sin 3(\tfrac{59\pi}{6}) \approx \underline{\underline{-\tfrac{1}{18}}}.$$

30. Find the general solution to the differential equation (9)

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2\cos t - \sin t.$$

Solution

Complementary function: The characteristic equation is

$$\begin{aligned} m^2 + 5m + 6 = 0 &\Rightarrow (m+2)(m+3) = 0 \\ &\Rightarrow m = -2 \text{ or } m = -3 \end{aligned}$$

and so the complementary function is $x = Ae^{-3t} + Be^{-2t}$.
Particular integral: try $x = C\cos t + D\sin t$:

$$\frac{dx}{dt} = -C\sin t + D\cos t \text{ and } \frac{d^2x}{dt^2} = -C\cos t - D\sin t.$$

Now,

$$\begin{aligned} \underline{\sin t}: &\quad -D - 5C + 6D = -1 \Rightarrow -5C + 5D = -1, \\ \underline{\cos t}: &\quad -C + 5D + 6C = 2 \Rightarrow 5C + 5D = 2, \end{aligned}$$

and this gives $C = \frac{3}{10}, D = \frac{1}{10}$. Thus,

$$x = \tfrac{3}{10}\cos t + \tfrac{1}{10}\sin t.$$

General solution: The general solution is

$$\underline{\underline{x = Ae^{-3t} + Be^{-2t} + \tfrac{3}{10}\cos t + \tfrac{1}{10}\sin t.}}$$

31. (a) Find the value of λ for which $\lambda t^2 e^{3t}$ is a particular integral of the differential equation (5)

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 6e^{3t}, t \geqslant 0.$$

Solution

Particular integral:

$$\frac{dy}{dt} = 2\lambda t e^{3t} + 3\lambda t^2 e^{3t}, \frac{d^2y}{dt^2} = 2\lambda e^{3t} + 12\lambda t e^{3t} + 9\lambda t^2 e^{3t}$$

If we substitute into the differential equation and compare coefficients:

$$\begin{aligned} t^2e^{3t} &: \quad 9\lambda - 18\lambda + 9\lambda = 0 \text{ (and this tells us nothing)} \\ te^{3t} &: \quad 12\lambda - 12\lambda = 0 \text{ (and this, again, tells us nothing)} \\ e^{3t} &: \quad 2\lambda = 6 \Rightarrow \underline{\underline{\lambda = 3}}. \end{aligned}$$

(b) Hence find the general solution of the differential equation. (3)

Solution

Complementary function: The characteristic equation is

$$m^2 - 6m + 9 = 0 \Rightarrow (m-3)^2 = 0$$

and so the complementary function is $y = e^{3t}(A + Bt)$.

General solution: Hence the general solution is

$$\underline{\underline{y = e^{3t}(A + Bt + 3t^2)}}.$$

Given that when $t = 0$, $y = 5$ and $\dfrac{dy}{dt} = 4$,

(c) find the particular solution of this differential equation, giving your solution in the form $y = f(t)$. (5)

Solution

$t = 0, y = 5 \Rightarrow A = 5.$

$$\frac{dy}{dt} = 3e^{3t}(5 + Bt + 3t^2) + e^{3t}(B + 6t)$$

and so

$$t = 0, \frac{dy}{dt} = 4 \Rightarrow 15 + B = 4 \Rightarrow B = -11.$$

Hence the particular solution is

$$\underline{\underline{y = \mathrm{e}^{3t}(5 - 11t + 3t^2)}}.$$

32. (a) Show that the transformation $y = vx$ transforms the equation (6)

$$4x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 8x\frac{\mathrm{d}y}{\mathrm{d}x} + (8 + 4x^2)y = x^4 \qquad (\dagger)$$

into the equation

$$4\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 4v = x. \qquad (\ddagger)$$

Solution

$$y = vx \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\frac{\mathrm{d}v}{\mathrm{d}x} + x\frac{\mathrm{d}^2v}{\mathrm{d}x^2}.$$

Substitute into ($\dagger$):

$$\begin{aligned} & 4x^2\left(2\frac{\mathrm{d}v}{\mathrm{d}x} + x\frac{\mathrm{d}^2v}{\mathrm{d}x^2}\right) - 8x\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (8 + 4x^2)(vx) = x^4 \\ \Rightarrow\quad & 8x^2\frac{\mathrm{d}v}{\mathrm{d}x} + 4x^3\frac{\mathrm{d}^2v}{\mathrm{d}x^2} - 8vx - 8x^2\frac{\mathrm{d}v}{\mathrm{d}x} + 8vx + 4vx^3 = x^4 \\ \Rightarrow\quad & 4x^3\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 4vx^3 = x^4 \\ \Rightarrow\quad & \underline{\underline{4\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 4v = x.}} \end{aligned}$$

(b) Solve the differential equation ($\ddagger$) to find v as a function of x. (6)

Solution

Complementary function: The characteristic equation is

$$4m^2 + 4 = 0 \Rightarrow m = \pm\mathrm{i}$$

and so the complementary function is

$$v = A\sin x + B\cos x.$$

Particular integral: For the particular solution, try $v = Cx + D$:

$$\frac{\mathrm{d}v}{\mathrm{d}x} = C, \frac{\mathrm{d}^2v}{\mathrm{d}x^2} = 0$$

and substitute:

$$0 + 4(Cx + D) \equiv x \Rightarrow C = \tfrac{1}{4}, D = 0.$$

General solution: Hence the general solution of (‡) is

$$\underline{\underline{v = A\sin x + B\cos x + \tfrac{1}{4}x}}.$$

(c) Hence state the general solution of the differential equation (†). (1)

Solution

$\underline{\underline{y = x(A\sin x + B\cos x + \tfrac{1}{4}x)}}$.

33. (a) Show that the substitution $x = \mathrm{e}^z$ transforms the differential equation (7)

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 3\ln x, x > 0, \qquad (\dagger)$$

into the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + \frac{\mathrm{d}y}{\mathrm{d}z} - 2y = 3z. \qquad (\ddagger)$$

Solution

This is a standard substitution and you need to know how to do this both correctly and quickly:

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}z} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \mathrm{e}^z = x\frac{\mathrm{d}y}{\mathrm{d}x}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}z^2} &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right) = \frac{\mathrm{d}}{\mathrm{d}z}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}\right) \times \frac{\mathrm{d}x}{\mathrm{d}z} \\ &= \left(\frac{\mathrm{d}y}{\mathrm{d}x} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) \times x \\ &= x\frac{\mathrm{d}y}{\mathrm{d}x} + x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}.\end{aligned}$$

Hence we can rewrite (†) as

$$\begin{aligned}x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 3\ln x &\Rightarrow \left[x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x}\right] + x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 3\ln x \\ &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}z^2} + \frac{\mathrm{d}y}{\mathrm{d}z} - 2y = 3\ln(\mathrm{e}^z) \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + \frac{\mathrm{d}y}{\mathrm{d}z} - 2y = 3z}}.\end{aligned}$$

(b) Find the general solution of the differential equation (‡). (6)

Solution

Complementary function: The characteristic equation is

$$m^2 + m - 2 = 0 \Rightarrow (m+2)(m-1) = 0 \Rightarrow m = -2 \text{ or } 1$$

and hence the complementary function is

$$y = A\mathrm{e}^{-2z} + B\mathrm{e}^{z}.$$

Particular integral: For the particular integral, try

$$y = Cz + D \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = C \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}z^2} = 0$$

and substitute into (‡):

$$0 + C - 2(Cz + D) = 3z \Rightarrow C = -\tfrac{3}{2}, D = -\tfrac{3}{4}.$$

General solution: Hence the general solution is

$$\underline{\underline{y = A\mathrm{e}^{-2z} + B\mathrm{e}^{z} - \tfrac{3}{2}z - \tfrac{3}{4}}}.$$

(c) Hence obtain the general solution of the differential equation (†) giving your answer in the form $y = \mathrm{f}(x)$. (1)

Solution

$$\begin{aligned} y &= A\mathrm{e}^{-2z} + B\mathrm{e}^{z} - \tfrac{3}{2}z - \tfrac{3}{4} \\ &= A\left(\mathrm{e}^{z}\right)^{-2} + B\left(\mathrm{e}^{z}\right) - \tfrac{3}{2}z - \tfrac{3}{4} \\ &= \underline{\underline{Ax^{-2} + Bx - \tfrac{3}{2}\ln x - \tfrac{3}{4}}}. \end{aligned}$$

34. (a) Find the general solution of the differential equation (6)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 27\mathrm{e}^{-x}.$$

Solution

Complementary function: The characteristic equation is

$$m^2 + 2m + 10 = 0 \Rightarrow m = -1 + 3\mathrm{i}$$

and hence the complementary function is

$$y = \mathrm{e}^{-x}(A\sin 3x + B\cos 3x).$$

Particular integral: For the particular integral, try $y = C\mathrm{e}^{-x}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -C\mathrm{e}^{-x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = C\mathrm{e}^{-x}$$

and substitute this into the differential equation:

$$C\mathrm{e}^{-x} - 2C\mathrm{e}^{-x} + 10C\mathrm{e}^{-x} = 27\mathrm{e}^{-x}$$

and hence $C = 3$. So the general solution is

$$\underline{\underline{y = \mathrm{e}^{-x}(A\sin 3x + B\cos 3x) + 3\mathrm{e}^{-x}}}.$$

(b) Find the particular solution that satisfies $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$. (6)

Solution

$x = 0 \Rightarrow B + 3 = 0 \Rightarrow B = -3.$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{e}^{-x}(A\sin 3x - 3\cos 3x) + \mathrm{e}^{-x}(3A\cos 3x + 3\sin 3x) - 3\mathrm{e}^{-x}$$

and so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 3 + 3A - 3 = 0 \Rightarrow A = 0$. So the particular solution is

$$\underline{\underline{y = 3\mathrm{e}^{-x}\cos 3x + 3\mathrm{e}^{-x}}}.$$

35. (a) Show that the transformation $x = \mathrm{e}^u$ transforms the differential equation (6)

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 7x\frac{\mathrm{d}y}{\mathrm{d}x} + 16y = 2\ln x,\ x > 0, \qquad \text{(I)}$$

into the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}u^2} - 8\frac{\mathrm{d}y}{\mathrm{d}u} + 16y = 2u \qquad \text{(II)}.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}u} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \mathrm{e}^u = x\frac{\mathrm{d}y}{\mathrm{d}x}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}u^2} &= \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathrm{d}y}{\mathrm{d}u}\right) = \frac{\mathrm{d}}{\mathrm{d}u}\left(x\frac{\mathrm{d}y}{\mathrm{d}u}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}\right) \times \frac{\mathrm{d}x}{\mathrm{d}u} \\ &= \left(\frac{\mathrm{d}y}{\mathrm{d}x} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) \times x \\ &= x\frac{\mathrm{d}y}{\mathrm{d}x} + x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}.\end{aligned}$$

Finally, $x = \mathrm{e}^u$ and so $u = \ln x$. Hence

$$\begin{aligned} & x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 7x \frac{\mathrm{d}y}{\mathrm{d}x} + 16y = 2 \ln x \\ \Rightarrow & \left(x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x \frac{\mathrm{d}y}{\mathrm{d}x} \right) - 8x \frac{\mathrm{d}y}{\mathrm{d}x} + 16y = 2 \ln x \\ \Rightarrow & \underline{\underline{\frac{\mathrm{d}^2 y}{\mathrm{d}u^2} - 8 \frac{\mathrm{d}y}{\mathrm{d}u} + 16y = 2u,}} \end{aligned}$$

as required.

(b) Find the general solution of the differential equation (II), expressing y as a function of u. (7)

Solution

Complementary function: The auxiliary equation is

$$m^2 - 8m + 16 = 0 \Rightarrow (m-4)^2 = 0 \Rightarrow m = 4,$$

and hence the complementary function is

$$y = (A + Bu)\mathrm{e}^{4u}.$$

Particular integral: For the particular integral, try

$$y = C + Du \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}u} = D \Rightarrow \frac{\mathrm{d}^2 y}{\mathrm{d}u^2} = 0.$$

Substitute into the differential equation:

$$0 - 8D + 16(C + Du) = 2u \Rightarrow D = \tfrac{1}{8}, C = \tfrac{1}{16}.$$

So the particular integral is $y = \frac{1}{4} + \frac{1}{2}u$.

General solution: Hence the general solution is

$$\underline{\underline{y = (A + Bu)\mathrm{e}^{4u} + \tfrac{1}{16} + \tfrac{1}{8}u.}}$$

(c) Hence obtain the general solution of the differential equation (I). (1)

Solution

$$y = (A + B\ln x)e^{4\ln x} + \tfrac{1}{16} + \tfrac{1}{8}\ln x = \underline{\underline{x^4(A + B\ln x) + \tfrac{1}{16} + \tfrac{1}{8}\ln x}}.$$

36. (a) Show that the transformation $x = e^u$ transforms the differential equation (6)

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = -x^{-2},\ x > 0, \qquad \text{(I)}$$

into the differential equation

$$\frac{d^2y}{du^2} - 3\frac{dy}{du} + 2y = -e^{-2u} \qquad \text{(II)}.$$

Solution

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = \frac{dy}{dx} \times e^u = x\frac{dy}{dx}$$

and

$$\begin{aligned}\frac{d^2y}{du^2} &= \frac{d}{du}\left(\frac{dy}{du}\right) = \frac{d}{du}\left(x\frac{dy}{du}\right)\\ &= \frac{d}{dx}\left(x\frac{dy}{dx}\right) \times \frac{dx}{du}\\ &= \left(\frac{dy}{dx} + x\frac{d^2y}{dx^2}\right) \times x\\ &= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}.\end{aligned}$$

Finally, $x = e^u$ and so $u = \ln x$. Hence

$$\begin{aligned}&x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = -x^{-2}\\ \Rightarrow &\left(x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}\right) - 3x\frac{dy}{dx} + 2y = -(e^u)^2\\ \Rightarrow &\underline{\underline{\frac{d^2y}{du^2} - 3\frac{dy}{du} + 2y = -e^{-2u}}},\end{aligned}$$

as required.

(b) Find the general solution of the differential equation (II). (7)

Solution

Complementary function: The characteristic equation is

$$\begin{aligned} m^2 - 3m + 2 = 0 &\Rightarrow (m-1)(m-2) = 0 \\ &\Rightarrow m = 1 \text{ or } m = 2 \end{aligned}$$

and so the complementary function is $y = A\mathrm{e}^{u} + B\mathrm{e}^{2u}$.
Particular integral: try $y = C\mathrm{e}^{-2u}$:

$$\frac{\mathrm{d}y}{\mathrm{d}u} = -2C\mathrm{e}^{-2u} \text{ and } \frac{\mathrm{d}y}{\mathrm{d}u} = 4C\mathrm{e}^{-2u}$$

and substitute into (‡):

$$(4C + 6C + 2C)\,\mathrm{e}^{-2u} = -\mathrm{e}^{-2u} \Rightarrow C = -\tfrac{1}{12}$$

and the particular integral is

$$y = -\tfrac{1}{12}\mathrm{e}^{-2u}.$$

General solution: Hence the general solution is

$$\underline{\underline{y = A\mathrm{e}^{u} + B\mathrm{e}^{2u} - \tfrac{1}{12}\mathrm{e}^{-2u}}}.$$

(c) Hence obtain the general solution of the differential equation (I) giving your answer in the form $y = \mathrm{f}(x)$. (1)

Solution

$$y = A\mathrm{e}^{u} + B\mathrm{e}^{2u} - \tfrac{1}{12}\mathrm{e}^{-2u} = \underline{\underline{Ax + Bx^2 - \tfrac{1}{12}x^{-2}}}.$$

37. (a) Find the general solution of the differential equation (8)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} = 26\sin 3x.$$

Solution

Complementary function: The characteristic equation is

$$m^2 - 2m = 0 \Rightarrow m(m-2) = 0 \Rightarrow m = 0 \text{ or } 2$$

and hence the complementary function is

$$y = A + Be^{2x}.$$

<u>Particular integral</u>: try $y = C\cos 3x + D\sin 3x$:

$$\frac{dy}{dx} = -3C\sin 3x + 3D\cos 3x \text{ and } \frac{d^2y}{dx^2} = -9C\cos 3x - 9D\sin 3x.$$

Now,

$$\begin{aligned} \underline{\sin 3x}: &\quad -9D + 6C = 26 \\ \underline{\cos 3x}: &\quad -9C - 6D = 0, \end{aligned}$$

and this gives $C = \frac{4}{3}$ and $D = -2$. The particular integral is

$$y = \tfrac{4}{3}\cos 3x - 2\sin 3x.$$

<u>General solution</u>: Hence the general solution is

$$\underline{\underline{y = A + Be^{2x} + \tfrac{4}{3}\cos 3x - 2\sin 3x.}}$$

(b) Find the particular solution of this differential equation for which $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$. (5)

Solution

$$x = 0, y = 0 \Rightarrow 0 = A + B + \tfrac{4}{3}$$

and

$$\begin{aligned} y = A + Be^{2x} + \tfrac{4}{3}\cos 3x - 2\sin 3x \Rightarrow \frac{dy}{dx} &= 2Be^{2x} - 4\sin 3x - 6\cos 3x \\ &\Rightarrow 0 = 2B - 6. \end{aligned}$$

Sove:

$$B = 3 \text{ and } A = -\tfrac{13}{3}$$

and the particular solution is

$$\underline{\underline{y = -\tfrac{13}{3} + 3e^{2x} + \tfrac{4}{3}\cos 3x - 2\sin 3x.}}$$