Dr Oliver Mathematics
Mathematics
Differentiation Part 3
Past Examination Questions

This booklet consists of 53 questions across a variety of examination topics.
The total number of marks available is 439.

1. (a) Differentiate with respect to x:

   (i) $3\sin^2 x + \sec 2x$, (3)

   Solution

   $$\frac{\mathrm{d}}{\mathrm{d}x}(3\sin^2 x + \sec 2x) = \underline{\underline{6\sin x\cos x + 2\sec 2x\tan 2x}}.$$

   (ii) $[x + \ln(2x)]^3$. (3)

   Solution

   $$\frac{\mathrm{d}}{\mathrm{d}x}\left([x + \ln(2x)]^3\right) = \underline{\underline{3\,[x + \ln(2x)]^2\left(1 + \frac{1}{x}\right)}}.$$

   Given that $y = \dfrac{5x^2 - 10x + 9}{(x-1)^2}$, $x \neq 1$,

   (b) show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{8}{(x-1)^3}$. (6)

   Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(x-1)^2 \times (10x-10) - (5x^2-10x+9) \times 2(x-1)}{(x-1)^4} \\ &= \frac{10(x-1)^3 - 2(x-1)(5x^2-10x+9)}{(x-1)^4} \\ &= \frac{10(x-1)^2 - 2(5x^2-10x+9)}{(x-1)^3} \\ &= \frac{10(x^2-2x+1) - (10x^2-20x+18)}{(x-1)^3} \\ &= \frac{(10x^2-20x+10) - (10x^2-20x+18)}{(x-1)^3} \\ &= -\frac{8}{(x-1)^3}.\end{aligned}$$

2. (3)

$$\mathrm{f}(x) = 3\mathrm{e}^x - \tfrac{1}{2}\ln x - 2,\ x > 0.$$

Differentiate to find $\mathrm{f}'(x)$.

Solution

$$\mathrm{f}'(x) = 3\mathrm{e}^x - \frac{1}{2x}.$$

3. (a) Differentiate with respect to x:

(i) $x^2\mathrm{e}^{3x+2}$, (4)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2\mathrm{e}^{3x+2}) = 2x\mathrm{e}^{3x+2} + 3x^2\mathrm{e}^{3x+2}.$$

(ii) $\dfrac{\cos(2x^3)}{3x}$. (4)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\cos(2x^3)}{3x}\right] = \frac{-18x^3\sin(2x^3) - 3\cos(2x^3)}{9x^2}.$$

(b) Given that $x = 4\sin(2y+6)$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x. (5)

Solution

$$\begin{aligned} x = 4\sin(2y+6) &\Rightarrow 1 = 8\cos(2y+6)\,\frac{\mathrm{d}y}{\mathrm{d}x} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{8\cos(2y+6)} \\ &= \pm\frac{1}{8\sqrt{\cos^2(2y+6)}} \\ &= \pm\frac{1}{8\sqrt{1-\sin^2(2y+6)}} \\ &= \pm\frac{1}{8\sqrt{1-\frac{x^2}{16}}} \\ &= \pm\frac{1}{8\sqrt{\frac{1}{16}(16-x^2)}} \\ &= \underline{\underline{\pm\frac{1}{2\sqrt{16-x^2}}}}. \end{aligned}$$

4. Differentiate with respect to x:

(a) $\mathrm{e}^{3x} + \ln 2x$, (3)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{3x} + \ln 2x) = \underline{\underline{3\mathrm{e}^{3x} + \frac{1}{x}}}.$$

(b) $(5+x^2)^{\frac{3}{2}}$. (3)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(5+x^2)^{\frac{3}{2}}\right] = \underline{\underline{3x(5+x^2)^{\frac{1}{2}}}}.$$

5. The curve

$$y = (2x-1)\tan 2x,\ 0 \leqslant x < \tfrac{\pi}{4},$$

has a minimum at the point P. The x-coordinate of P is k.

Show that k satisfies the equation

$$4k + \sin 4k - 2 = 0.$$

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 2(2x-1)\sec^2 2x + 2\tan 2x = 0\\ &\Rightarrow \frac{2(2x-1)}{\cos^2 2x} + \frac{2\sin 2x}{\cos 2x} = 0\\ &\Rightarrow \frac{2(2x-1) + 2\sin 2x\cos 2x}{\cos^2 2x} = 0\\ &\Rightarrow 2(2x-1) + 2\sin 2x \cos 2x = 0\\ &\Rightarrow \underline{\underline{4k + \sin 4k - 2 = 0}}.\end{aligned}$$

6. The curve C has equation $x = 2\sin y$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{2}}$ at $P\left(\sqrt{2}, \frac{\pi}{4}\right)$. (4)

Solution

$$\begin{aligned}x = 2\sin y &\Rightarrow 1 = 2\cos y \frac{\mathrm{d}y}{\mathrm{d}x}\\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\cos y}.\end{aligned}$$

Now, at $P\left(\sqrt{2}, \frac{\pi}{4}\right)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\cos\frac{\pi}{4}} = \underline{\underline{\frac{1}{\sqrt{2}}}}.$$

(b) Find an equation of the normal to C at P. Give your answer in the form $y = mx + c$, where m and c are exact constants. (4)

Solution

$m' = -\sqrt{2}$ and we have

$$\begin{aligned} y - \frac{\pi}{4} = -\sqrt{2}(x - \sqrt{2}) &\Rightarrow y - \frac{\pi}{4} = -\sqrt{2}x + 2 \\ &\Rightarrow \underline{\underline{y = -\sqrt{2}x + 2 + \frac{\pi}{4}}}. \end{aligned}$$

7. (a) The curve C has equation (6)

$$y = \frac{x}{9 + x^2}.$$

Use calculus to find the coordinates of the turning points of C.

Solution

$$\begin{aligned} y = \frac{x}{9 + x^2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(9 + x^2) - 2x^2}{(9 + x^2)^2} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{9 - x^2}{(9 + x^2)^2} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 9 - x^2. \end{aligned}$$

Now,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 9 - x^2 = 0 \\ &\Rightarrow x^2 = 9 \\ &\Rightarrow x = \pm 3. \end{aligned}$$

So, the turning points of C are $\underline{\underline{(3, \frac{1}{6})}}$ and $\underline{\underline{(-3, -\frac{1}{6})}}$.

(b) Given that (5)

$$y = (1 + \mathrm{e}^{2x})^{\frac{3}{2}},$$

find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = \frac{1}{2}\ln 3$.

Solution

$$y = (1 + \mathrm{e}^{2x})^{\frac{3}{2}} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3\mathrm{e}^{2x}(1 + \mathrm{e}^{2x})^{\frac{1}{2}}$$

and

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=\frac{1}{2}\ln 3} = 3\mathrm{e}^{2(\frac{1}{2}\ln 3)}(1+\mathrm{e}^{2(\frac{1}{2}\ln 3)})^{\frac{1}{2}}$$
$$= 3\mathrm{e}^{\ln 3}(1+\mathrm{e}^{\ln 3})^{\frac{1}{2}}$$
$$= 3(3)(1+3)^{\frac{1}{2}}$$
$$= \underline{\underline{18}}.$$

8.

$$\mathrm{f}(x) = \frac{2x+3}{x+2} - \frac{9+2x}{2x^2+3x-2},\ x > \tfrac{1}{2}.$$

(a) Show that $\mathrm{f}(x) = \dfrac{4x-6}{2x-1}$. (7)

Solution

$$\begin{aligned}
\mathrm{f}(x) &= \frac{2x+3}{x+2} - \frac{9+2x}{2x^2+3x-2} \\
&= \frac{2x+3}{x+2} - \frac{9+2x}{(2x-1)(x+2)} \\
&= \frac{(2x+3)(2x-1)-(9+2x)}{(2x-1)(x+2)} \\
&= \frac{(4x^2+4x-3)-(9+2x)}{(2x-1)(x+2)} \\
&= \frac{4x^2+2x-12}{(2x-1)(x+2)} \\
&= \frac{(4x-6)(x+2)}{(2x-1)(x+2)} \\
&= \underline{\underline{\frac{4x-6}{2x-1}}}.
\end{aligned}$$

(b) Hence, or otherwise, find $\mathrm{f}'(x)$ in its simplest form. (3)

Solution

$$f(x) = \frac{2x+3}{x+2} - \frac{9+2x}{2x^2+3x-2} \Rightarrow f(x) = \frac{4x-6}{2x-1}$$
$$\Rightarrow f'(x) = \frac{4(2x-1) - 2(4x-6)}{(2x-1)^2}$$
$$\Rightarrow \underline{\underline{f'(x) = \frac{8}{(2x-1)^2}}}.$$

9. A curve C has equation
$$y = x^2 e^x.$$

(a) Find $\frac{dy}{dx}$, using the product rule for differenitation. (3)

Solution

$$y = x^2 e^x \Rightarrow \frac{dy}{dx} = x^2 e^x + 2xe^x = \underline{\underline{x(x+2)e^x}}.$$

(b) Hence find the coordinates of the turning points of C. (3)

Solution

$$\frac{dy}{dx} = 0 \Rightarrow x(x+2)e^x = 0$$
$$\Rightarrow x = -2 \text{ or } x = 0,$$

and we have $\underline{\underline{(-2, 4e^{-2})}}$ and $\underline{\underline{(0,0)}}$ as the turning points.

(c) Find $\frac{d^2y}{dx^2}$. (2)

Solution

$$\frac{dy}{dx} = x^2 e^x + 2xe^x \Rightarrow \frac{d^2y}{dx^2} = x^2 e^x + 2xe^x + 2xe^x + 2e^x$$
$$\Rightarrow \underline{\underline{\frac{d^2y}{dx^2} = (x^2 + 4x + 2)e^x}}.$$

(d) Determine the nature of each turning point of the curve C. (2)

Solution

If $x = -2$, then $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2\mathrm{e}^{-2} < 0$ and we have a $\underline{\underline{\text{maximum}}}$.

If $x = 0$, then $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2 > 0$ and we have a $\underline{\underline{\text{minimum}}}$.

10. (a) A curve C has equation (6)

$$y = \mathrm{e}^{2x}\tan x,\ x \neq (2n+1)\frac{\pi}{2}.$$

Show that the turning points on C where $\tan x = -1$.

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 2\mathrm{e}^{2x}\tan x + \mathrm{e}^{2x}\sec^2 x = 0\\ &\Rightarrow 2\tan x + \sec^2 x = 0\\ &\Rightarrow 2\tan x + (\tan^2 x + 1) = 0\\ &\Rightarrow (\tan x + 1)^2 = 0\\ &\Rightarrow \tan x + 1 = 0\\ &\Rightarrow \underline{\underline{\tan x = -1}}.\end{aligned}$$

(b) Find an equation of the tangent to C at the point where $x = 0$. (2)

Solution

$$x = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

and so we have

$$y - 0 = 1(x - 0) \Rightarrow \underline{\underline{y = x}}.$$

11. The point P lies on the curve with equation

$$y = 4\mathrm{e}^{2x+1}.$$

The y-coordinate of P is 8.

(a) Find, in terms of $\ln 2$, the x-coordinate of P. (2)

Solution

$$\begin{aligned} 4e^{2x+1} = 8 &\Rightarrow e^{2x+1} = 2 \\ &\Rightarrow 2x + 1 = \ln 2 \\ &\Rightarrow 2x = \ln 2 - 1 \\ &\Rightarrow \underline{\underline{x = \tfrac{1}{2}(\ln 2 - 1)}}. \end{aligned}$$

(b) Find an equation of the tangent to the curve at the point P in the form $y = ax + b$, where a and b are exact constants to be found. (4)

Solution

$$y = 4e^{2x+1} \Rightarrow \frac{dy}{dx} = 8e^{2x+1},$$

and, $x = \frac{1}{2}(\ln 2 - 1)$,

$$\frac{dy}{dx} = 8e^{2(\frac{1}{2}(\ln 2 - 1))+1} = 16.$$

Now,

$$\begin{aligned} y - 8 = 16(x - \tfrac{1}{2}(\ln 2 - 1) &\Rightarrow y - 8 = 16x - 8\ln 2 + 8 \\ &\Rightarrow \underline{\underline{y = 16x - 8\ln 2 + 16}}. \end{aligned}$$

12. (a) Differentiate with respect to x:

(i) $e^{3x}(\sin x + 2\cos x)$, (3)

Solution

$$\begin{aligned} \frac{d}{dx}\left[e^{3x}(\sin x + 2\cos x)\right] &= 3e^{3x}(\sin x + 2\cos x) + e^{3x}(\cos x - 2\sin x) \\ &= \underline{\underline{e^{3x}(\sin x + 7\cos x)}}. \end{aligned}$$

(ii) $x^3 \ln(5x + 2)$. (3)

Solution

$$\frac{d}{dx}\left[x^3 \ln(5x + 2)\right] = \underline{\underline{3x^2 \ln(5x + 2) + \frac{5x^3}{5x + 2}}}.$$

Given that $y = \dfrac{3x^2 + 6x - 7}{(x+1)^2}$, $x \neq -1$,,

(b) show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{20}{(x+1)^3}$. (5)

Solution

$$\begin{aligned} y = \frac{3x^2+6x-7}{(x+1)^2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(6x+6)(x+1)^2 - 2(x+1)(3x^2+6x-7)}{(x+1)^4} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(6x+6)(x+1) - 2(3x^2+6x-7)}{(x+1)^3} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(6x^2+12x+6) - (6x^2+12x-14)}{(x+1)^3} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{20}{(x+1)^3}. \end{aligned}$$

(c) Hence find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and the real values of x for which $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\dfrac{15}{4}$. (3)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{20}{(x+1)^3} \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{60}{(x+1)^4}$$

and

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{15}{4} &\Rightarrow -\frac{60}{(x+1)^4} = -\frac{15}{4} \\ &\Rightarrow (x+1)^4 = 16 \\ &\Rightarrow x+1 = \pm 2 \\ &\Rightarrow x = -3 \text{ or } x = 1. \end{aligned}$$

13. Find the equation of the tangent to the curve $x = \cos(2y + \pi)$ at $\left(0, \frac{\pi}{4}\right)$. (6)

Give your answer in the form $y = ax + b$, where a and b are constant to be found.

Solution

$$x = \cos(2y + \pi) \Rightarrow 1 = -2\sin(2y + \pi)\,\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2\sin(2y + \pi)}.$$

Now,

$$x = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{2}$$

and we have

$$y - \tfrac{\pi}{4} = \tfrac{1}{2}(x - 0) \Rightarrow \underline{\underline{y = \tfrac{1}{2}x + \tfrac{\pi}{4}}}.$$

14. (a) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point where $x = 2$ on the curve with equation (6)

$$y = x^2\sqrt{5x - 1}.$$

Solution

$$y = x^2\sqrt{5x - 1} \Rightarrow y = x^2(5x - 1)^{\frac{1}{2}}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2x(5x - 1)^{\frac{1}{2}} + \tfrac{5}{2}x^2(5x - 1)^{-\frac{1}{2}}.$$

When $x = 2$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2(2)(10 - 1)^{\frac{1}{2}} + \tfrac{5}{2}(2^2)(10 - 1)^{-\frac{1}{2}} = \underline{\underline{15\tfrac{1}{3}}}.$$

(b) Differentiate $\frac{\sin 2x}{x^2}$ with respect to x. (4)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin 2x}{x^2}\right] = \frac{2x^2\cos 2x - 2x\sin 2x}{x^4}$$
$$= \underline{\underline{\frac{2x\cos 2x - 2\sin 2x}{x^3}}}.$$

15. (5)

$$\mathrm{f}(x) = 3x\mathrm{e}^x - 1.$$

The curve with equation $y = \mathrm{f}(x)$ has a turning point P.

Find the exact coordinates of P.

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 3\mathrm{e}^x + 3x\mathrm{e}^x = 0\\ &\Rightarrow 3(1+x)\mathrm{e}^x = 0\\ &\Rightarrow 1 + x = 0\\ &\Rightarrow x = -1\\ &\Rightarrow y = -3\mathrm{e}^{-1} - 1,\end{aligned}$$

and the turning point is at $\underline{\underline{(-1, -3\mathrm{e}^{-1} - 1)}}$.

16.
$$\mathrm{f}(x) = \frac{2x+2}{x^2-2x-3} - \frac{x+1}{x-3}.$$

(a) Express $\mathrm{f}(x)$ as a single fraction in its simplest form. (4)

Solution

$$\begin{aligned}\mathrm{f}(x) &= \frac{2x+2}{x^2-2x-3} - \frac{x+1}{x-3}\\ &= \frac{2x+2}{(x-3)(x+1)} - \frac{x+1}{x-3}\\ &= \frac{2x+2-(x+1)^2}{(x-3)(x+1)}\\ &= \frac{2x+2-(x^2+2x+1)}{(x-3)(x+1)}\\ &= \frac{1-x^2}{(x-3)(x+1)}\\ &= \frac{(1-x)(1+x)}{(x-3)(x+1)}\\ &= \underline{\underline{\frac{1-x}{x-3}}}.\end{aligned}$$

(b) Hence show that $\mathrm{f}'(x) = \dfrac{2}{(x-3)^2}$. (3)

Solution

$$\mathrm{f}(x) = \frac{1-x}{x-3} \Rightarrow \mathrm{f}'(x) = \frac{-(x-3)-(1-x)}{(x-3)^2}$$
$$\Rightarrow \mathrm{f}'(x) = \underline{\underline{\frac{2}{(x-3)^2}}}.$$

17. (a) Differentiate with respect to x:

(i) $x^2 \cos 3x$, (3)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 \cos 3x) = \underline{\underline{2x \cos 3x - 3x^2 \sin 3x}}.$$

(ii) $\dfrac{\ln(x^2+1)}{x^2+1}$. (4)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\ln(x^2+1)}{x^2+1}\right] = \frac{(x^2+1) \times \frac{2x}{x^2+1} - 2x \times \ln(x^2+1)}{(x^2+1)^2}$$
$$= \underline{\underline{\frac{2x - 2x\ln(x^2+1)}{(x^2+1)^2}}}.$$

(b) A curve C has the equation (6)

$$y = \sqrt{4x+1},\ x > -\tfrac{1}{4},\ y > 0.$$

The point P on the curve has x-coordinate 2. Find an equation of the tangent to C at P in the form $ax + by + c = 0$, where a, b, and c are integers.

Solution

$$y = \sqrt{4x+1} \Rightarrow y = (4x+1)^{\frac{1}{2}}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2(4x+1)^{-\frac{1}{2}},$$

and, when $x = 2$, we have $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{3}$ in which case the equation is

$$y - 3 = \tfrac{2}{3}(x - 2) \Rightarrow 3y - 9 = 2x - 4$$
$$\Rightarrow \underline{\underline{2x - 3y + 5 = 0}}.$$

18. The function g is defined by

$$\mathrm{g}(x) = \frac{\mathrm{e}^x - 3}{\mathrm{e}^x - 2},\ x \in \mathbb{R},\ x \neq \ln 2.$$

(a) Differentiate $\mathrm{g}(x)$ to show that $\mathrm{g}'(x) = \frac{\mathrm{e}^x}{(\mathrm{e}^x - 2)^2}$. (3)

Solution

$$\begin{aligned}\mathrm{g}'(x) &= \frac{\mathrm{e}^x(\mathrm{e}^x - 2) - \mathrm{e}^x(\mathrm{e}^x - 3)}{(\mathrm{e}^x - 2)^2} \\ &= \underline{\underline{\frac{\mathrm{e}^x}{(\mathrm{e}^x - 2)^2}}}.\end{aligned}$$

(b) Find the exact values of x for which $\mathrm{g}'(x) = 1$. (4)

Solution

$$\begin{aligned}\mathrm{g}'(x) = 1 &\Rightarrow \frac{\mathrm{e}^x}{(\mathrm{e}^x - 2)^2} = 1 \\ &\Rightarrow \mathrm{e}^x = (\mathrm{e}^x - 2)^2 \\ &\Rightarrow \mathrm{e}^x = \mathrm{e}^{2x} - 4\mathrm{e}^x + 4 \\ &\Rightarrow \mathrm{e}^{2x} - 5\mathrm{e}^x + 4 = 0 \\ &\Rightarrow (\mathrm{e}^x - 1)(\mathrm{e}^x - 4) = 0 \\ &\Rightarrow \mathrm{e}^x - 1 = 0 \text{ or } \mathrm{e}^x - 4 = 0 \\ &\Rightarrow \mathrm{e}^x = 1 \text{ or } \mathrm{e}^x = 4 \\ &\Rightarrow \underline{\underline{x = 0}} \text{ or } \underline{\underline{x = \ln 4}}.\end{aligned}$$

19. (a) Given that $y = \frac{\ln(x^2 + 1)}{x}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$. (4)

Solution

$$y = \frac{\ln(x^2+1)}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x \times \frac{2x}{1+x^2} - \ln(x^2+1) \times 1}{x^2}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{2x^2}{1+x^2} - \ln(x^2+1)}{x^2}$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x^2 - (1+x^2)\ln(x^2+1)}{x^2(1+x^2)}}}.$$

(b) Given that $x = \tan y$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}$. (5)

Solution

$$x = \tan y \Rightarrow 1 = \sec^2 y \, \frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec^2 y}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+\tan^2 y}$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}}}.$$

20. (a) By writing $\sec x$ as $\frac{1}{\cos x}$, show that $\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$. (3)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\cos x}\right]$$
$$= \frac{0 - 1 \times (-\sin x)}{\cos^2 x}$$
$$= \frac{\sin x}{\cos^2 x}$$
$$= \frac{1}{\cos x}\,\frac{\sin x}{\cos x}$$
$$= \underline{\underline{\sec x \tan x}}.$$

Given that $y = \mathrm{e}^{2x} \sec 3x$,

(b) find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. (4)

Solution

$$y = \mathrm{e}^{2x} \sec 3x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^{2x} \sec 3x + 3\mathrm{e}^{2x} \sec 3x \tan 3x$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{2x} \sec 3x(2 + 3\tan 3x).$$

The curve with equation $y = \mathrm{e}^{2x} \sec 3x$, $-\frac{\pi}{6} < x < \frac{\pi}{6}$, has a minimum turning point at (a, b).

(c) Find the values of the constants a and b, giving your answers to 3 significant figures. (4)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \mathrm{e}^{2x} \sec 3x(2 + 3\tan 3x) = 0$$
$$\Rightarrow 2 + 3\tan 3x = 0$$
$$\Rightarrow \tan 3x = -\tfrac{2}{3}$$
$$\Rightarrow 3x = \arctan(-\tfrac{2}{3})$$
$$\Rightarrow x = \tfrac{1}{3}\arctan(-\tfrac{2}{3})$$
$$\Rightarrow x = -0.196\,000\,867\,8 \text{ (FCD)}$$
$$\Rightarrow y = 0.812\,093\,867\,1 \text{ (FCD)},$$

and so it is $(-0.196, 0.812)$ (3 sf).

21. A curve C has equation (7)

$$y = \frac{3}{(5 - 3x)^2},\ x \neq \tfrac{5}{3}.$$

The point P on C has x-coordinate 2. Find an equation of the normal to C at P in the form $ax + by + c = 0$, where a, b, and c are integers.

Solution

$$y = \frac{3}{(5-3x)^2} \Rightarrow y = 3(5-3x)^{-2}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 18(5-3x)^{-3}.$$

Now,

$$x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -18 \Rightarrow m = \tfrac{1}{18}$$

and we have

$$y - 3 = \tfrac{1}{18}(x-2) \Rightarrow 18y - 54 = x - 2$$
$$\Rightarrow \underline{\underline{x - 18y + 52 = 0}}.$$

22. Figure 1 shows a sketch of the curve with the equation $y = (2x^2 - 5x + 2)\mathrm{e}^{-x}$.

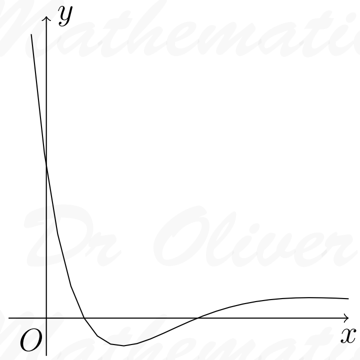

Figure 1: $y = (2x^2 - 5x + 2)\mathrm{e}^{-x}$

(a) Find the coordinates of the point C where C crosses the y-axis. (1)

Solution

$\underline{\underline{(0, 2)}}$.

(b) Show that C crosses the x-axis at $x = 2$ and find the x-coordinate of the other point where C crosses the x-axis. (3)

Solution

$$
\begin{aligned}
y = 0 &\Rightarrow (2x^2 - 5x + 2)\mathrm{e}^{-x} = 0\\
&\Rightarrow 2x^2 - 5x + 2 = 0\\
&\Rightarrow (2x - 1)(x - 2) = 0\\
&\Rightarrow 2x - 1 = 0 \text{ or } x - 2 = 0\\
&\Rightarrow \underline{\underline{x = \tfrac{1}{2}}} \text{ or } \underline{\underline{x = 2}}.
\end{aligned}
$$

(c) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. (3)

Solution

$$
\frac{\mathrm{d}y}{\mathrm{d}x} = -(2x^2 - 5x + 2)\mathrm{e}^{-x} + (4x - 5)\mathrm{e}^{-x} = \underline{\underline{(-2x^2 + 9x - 7)\mathrm{e}^{-x}}}.
$$

(d) Hence find the exact coordinates of the turning points of C. (5)

Solution

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow (-2x^2 + 9x - 7)\mathrm{e}^{-x} = 0\\
&\Rightarrow -2x^2 + 9x - 7 = 0\\
&\Rightarrow 2x^2 - 9x + 7 = 0\\
&\Rightarrow (2x - 7)(x - 1) = 0\\
&\Rightarrow 2x - 7 = 0 \text{ or } x - 1 = 0\\
&\Rightarrow x = \tfrac{7}{2} \text{ or } x = 1,
\end{aligned}
$$

and we have $\underline{\underline{(1, -\mathrm{e}^{-1})}}$ and $\underline{\underline{(\tfrac{7}{2}, 9\mathrm{e}^{-\frac{7}{2}})}}$.

23. Given that

$$
\mathrm{f}(x) = \frac{4x - 1}{2(x - 1)} - \frac{3}{2(x - 1)(2x - 1)} - 2,\ x > 1,
$$

(a) show that (6)

$$
\mathrm{f}(x) = \frac{3}{2x - 1}.
$$

Solution

$$\begin{aligned} f(x) &= \frac{4x-1}{2(x-1)} - \frac{3}{2(x-1)(2x-1)} - 2 \\ &= \frac{(4x-1)(2x-1) - 3 - 4(x-1)(2x-1)}{2(x-1)(2x-1)} \\ &= \frac{(8x^2-6x+1) - 3 - (8x^2-12x+4)}{2(x-1)(2x-1)} \\ &= \frac{6x-6}{2(x-1)(2x-1)} \\ &= \frac{6(x-1)}{2(x-1)(2x-1)} \\ &= \underline{\underline{\frac{3}{2x-1}}}. \end{aligned}$$

(b) Hence differentiate $f(x)$ and find $f'(2)$. (3)

Solution

$$f(x) = \frac{3}{2x-1} \Rightarrow f'(x) = \frac{0 - 3 \times 2}{(2x-1)^2} \Rightarrow f'(x) = -\frac{6}{(2x-1)^2},$$

and

$$x = 2 \Rightarrow \underline{\underline{f'(2) = -\tfrac{2}{3}}}.$$

24. (3)

$$f(x) = (8-x)\ln x, \; x > 0.$$

Find $f'(x)$.

Solution

$$f(x) = (8-x)\ln x \Rightarrow \underline{\underline{f'(x) = \frac{8-x}{x} - \ln x}}.$$

25. The curve C has equation

$$y = \frac{3 + \sin 2x}{2 + \cos 2x}.$$

(a) Show that (4)

$$\frac{dy}{dx} = \frac{6\sin 2x + 4\cos 2x + 2}{(2+\cos 2x)^2}.$$

Solution

$$\begin{aligned} y = \frac{3+\sin 2x}{2+\cos 2x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(2+\cos 2x)(2\cos 2x) - (3+\sin 2x)(-2\sin 2x)}{(2+\cos 2x)^2} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(4\cos 2x + 2\cos^2 2x) - (-6\sin 2x - 2\sin^2 2x)}{(2+\cos 2x)^2} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4\cos 2x + 2\cos^2 2x + 6\sin 2x + 2\sin^2 2x}{(2+\cos 2x)^2} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6\sin 2x + 4\cos 2x + 2(\cos^2 2x + \sin^2 2x)}{(2+\cos 2x)^2} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6\sin 2x + 4\cos 2x + 2}{(2+\cos 2x)^2}. \end{aligned}$$

(b) Find an equation of the tangent to C at the point on C where $x = \frac{\pi}{2}$. (4)
Writhe your answer in the form $y = ax + b$, where a and b are exact constants.

Solution

$$x = \tfrac{\pi}{2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{0-4+2}{(2-1)^2} = -2.$$

Now, $x = \frac{\pi}{2} \Rightarrow y = 3$ and we have

$$\begin{aligned} y - 3 = -2(x - \tfrac{\pi}{2}) &\Rightarrow y - 3 = -2x + \pi \\ &\Rightarrow y = -2x + \pi + 3. \end{aligned}$$

26. (a) Given that (3)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x,$$

show that $\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$.

Solution

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\cos x}\right] \\ &= \frac{0 - 1 \times (-\sin x)}{\cos^2 x} \\ &= \frac{\sin x}{\cos^2 x} \\ &= \frac{1}{\cos x}\,\frac{\sin x}{\cos x} \\ &= \underline{\underline{\sec x \tan x}}.\end{aligned}$$

Given that

$$x = \sec 2y,$$

(b) find $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y. (2)

Solution

$$x = \sec 2y \Rightarrow \underline{\underline{\frac{\mathrm{d}x}{\mathrm{d}y} = 2\sec 2y \tan 2y}}.$$

(c) Hence find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x. (4)

Solution

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}y} = 2\sec 2y \tan 2y \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2\sec 2y \tan 2y} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2\sec 2y\sqrt{\tan^2 2y}} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2\sec 2y\sqrt{\sec^2 2y - 1}} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \underline{\underline{\frac{1}{2x\sqrt{x^2 - 1}}}}.\end{aligned}$$

27. Differentiate with respect to x:

(a) $\ln(x^2 + 3x + 5)$, (2)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(x^2+3x+5)\right] = \underline{\underline{\frac{2x+3}{x^2+3x+5}}}.$$

(b) $\dfrac{\cos x}{x^2}$. (3)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\cos x}{x^2}\right] = \frac{-x^2\sin x - 2x\cos x}{x^4} = \underline{\underline{\frac{-x\sin x - 2\cos x}{x^3}}}.$$

28.

$$\mathrm{f}(x) = \frac{4x-5}{(2x+1)(x-3)} - \frac{2x}{x^2-9},\ x \neq \pm 3,\ x \neq -\tfrac{1}{2}.$$

(a) Show that (5)

$$\mathrm{f}(x) = \frac{5}{(2x+1)(x+3)}.$$

Solution

$$\begin{aligned}\mathrm{f}(x) &= \frac{4x-5}{(2x+1)(x-3)} - \frac{2x}{x^2-9}\\ &= \frac{4x-5}{(2x+1)(x-3)} - \frac{2x}{(x-3)(x+3)}\\ &= \frac{(4x-5)(x+3)-2x(2x+1)}{(2x+1)(x-3)(x+3)}\\ &= \frac{(4x^2+7x-15)-(4x^2+2x)}{(2x+1)(x-3)(x+3)}\\ &= \frac{5x-15}{(2x+1)(x-3)(x+3)}\\ &= \frac{5(x-3)}{(2x+1)(x-3)(x+3)}\\ &= \underline{\underline{\frac{5}{(2x+1)(x+3)}}}.\end{aligned}$$

The curve C has equation $y = \mathrm{f}(x)$. The point $P(-1, -\frac{5}{2})$ lies on C.

(b) Find an equation of the normal to C at P. (8)

Solution

$$\mathrm{f}(x) = \frac{5}{2x^2 + 7x + 3} \Rightarrow \mathrm{f}'(x) = -\frac{5(4x+7)}{(2x^2 + 7x + 3)^2},$$

and, $x = -1$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\tfrac{15}{4} \Rightarrow m' = \tfrac{4}{15}.$$

Now,

$$y + \tfrac{5}{2} = \tfrac{4}{15}(x+1) \Rightarrow \underline{\underline{y = \tfrac{4}{15}x - \tfrac{67}{30}}}.$$

29. Differentiate with respect to x, giving your answer in its simplest form:

(a) $x^2 \ln(3x)$, (4)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 \ln(3x)\right] = \underline{\underline{2x\ln(3x) + x}}.$$

(b) $\dfrac{\sin 4x}{x^3}$. (5)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin 4x}{x^3}\right] = \frac{4x^3 \cos 4x - 3x^2 \sin 4x}{x^6} = \underline{\underline{\frac{4x\cos 4x - 3\sin 4x}{x^4}}}.$$

30. The point P is the point on the curve $x = 2\tan(y + \frac{\pi}{12})$ with y-coordinate $\frac{\pi}{4}$. (7)

Find an equation of the normal to the curve at P.

Solution

$y = \frac{\pi}{4} \Rightarrow x = 2\sqrt{3}$. Now,

$$x = 2\tan(y + \tfrac{\pi}{12}) \Rightarrow 1 = 2\sec^2(y + \tfrac{\pi}{12})\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sec^2(y + \frac{\pi}{12})}.$$

When $y = \frac{\pi}{4}$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{8} \Rightarrow m' = -8$$

and so we have

$$y - \frac{\pi}{4} = -8(x - 2\sqrt{3}) \Rightarrow \underline{\underline{y = -8x + 16\sqrt{3} + \frac{\pi}{4}}}.$$

31. (4)

$$f(x) = x^2 - 3x + 2\cos(\tfrac{1}{2}x),\ 0 \leqslant x \leqslant \pi.$$

The curve with equation $y = f(x)$ has a minimum point P.

Show that the x-coordinate of P is the solution of the equation

$$x = \frac{3 + \sin(\frac{1}{2}x)}{2}.$$

Solution

$$\begin{aligned} \frac{dy}{dx} = 0 &\Rightarrow 2x - 3 - \sin(\tfrac{1}{2}x) = 0 \\ &\Rightarrow 2x = 3 + \sin(\tfrac{1}{2}x) \\ &\Rightarrow \underline{\underline{x = \frac{3 + \sin(\frac{1}{2}x)}{2}}}. \end{aligned}$$

32. Figure 2 shows a sketch of the curve C which has equation

$$y = e^{x\sqrt{3}} \sin 3x,\ -\tfrac{\pi}{3} \leqslant x \leqslant \tfrac{\pi}{3}.$$

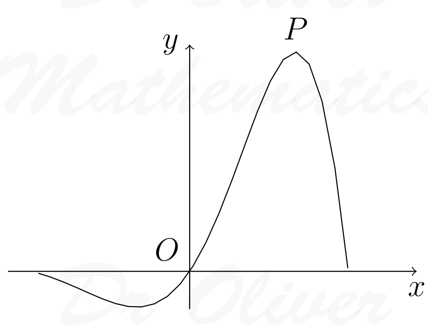

Figure 2: $y = e^{x\sqrt{3}} \sin 3x$

(a) Find the x-coordinate of the turning point P on C, for which $x > 0$. (6)
Give your answer as a multiple of π.

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow \sqrt{3}\mathrm{e}^{x\sqrt{3}}\sin 3x + 3\mathrm{e}^{x\sqrt{3}}\cos 3x = 0\\ &\Rightarrow \sqrt{3}\sin 3x = -3\cos 3x\\ &\Rightarrow \tan 3x = -\sqrt{3}\\ &\Rightarrow 3x = \tfrac{2\pi}{3}\\ &\Rightarrow \underline{\underline{x = \tfrac{2\pi}{9}}},\end{aligned}$$

as $3x = -\frac{\pi}{3}$ is not sensible.

(b) Find an equation of the normal to C at the point where $x = 0$. (3)

Solution

$$x = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3 \Rightarrow m' = -\tfrac{1}{3}$$

and

$$y - 0 = -\tfrac{1}{3}(x - 0) \Rightarrow \underline{\underline{y = -\tfrac{1}{3}x}}.$$

33. (a) Differentiate with respect to x:

(i) $x^{\frac{1}{2}}\ln(3x)$, (3)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{\frac{1}{2}}\ln(3x)\right] = \underline{\underline{\tfrac{1}{2}x^{-\frac{1}{2}}\ln(3x) + x^{-\frac{1}{2}}}}.$$

(ii) $\dfrac{1-10x}{(2x-1)^5}$, giving your answer in its simplest form. (3)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1-10x}{(2x-1)^5}\right] = \frac{-10(2x-1)^5 - 10(2x-1)^4(1-10x)}{(2x-1)^{10}}$$
$$= \frac{-10(2x-1) - 10(1-10x)}{(2x-1)^6}$$
$$= \underline{\underline{\frac{80x}{(2x-1)^6}}}.$$

(b) Given that $x = 3\tan 2y$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x. (5)

Solution

$$x = 3\tan 2y \Rightarrow 1 = 6\sec^2 2y \frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{6\sec^2 2y}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{6(\tan^2 2y + 1)}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{6\left[\left(\frac{x}{3}\right)^2 + 1\right]}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{6\left[\frac{x^2}{9} + 1\right]}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{2}{3}x^2 + 6}$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2x^2 + 18}}}.$$

34. The curve C has equation

$$y = (2x-3)^5.$$

The point P lies on C and has coordinates $(w, -32)$.

Find

(a) the value of w, (2)

Solution

$$(2x-3)^5 = -32 \Rightarrow 2x-3 = -2$$
$$\Rightarrow 2x = 1$$
$$\Rightarrow \underline{\underline{x = \tfrac{1}{2}}}.$$

(b) the equation of the tangent to C at the point P in the form $y = mx + c$, where m and c are constants. (5)

Solution

$$y = (2x-3)^5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 10(2x-3)^4,$$

and, when $x = \frac{1}{2}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 160$ and we have

$$y + 32 = 160(x - \tfrac{1}{2}) \Rightarrow \underline{\underline{y = 160x - 112}}.$$

35. (a) Differentiate with respect to x:

(i) $y = x^3 \ln 2x$, (3)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^3 \ln 2x) = \underline{\underline{3x^2 \ln 2x + x^2}}.$$

(ii) $(x + \sin 2x)^3$. (3)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(x + \sin 2x)^3\right] = \underline{\underline{3(x + \sin 2x)^2(1 + 2\cos 2x)}}.$$

Given that $x = \cot y$,

(b) show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{1+x^2}$. (5)

Solution

$$\begin{aligned}
x = \cot y \Rightarrow 1 &= -\operatorname{cosec}^2 y \,\frac{\mathrm{d}y}{\mathrm{d}x} \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{1}{\operatorname{cosec}^2 y} \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{1}{1+\cot^2 y} \\
\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{1+x^2}}}.
\end{aligned}$$

36.

$$\mathrm{f}(x) = \frac{2}{x+2} + \frac{4}{x^2+5} - \frac{18}{(x^2+5)(x+2)},\ x \geqslant 0.$$

(a) Show that $\mathrm{f}(x) = \dfrac{2x}{x^2+5}$. (4)

Solution

$$\begin{aligned}
\mathrm{f}(x) &= \frac{2}{x+2} + \frac{4}{x^2+5} - \frac{18}{(x^2+5)(x+2)} \\
&= \frac{2(x^2+5) + 4(x+2) - 18}{(x^2+5)(x+2)} \\
&= \frac{(2x^2+10) + (4x+8) - 18}{(x^2+5)(x+2)} \\
&= \frac{2x^2+4x}{(x^2+5)(x+2)} \\
&= \frac{2x(x+2)}{(x^2+5)(x+2)} \\
&= \underline{\underline{\frac{2x}{x^2+5}}}.
\end{aligned}$$

(b) Hence, or otherwise, find $\mathrm{f}'(x)$ in its simplest form. (3)

Solution

$$\begin{aligned} f(x) &= \frac{2}{x+2} + \frac{4}{x^2+5} - \frac{18}{(x^2+5)(x+2)} \\ \Rightarrow \quad f(x) &= \frac{2x}{x^2+5} \\ \Rightarrow \quad f'(x) &= \frac{2(x^2+5) - 4x^2}{(x^2+5)^2} \\ \Rightarrow \quad \underline{\underline{f'(x) = \frac{10-2x^2}{(x^2+5)^2}}}. \end{aligned}$$

Figure 3 shows a graph of the curve with equation $y = f(x)$.

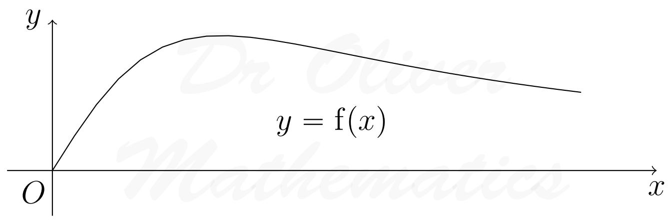

Figure 3: $f(x) = \frac{2}{x+2} - \frac{4}{x^2+5} - \frac{18}{(x^2+5)(x+2)}$

(c) Calculate the range of $f(x)$. (5)

Solution

$$\begin{aligned} f'(x) = 0 &\Rightarrow \frac{10-2x^2}{(x^2+5)^2} = 0 \\ &\Rightarrow 10 - 2x^2 = 0 \\ &\Rightarrow 2x^2 = 10 \\ &\Rightarrow x^2 = 5 \\ &\Rightarrow x = \sqrt{5} \text{ (as } x = -\sqrt{5} \text{ is not a valid solution)} \\ &\Rightarrow y = \tfrac{\sqrt{5}}{5}, \end{aligned}$$

and the range is $\underline{\underline{0 \leqslant f(x) \leqslant \tfrac{\sqrt{5}}{5}}}$.

37. (5)

$$f(x) = 25x^2 e^{2x} - 16, \; x \in \mathbb{R}.$$

Using calculus, find the exact coordinates of the turning points on the curve with equation $y = \mathrm{f}(x)$.

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 50x\mathrm{e}^{2x} + 50x^2\mathrm{e}^{2x} = 0\\ &\Rightarrow 50x(1+x)\mathrm{e}^{2x} = 0\\ &\Rightarrow x(1+x) = 0\\ &\Rightarrow x = -1 \text{ or } x = 0,\end{aligned}$$

and the turning points are $\underline{\underline{(-1, 25\mathrm{e}^{-2} - 16)}}$ and $\underline{\underline{(0, -16)}}$.

38. Given that

$$x = \sec^2 3y,\ 0 < y < \tfrac{\pi}{6},$$

(a) find $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ in terms of y. (2)

Solution

$$x = \sec^2 3y \Rightarrow \underline{\underline{\frac{\mathrm{d}x}{\mathrm{d}y} = 6\sec^2 3y \tan 3y}}.$$

(b) Hence show that (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{6x(x-1)^{\frac{1}{2}}}.$$

Solution

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}y} = 6\sec^2 3y \tan 3y &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{6\sec^2 3y \tan 3y}\\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{6\sec^2 3y\sqrt{\tan^2 3y}}\\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{6\sec^2 3y\sqrt{\sec^2 3y - 1}}\\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{6x(x-1)^{\frac{1}{2}}}}}.\end{aligned}$$

(c) Find an expression for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of x. Give your answer in its simplest form. (4)

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{6x(x-1)^{\frac{1}{2}}} \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{0 - \left[6(x-1)^{\frac{1}{2}} + 3x(x-1)^{-\frac{1}{2}}\right]}{36x^2(x-1)} \\ \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\frac{6(x-1)+3x}{36x^2(x-1)^{\frac{1}{2}}} \\ \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\frac{9x-6}{36x^2(x-1)^{\frac{1}{2}}} \\ \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{6-9x}{36x^2(x-1)^{\frac{1}{2}}} \\ \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{2-3x}{12x^2(x-1)^{\frac{1}{2}}}.\end{aligned}$$

39. (a) Differentiate (3)

$$\frac{\cos 2x}{\sqrt{x}}$$

with respect to x.

Solution

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\cos 2x}{\sqrt{x}}\right] &= \frac{-2x^{\frac{1}{2}}\sin 2x - \frac{1}{2}x^{-\frac{1}{2}}\cos 2x}{x} \\ &= \frac{-4x\sin 2x - \cos 2x}{2x^{\frac{3}{2}}}.\end{aligned}$$

(b) Show that $\frac{\mathrm{d}}{\mathrm{d}x}(\sec^2 3x)$ can be written in the form (3)

$$\mu(\tan 3x + \tan^3 3x)$$

where μ is a constant.

Solution

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\sec^2 3x) &= 6\sec^2 3x \tan 3x \\ &= 6(\tan^2 3x + 1)\tan 3x \\ &= \underline{\underline{6(\tan 3x + \tan^3 3x)}}.\end{aligned}$$

(c) Given that $x = 2\sin\left(\frac{y}{3}\right)$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x, simplifying your answer. (4)

Solution

$$\begin{aligned}x = 2\sin\left(\frac{y}{3}\right) &\Rightarrow 1 = \tfrac{2}{3}\cos\left(\frac{y}{3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2\cos\left(\frac{y}{3}\right)} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2\sqrt{1-\sin^2\left(\frac{y}{3}\right)}} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2\sqrt{1-(\frac{x}{2})^2}} \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{\sqrt{4-x^2}}}}.\end{aligned}$$

40. Figure 4 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = (x^2 + 3x + 1)\mathrm{e}^{x^2}.$$

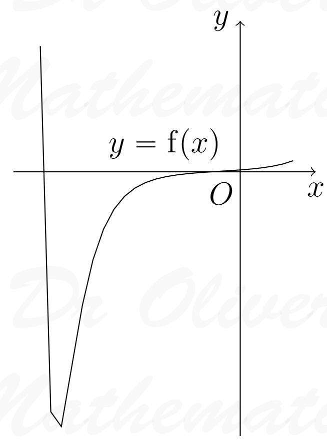

Figure 4: $f(x) = (x^2 + 3x + 1)e^{x^2}$

The curve cuts the x-axis at the points A and B.

(a) Calculate the x-coordinate of A and the x-coordinate of B, giving your answers to 3 decimal places. (2)

Solution

$$\begin{aligned} f(x) = 0 &\Rightarrow (x^2 + 3x + 1)e^{x^2} = 0 \\ &\Rightarrow x^2 + 3x + 1 = 0 \\ &\Rightarrow x^2 + 3x + 2.25 = 1.25 \\ &\Rightarrow (x + 1.5)^2 = 1.25 \\ &\Rightarrow x = \frac{-3 \pm \sqrt{5}}{2} \\ &\Rightarrow \underline{\underline{x = -2.618\ -0.382\ (3\text{ dp})}}. \end{aligned}$$

(b) Find $f'(x)$. (3)

Solution

$$\begin{aligned} f(x) = (x^2 + 3x + 1)e^{x^2} &\Rightarrow f'(x) = (2x + 3)e^{x^2} + 2x(x^2 + 3x + 1)e^{x^2} \\ &\Rightarrow \underline{\underline{f'(x) = (2x^3 + 6x^2 + 4x + 3)e^{x^2}}}. \end{aligned}$$

The curve has a minimum turning point at the point P.

(c) Show that the x-coordinate of P is the solution of (3)

$$x = -\frac{3(2x^2+1)}{2(x^2+2)}.$$

Solution

$$\begin{aligned} f'(x) = 0 &\Rightarrow (2x^3 + 6x^2 + 4x + 3)e^{x^2} = 0 \\ &\Rightarrow 2x^3 + 6x^2 + 4x + 3 = 0 \\ &\Rightarrow 2x^3 + 4x = -6x^2 - 3 \\ &\Rightarrow 2x(x^2+2) = -3(2x^2+1) \\ &\Rightarrow \underline{\underline{x = -\frac{3(2x^2+1)}{2(x^2+2)}}}. \end{aligned}$$

41. The curve C has equation $y = f(x)$ where

$$f(x) = \frac{4x+1}{x-2},\ x > 2.$$

(a) Show that (3)

$$f'(x) = -\frac{9}{(x-2)^2}.$$

Solution

$$\begin{aligned} f(x) = \frac{4x+1}{x-2} &\Rightarrow f'(x) = \frac{4(x-2)-(4x+1)}{(x-2)^2} \\ &\Rightarrow \underline{\underline{f'(x) = -\frac{9}{(x-2)^2}}}. \end{aligned}$$

Given that P is a point on C such that $f'(x) = -1$,

(b) find the coordinates of P. (3)

Solution

$$
\begin{aligned}
f'(x) = -1 &\Rightarrow -\frac{9}{(x-2)^2} = -1 \\
&\Rightarrow (x-2)^2 = 9 \\
&\Rightarrow x - 2 = 3 \\
&\Rightarrow x = 5 \\
&\Rightarrow y = 7,
\end{aligned}
$$

and so we have $\underline{\underline{(5, 7)}}$.

42. The curve C has equation $x = 8y \tan 2y$.

The point P has coordinates $(\pi, \frac{\pi}{8})$.

(a) Verify that P lies on C. (1)

Solution

When $y = \frac{\pi}{8}$, $x = 8 \times \frac{\pi}{8} \times \tan \frac{\pi}{4} = \underline{\underline{\pi}}$.

(b) Find the equation of the tangent to C at P in the form $ay = x + b$, where the constants a and b are to be found in terms of π. (7)

Solution

$$
\begin{aligned}
x = 8y \tan 2y &\Rightarrow 1 = 8 \tan 2y \frac{\mathrm{d}y}{\mathrm{d}x} + 16y \sec^2 2y \frac{\mathrm{d}y}{\mathrm{d}x} \\
&\Rightarrow 1 = (8 \tan 2y + 16y \sec^2 2y) \frac{\mathrm{d}y}{\mathrm{d}x} \\
&\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{8(\tan 2y + 2y \sec^2 2y)}.
\end{aligned}
$$

Now,

$$
x = \pi \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{8(1 + \frac{\pi}{4} \sec^2 \frac{\pi}{4})} = \frac{1}{8(1 + \frac{\pi}{2})} = \frac{1}{8 + 4\pi},
$$

and

$$
\begin{aligned}
y - \tfrac{\pi}{8} = \frac{1}{8 + 4\pi}(x - \pi) &\Rightarrow (8 + 4\pi)(y - \tfrac{\pi}{8}) = x - \pi \\
&\Rightarrow (8 + 4\pi)y - \pi - \frac{\pi^2}{2} = x - \pi \\
&\Rightarrow \underline{\underline{(8 + 4\pi)y = x + \frac{\pi^2}{2}}}.
\end{aligned}
$$

43. The curve C has equation (4)

$$y = 2\cos(\tfrac{1}{2}x^2) + x^3 - 3x - 2$$

and has a minimum turning point P.

Show that the x-coordinate of P is a solution of the equation

$$x = \sqrt{1 + \tfrac{2}{3}x\sin(\tfrac{1}{2}x^2)}.$$

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow -2x\sin(\tfrac{1}{2}x^2) + 3x^2 - 3 = 0\\ &\Rightarrow 3x^2 = 3 + 2x\sin(\tfrac{1}{2}x^2)\\ &\Rightarrow x^2 = 1 + \tfrac{2}{3}x\sin(\tfrac{1}{2}x^2)\\ &\Rightarrow \underline{\underline{x = \sqrt{1 + \tfrac{2}{3}x\sin(\tfrac{1}{2}x^2)}}}.\end{aligned}$$

44. A curve C has equation $y = \mathrm{e}^{4x} + x^4 + 8x + 5$.

(a) Show that the x-coordinate of any turning point of C satisfies the equation (3)

$$x^3 = -2 - \mathrm{e}^{4x}.$$

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 4\mathrm{e}^{4x} + 4x^3 + 8 = 0\\ &\Rightarrow \mathrm{e}^{4x} + x^3 + 2 = 0\\ &\Rightarrow \underline{\underline{x^3 = -2 - \mathrm{e}^{4x}}}.\end{aligned}$$

(b) Sketch, on a single diagram, the curves with equation (4)

(i) $y = x^3$,

(ii) $y = -2 - \mathrm{e}^{4x}$. (2)

On your diagram give the coordinates of the points where each curve crosses the y-axis and state the equation of any asymptotes.

Solution

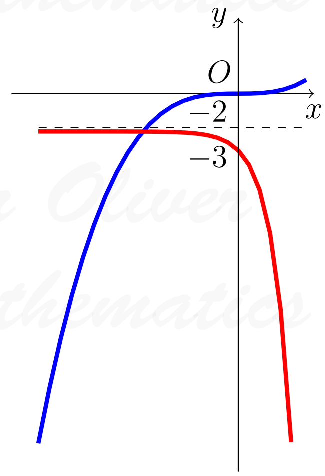

Figure 5: $y = x^3$ and $y = -2 - \mathrm{e}^{4x}$

(c) Explain how your diagram illustrates that the equation $x^3 = -2 - \mathrm{e}^{4x}$ has only one root. (1)

Solution

They only intersect at one point so clearly <u>there is one root</u>.

45. (a) Given that (4)

$$x = \sec 2y,\ 0 < y < \tfrac{\pi}{4},$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{4x\sqrt{x-1}}.$$

Solution

$$
\begin{aligned}
x = \sec 2y \Rightarrow 1 &= 4\sec 2y \tan 2y \frac{\mathrm{d}y}{\mathrm{d}x} \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{4\sec 2y \tan 2y} \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{4\sec 2y\sqrt{\tan^2 2y}} \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{4\sec 2y\sqrt{\sec^2 2y - 1}} \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{4x\sqrt{x-1}}.
\end{aligned}
$$

(b) Given that (5)

$$y = (x^2 + x^3)\ln 2x,$$

find the exact value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = \frac{\mathrm{e}}{2}$, giving your answer in its simplest form.

Solution

$$
\begin{aligned}
y = (x^2 + x^3)\ln 2x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= (2x + 3x^2)\ln 2x + (x^2 + x^3)\frac{1}{x} \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= (2x + 3x^2)\ln 2x + x + x^2.
\end{aligned}
$$

Now,

$$
\begin{aligned}
x = \tfrac{\mathrm{e}}{2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= (\mathrm{e} + 3(\tfrac{\mathrm{e}}{2})^2)\ln e + \tfrac{\mathrm{e}}{2} + (\tfrac{\mathrm{e}}{2})^2 \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \mathrm{e} + \tfrac{3\mathrm{e}^2}{4} + \tfrac{\mathrm{e}}{2} + \tfrac{\mathrm{e}^2}{4} \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \mathrm{e}^2 + \tfrac{3\mathrm{e}}{2}.
\end{aligned}
$$

(c) Given that (3)

$$\mathrm{f}(x) = \frac{3\cos x}{(x+1)^{\frac{1}{3}}},\ x \neq -1,$$

show that

$$\mathrm{f}'(x) = \frac{\mathrm{g}(x)}{(x+1)^{\frac{4}{3}}},\ x \neq -1,$$

where $g(x)$ is an expression to be found.

Solution

$$\mathrm{f}(x) = \frac{3\cos x}{(x+1)^{\frac{1}{3}}} \Rightarrow \mathrm{f}'(x) = \frac{-3(x+1)^{\frac{1}{3}}\sin x - (x+1)^{-\frac{2}{3}}\cos x}{(x+1)^{\frac{2}{3}}}$$

$$\Rightarrow \mathrm{f}'(x) = \frac{-3(x+1)\sin x - \cos x}{(x+1)^{\frac{4}{3}}}.$$

46. The point P lies on the curve with equation

$$x = (4y - \sin 2y)^2.$$

Given that P has (x, y) coordinates $(p, \frac{\pi}{2})$, where p is a constant,

(a) find the exact value of p. (1)

Solution

$$y = \tfrac{\pi}{2} \Rightarrow x = (2\pi - \sin\pi)^2 = \underline{\underline{4\pi^2}}.$$

The tangent to the curve at P cuts the y-axis at the point A.

(b) Use calculus to find the coordinates of A. (6)

Solution

$$x = (4y - \sin 2y)^2 \Rightarrow 1 = 2(4y - \sin 2y)(4 - 2\cos 2y)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2(4y - \sin 2y)(4 - 2\cos 2y)}.$$

Now,

$$\tfrac{\pi}{2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2(2\pi - \sin\pi)(4 - 2\cos\pi)} = \frac{1}{24\pi}$$

and

$$y - \tfrac{\pi}{2} = \frac{1}{24\pi}(x - 4\pi^2).$$

Finally,

$$x = 0 \Rightarrow y - \tfrac{\pi}{2} = -\tfrac{\pi}{6} \Rightarrow \underline{\underline{y = \tfrac{\pi}{3}}}.$$

47. Figure 6 shows a sketch of part of the curve with equation

$$g(x) = x^2(1-x)e^{-2x},\ x \leqslant 0.$$

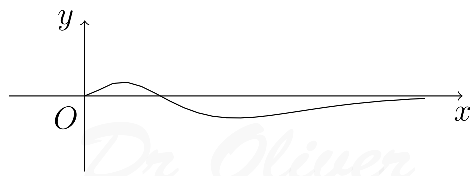

Figure 6: $g(x) = x^2(1-x)e^{-2x}$

(a) Show that $g'(x) = f(x)e^{-2x}$, where $f(x)$ is a cubic function to be found. (3)

Solution

We will apply a bit of Leibniz's Theorem!

$$\begin{aligned} g'(x) &= \left[\frac{d}{dx}(x^2)\right](1-x)e^{-2x} + x^2\left[\frac{d}{dx}(1-x)\right]e^{-2x} + x^2(1-x)\left[\frac{d}{dx}(e^{-2x})\right] \\ &= 2x(1-x)e^{-2x} - x^2e^{-2x} - 2x^2(1-x)e^{-2x} \\ &= \left[2x(1-x) - x^2 - 2x^2(1-x)\right]e^{-2x} \\ &= \left[(2x-2x^2) - x^2 - (2x^2-2x^3)\right]e^{-2x} \\ &= \underline{\underline{(2x^3-5x^2+2x)e^{-2x}}}. \end{aligned}$$

(b) Hence find the range of g. (6)

Solution

$$\begin{aligned} g'(x) = 0 &\Rightarrow (2x^3-5x^2+2x)e^{-2x} = 0 \\ &\Rightarrow 2x^3-5x^2+2x = 0 \\ &\Rightarrow x(2x^2-5x+2) = 0 \\ &\Rightarrow x(2x-1)(x-2) = 0 \\ &\Rightarrow x = 0, 2x-1 = 0, \text{ or } x-2 = 0 \\ &\Rightarrow x = 0, x = \tfrac{1}{2}, \text{ or } x = 2. \end{aligned}$$

Now,

$$g(0) = 0, g(\tfrac{1}{2}) = \tfrac{1}{8}e^{-1}, \text{ and } g(2) = -4e^{-4}$$

and hence

$$\underline{\underline{-4e^{-4} \leqslant g(x) \leqslant \tfrac{1}{8}e^{-1}}}.$$

(c) State a reason why the function $\mathrm{g}^{-1}(x)$ does not exist. (1)

Solution

Because $\underline{\underline{\mathrm{g}(x) \text{ is not a one-to-one function}}}$.

48. Given that k is a **negative** constant and that the function $\mathrm{f}(x)$ is defined by

$$\mathrm{f}(x) = 2 - \frac{(x-5k)(x-k)}{x^2 - 3kx + 2k^2},\ x \geqslant 0,$$

(a) show that $\mathrm{f}(x) = \dfrac{x+k}{x-2k}$. (3)

Solution

$$\begin{aligned}\mathrm{f}(x) &= 2 - \frac{(x-5k)(x-k)}{x^2 - 3kx + 2k^2} \\ &= 2 - \frac{(x-5k)(x-k)}{(x-2k)(x-k)} \\ &= 2 - \frac{x-5k}{x-2k} \\ &= \frac{2(x-2k)-(x-5k)}{x-2k} \\ &= \underline{\underline{\frac{x+k}{x-2k}}}.\end{aligned}$$

(b) Hence find $\mathrm{f}'(x)$, giving your answer in its simplest form. (3)

Solution

$$\mathrm{f}(x) = \frac{x+k}{x-2k} \Rightarrow \mathrm{f}'(x) = \frac{(x-2k)-(x+k)}{(x-2k)^2}$$
$$\Rightarrow \underline{\underline{\mathrm{f}'(x) = -\frac{3k}{(x-2k)^2}}}.$$

(c) State, with a reason, whether $\mathrm{f}(x)$ is an increasing or decreasing function. Justify your answer. (2)

Solution

$f(x)$ is an <u>increasing function</u> as $f(x) > 0$.

49.

$$y = \frac{4x}{x^2+5}.$$

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$, writing your answer as a single fraction in its simplest form. (4)

Solution

$$y = \frac{4x}{x^2+5} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4(x^2+5) - 8x^2}{(x^2+5)^2}$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{20-4x^2}{(x^2+5)^2}}}.$$

(b) Hence find the set of values of x for which $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$. (3)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} < 0 \Rightarrow \frac{20-4x^2}{(x^2+5)^2} < 0$$
$$\Rightarrow 20 - 4x^2 < 0$$
$$\Rightarrow x^2 > 5$$
$$\Rightarrow \underline{\underline{|x| > \sqrt{5}}}.$$

50. (a) Find, using calculus, the x-coordinate of the turning point of the curve with equation (5)

$$y = \mathrm{e}^{3x}\cos 4x,\ \tfrac{\pi}{4} \leqslant x < \tfrac{\pi}{2}.$$

Give your answer to 4 decimal places.

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 3\mathrm{e}^{3x}\cos 4x - 4\mathrm{e}^{3x}\sin 4x = 0 \\ &\Rightarrow 3\cos 4x = 4\sin 4x \\ &\Rightarrow \tan 4x = \tfrac{3}{4} \\ &\Rightarrow 4x = \pi + \arctan\tfrac{3}{4} \text{ (why?)} \\ &\Rightarrow x = \tfrac{1}{4}(\pi + \arctan\tfrac{3}{4}) \\ &\Rightarrow x = 0.946\,273\,440\,6 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 0.946\,3 \text{ (4 dp)}}}.\end{aligned}$$

(b) Given $x = \sin^2 2y$, $0 < y < \frac{\pi}{4}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a function of y. (5)

Write your answer in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p\operatorname{cosec}(qy),$$

where p and q are constants to be determined.

Solution

$$\begin{aligned}x = \sin^2 2y &\Rightarrow 1 = 4\sin 2y\cos 2y\,\frac{\mathrm{d}y}{\mathrm{d}x} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{4\sin 2y\cos 2y} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sin 4y} \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{2}\operatorname{cosec} 4y}}.\end{aligned}$$

51.

$$\mathrm{f}(x) = \frac{x^4 + x^3 - 3x^2 + 7x - 6}{x^2 + x - 6},\ x > 2,\ x \in \mathbb{R}.$$

(a) Given that (4)

$$\frac{x^4 + x^3 - 3x^2 + 7x - 6}{x^2 + x - 6} \equiv x^2 + A + \frac{b}{x - 2},$$

find the values of the constants A and B.

Solution

$$\begin{aligned}\frac{x^4+x^3-3x^2+7x-6}{x^2+x-6} &\equiv \frac{x^2(x^2+x-6)+3x^2+7x-6}{x^2+x-6}\\ &\equiv x^2+\frac{3x^2+7x-6}{x^2+x-6}\\ &\equiv x^2+\frac{3(x^2+x-6)+4x+12}{x^2+x-6}\\ &\equiv x^2+3+\frac{4x+12}{x^2+x-6}\\ &\equiv x^2+3+\frac{4(x+3)}{(x+3)(x-2)}\\ &\equiv \underline{\underline{x^2+3+\frac{4}{x-2}}}.\end{aligned}$$

(b) Hence, or otherwise, using calculus, find an equation of the normal to the curve with equation $y = \mathrm{f}(x)$ at the point where $x = 3$. (5)

Solution

$$\mathrm{f}(x) = x^2 + 3 + \frac{4}{x-2} \Rightarrow \mathrm{f}'(x) = 2x - \frac{4}{(x-2)^2}.$$

Now,

$$x = 3 \Rightarrow y = 16 \Rightarrow \mathrm{f}'(x) = 2 \Rightarrow m' = -\tfrac{1}{2}$$

and

$$y - 16 = -\tfrac{1}{2}(x-3) \Rightarrow \underline{\underline{y = -\tfrac{1}{2}x + \tfrac{35}{2}}}.$$

52. Figure 7 shows a sketch of part of the curve C with equation

$$y = 2\ln(2x+5) - \tfrac{3}{2}x,\ x > -2.5.$$

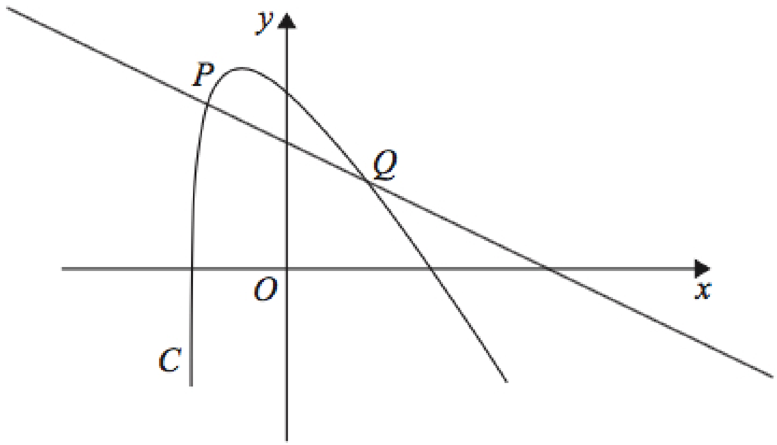

Figure 7: $y = 2\ln(2x+5) - \frac{3}{2}x$

The point P with x-coordinate -2 lies on C. Find an equation of the normal to C at P. Write your answer in the form $ax + by = c$, where a, b, and c are integers.

Solution

$$y = 2\ln(2x+5) - \tfrac{3}{2}x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4}{2x+5} - \tfrac{3}{2}$$
$$\Rightarrow \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=-2} = \tfrac{5}{2}$$

and so we have

$$m = -\tfrac{2}{5}.$$

Now,

$$x = -2 \Rightarrow y = 3$$

and

$$\begin{aligned} y - 3 = -\tfrac{2}{5}(x+2) &\Rightarrow 5y - 15 = -2(x+2) \\ &\Rightarrow 5y - 15 = -2x - 4 \\ &\Rightarrow \underline{\underline{2x + 5y = 11}}. \end{aligned}$$

53. (a) Given $y = 2x(x^2-1)^5$, show that

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{g}(x)(x^2-1)^4$, where $\mathrm{g}(x)$ is a function to be determined. (4)

Solution

$$\begin{aligned} y = 2x(x^2-1)^5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 2(x^2-1)^5 + 20x^2(x^2-1)^4 \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= (x^2-1)^4[2(x^2-1)+20x^2] \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= (22x^2-2)(x^2-1)^4. \end{aligned}$$

(ii) Hence find the set of values of x for which $\frac{\mathrm{d}y}{\mathrm{d}x} \geqslant 0$. (2)

Solution

The critical values are at $x = \pm 1$ (which can be dismissed — why?) and $x = \pm\frac{\sqrt{11}}{11}$ and

$$x \leqslant -\tfrac{\sqrt{11}}{11} \text{ or } x \geqslant \tfrac{\sqrt{11}}{11}.$$

(b) Given (4)

$$x = \ln(\sec 2y),\ 0 < y < \tfrac{\pi}{4},$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a function of x in its simplest form.

Solution

$$\begin{aligned} x = \ln(\sec 2y) \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} &= \frac{2\sec 2y \tan 2y}{\sec 2y} \\ \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} &= 2\tan 2y \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2\tan 2y} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2\sqrt{\sec^2 2y - 1}} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2\sqrt{(\mathrm{e}^x)^2 - 1}} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2\sqrt{\mathrm{e}^{2x} - 1}}. \end{aligned}$$