Core Mathematics 1

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Mathematics

The basics from GCSE

You are expected to be fluent in the following techniques from GCSE (you got at least an A grade after all . . .):

- multiplying out brackets where each may contain more that two terms,
- factorisation of algebraic expressions, including quadratics where the coefficient of x^2 is not 1, and spotting the difference of two squares when it occurs,
- writing a quadratic in completed square form,
- solving a quadratic equation using factorisation, the quadratic formula, completing the square, or using graphical methods,
- the laws of indices, including negative and fractional indices,
- surds, including simplifying them and rationalising the denominator,
- solving two linear equations in two variables by both algebraic and graphical methods,
- solving simultaneous equations in two variables where one equation is linear and the other is non-linear (such as a quadratic),
- solve linear and quadriatic inequalities, including simultaneous inequalities,
- sketch the graph of any quadratic,
- sketch the graph of any cubic that can be factorised into three (possibly repeated) brackets,
- sketch the graph of the reciprocal function,
- given the graph of $y = \mathrm{f}(x)$, sketch the graphs of the six related functions $y = a\,\mathrm{f}(x)$, $y = \mathrm{f}(ax)$, $y = \mathrm{f}(x + a)$, $y = \mathrm{f}(x) + a$, $y = -\,\mathrm{f}(x)$, and $y = \mathrm{f}(-x)$.

Discriminant of a quadratic

Given a quadratic expression $y = ax^2 + bx + c$ the *discriminant* is defined to be $b^2 - 4ac$. A quadratic will factorise with 'nice' numbers if $b^2 - 4ac$ is a perfect square but it also tells us about the number of solutions to the equation $ax^2 + bx + c = 0$:

$b^2 - 4ac > 0 \Rightarrow$ there are two real, distinct solutions
$b^2 - 4ac = 0 \Rightarrow$ there is one real, repeated solution
$b^2 - 4ac < 0 \Rightarrow$ there are no real solutions.

Straight lines

Although you will be familiar with straight lines of the form $y = mx+c$ where m is the gradient and c is the y-intercept this does not cover all cases (vertical lines — their gradient is undefined — are excluded). For this reason, you will often be asked to give a straight line in the form

$$ax + by + c = 0 \text{ or } ax + by = c,$$

often with the additional requirement that a, b, and c are integers.

Parallel and perpendicular lines

Two lines are parallel if and only if they have the same gradient; vertical lines, whose gradients are undefined, are obviously parallel. With the exception of horizontal and vertical lines (whose gradients are 0 and undefined respectively), if a line has gradient m then a line perpendicular to it has gradient $-\frac{1}{m}$.

Sequences

Given a formula for the n^{th} term of a sequence, perhaps as part of a recurrence relation, you should be able to find the numerical value of any term. Conversely, given a numerical value you should be able to determine whether this is a term in the sequence and, if so, which term it is.

Arithmetic sequence

An *arithmetic sequence* is one of the form a, $a+d$, $a + 2d$, . . . where a is the first term and d is the *common difference*. The n^{th} term of the series is

$$u_n = a + (n - 1)d$$

and the sum of the first n terms is

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + L),$$

where L is the last term in the sum. The proof of this results is a standard result and you must ensure that you can reproduce it in full in the examination. You may also be asked to find the minimum number of terms such that the sum first exceeds a given value, leading to a quadratic inequality.

Gradient of a curve

The *gradient of a curve* at a particular point is the gradient of the tangent to the curve at that point; you must be able to interpret this value as a rate of change. Graphical methods can give an approximation to this value but, for an exact value, we use differentiation.

Differentiation

Given a function $y = \mathrm{f}(x)$, the derivative, denoted by $f'(x)$ or $\frac{\mathrm{d}y}{\mathrm{d}x}$, is given by

$$\mathrm{f}'(x) = \lim_{h\to 0} \frac{\mathrm{f}(x + h) - f(x)}{h}.$$

An example: suppose $y = x^3$. In this case, the function is $\mathrm{f}(x) = x^3$ and hence

$$\begin{aligned}\mathrm{f}'(x) &= \lim_{h\to 0} \frac{(x + h)^3 - x^3}{h}\\ &= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}\\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^2}{h}\\ &= \lim_{h\to 0} (3x^2 + 3xh + h^2)\\ &= 3x^2.\end{aligned}$$

Hence, $\frac{\mathrm{d}}{\mathrm{d}x}(x^3) = 3x^2$. So, at any point (a, a^3) on the curve $y = x^3$, the gradient of the tangent to the curve is given by $3a^2$.

The rules for differentiation

There are three simple rules to follow and, possibly, one thing to do in order to interpret your answer in the context of the problem.

- begin by writing any expression in index form, i.e., as a linear combination of powers of x rather than as fractions, brackets, and so on,
- differentiate each part of the expression one term at a time: the derivative of a sum of terms is the sum of the derivatives of the individual terms,
- apply the rule $\frac{\mathrm{d}}{\mathrm{d}x}(ax^n) = nax^{n-1}$,
- if you calculate the derivative as a numerical value then you must be able to interpret that value as a rate of change (if required to do so).

Anti-differentiation

At the level of Core Mathematics 1, the *integral*

$$\int \mathrm{f}(x)\,\mathrm{d}x$$

is asking you to find a function whose derivative is $\mathrm{f}(x)$. For this reason, you will often see the term *anti-derivative* used to describe the result. An example: what is

$$\int 12x^2\,\mathrm{d}x?$$

We need to find a function whose derivative is $12x^2$. We know that the derivative of a cubic is a quadratic and, after some experimentation, we should be able to see that $\frac{\mathrm{d}}{\mathrm{d}x}(4x^3) = 12x^2$. But this answer is not unique:

$$\left.\begin{aligned}&\tfrac{\mathrm{d}}{\mathrm{d}x}(4x^3 + 5)\\ &\tfrac{\mathrm{d}}{\mathrm{d}x}(4x^3 - 17)\\ &\tfrac{\mathrm{d}}{\mathrm{d}x}(4x^3 + \pi)\end{aligned}\right\} = 12x^2,$$

and so we write

$$\int 12x^2\,\mathrm{d}x = 4x^3 + c$$

where c is an arbitrary constant.

The rules for integration

There are four simple rules to follow.

- begin by writing any expression in index form, i.e., as a linear combination of powers of x rather than as fractions, brackets, and so on,
- integrate each part of the expression one term at a time: the integral of a sum of terms is the sum of the integrals of the individual terms,
- apply the rule
$$\int ax^n\,\mathrm{d}x = \frac{a}{n+1}x^{n+1}\,(n \neq -1),$$
to each term, and
- then add on the arbitrary constant '$+c$'.

Differentiation and integration are not inverses

If you integrate a function and then differentiate the result then you will get back to where you started. If, on the other hand, you differentiate a function and the integrate the result you will only get to within an arbitrary constant of where you started.