# Dr Oliver Mathematics
# Advance Level Mathematics
# Core Mathematics 1: Non-Calculator
# 1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.

1. (a) Simplify (2)

$$\sqrt{48} - \frac{6}{\sqrt{3}}.$$

Write your answer in the form $a\sqrt{3}$, where $a$ is an integer to be found.

**Solution**

$$\begin{aligned}
\sqrt{48} - \frac{6}{\sqrt{3}} &= \sqrt{16 \times 3} - \frac{2 \times 3}{\sqrt{3}} \\
&= \sqrt{16} \times \sqrt{3} - \frac{2 \times 3}{\sqrt{3}} \\
&= 4\sqrt{3} - 2\sqrt{3} \\
&= \underline{\underline{2\sqrt{3}}}.
\end{aligned}$$

(b) Solve the equation (3)

$$3^{6x-3} = 81.$$

Write your answer as a rational number.

**Solution**

$$\begin{aligned}
3^{6x-3} = 81 &\Rightarrow 3^{6x-3} = 3^4 \\
&\Rightarrow 6x - 3 = 4 \\
&\Rightarrow 6x = 7 \\
&\Rightarrow \underline{\underline{x = \tfrac{7}{6}}}.
\end{aligned}$$

2. Given

$$y = 3\sqrt{x} - 6x + 4,\ x > 0,$$

(a) find $\int y \, \mathrm{d}x$, simplifying each term. (3)

**Solution**

$$\begin{aligned}\int (3\sqrt{x} - 6x + 4) \, \mathrm{d}x &= \int (3x^{\frac{1}{2}} - 6x + 4) \, \mathrm{d}x \\ &= \underline{\underline{2x^{\frac{3}{2}} - 3x^2 + 4x + c}}.\end{aligned}$$

(b) (i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. (4)

**Solution**

$$\frac{\mathrm{d}}{\mathrm{d}x}(3x^{\frac{1}{2}} - 6x + 4) = \underline{\underline{\tfrac{3}{2}x^{-\frac{1}{2}} - 6}}.$$

(ii) Hence find the value of $x$ such that $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

**Solution**

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow \tfrac{3}{2}x^{-\frac{1}{2}} - 6 = 0 \\ &\Rightarrow \tfrac{3}{2}x^{-\frac{1}{2}} = 6 \\ &\Rightarrow x^{-\frac{1}{2}} = 4 \\ &\Rightarrow x^{\frac{1}{2}} = \tfrac{1}{4} \\ &\Rightarrow \underline{\underline{x = \tfrac{1}{16}}}.\end{aligned}$$

3.

$$\mathrm{f}(x) = x^2 - 10x + 23.$$

(a) Express $\mathrm{f}(x)$ in the form (2)

$$(x + a)^2 + b,$$

where $a$ and $b$ are constants to be found.

**Solution**

$$\begin{aligned}x^2 - 10x + 23 &= (x^2 - 10x + 25) - 2 \\ &= \underline{\underline{(x - 5)^2 - 2}};\end{aligned}$$

hence, $\underline{\underline{a = -5}}$ and $\underline{\underline{b = -2}}$.

(b) Hence, or otherwise, find the exact solutions to the equation (2)

$$x^2 - 10x + 23 = 0.$$

**Solution**

$$\begin{aligned} x^2 - 10x + 23 = 0 &\Rightarrow (x-5)^2 - 2 = 0 \\ &\Rightarrow (x-5)^2 = 2 \\ &\Rightarrow x - 5 = \pm\sqrt{2} \\ &\Rightarrow \underline{\underline{x = 5 \pm \sqrt{2}}}. \end{aligned}$$

(c) Use your answer to part (b) to find the larger solution to the equation (2)

$$y - 10y^{0.5} + 23 = 0.$$

Write your solution in the form $p + q\sqrt{r}$, where $p$, $q$, and $r$ are integers.

**Solution**

$$\begin{aligned} (5+\sqrt{2})^2 &= 25 + 10\sqrt{2} + 2 \\ &= \underline{\underline{27 + 10\sqrt{2}}}; \end{aligned}$$

hence, $\underline{\underline{p = 27}}$, $\underline{\underline{p = 10}}$, and $\underline{\underline{r = 2}}$.

4. Each year, Andy pays into a savings scheme. In year one he pays in £600. His payments increase by £120 each year so that he pays £720 in year two, £840 in year three and so on, so that his payments form an arithmetic sequence.

(a) Find out how much Andy pays into the savings scheme in year ten. (2)

**Solution**

$a = 600$ and $d = 120$:

$$\begin{aligned} \text{Year 10} &= 600 + 9 \times 120 \\ &= 600 + 1\,080 \\ &= \underline{\underline{\pounds 1\,680}}. \end{aligned}$$

Kim starts paying money into a different savings scheme at the same time as Andy. In year one she pays in £130. Her payments increase each year so that she pays £210 in year two, £290 in year three and so on, so that her payments form a different arithmetic sequence.

At the end of year $N$, Andy has paid, in total, twice as much money into his savings scheme as Kim has paid, in total, into her savings scheme.

(b) Find the value of $N$. (5)

**Solution**

$$\begin{aligned}
& \tfrac{1}{2}N[2 \times 600 + 120(N-1)] = 2 \times \tfrac{1}{2}N[2 \times 130 + 80(N-1)] \\
\Rightarrow\ & 600N + 60N(N-1) = 260N + 80N(N-1) \\
\Rightarrow\ & 600N + 60N^2 - 60N = 260N + 80N^2 - 80N \\
\Rightarrow\ & 540N + 60N^2 = 180N + 80N^2 \\
\Rightarrow\ & 20N^2 - 360N = 0 \\
\Rightarrow\ & 20N(N-18) = 0 \\
\Rightarrow\ & N = 0 \text{ or } \underline{\underline{N = 18 \text{ years}}}.
\end{aligned}$$

5. Figure 1 shows the sketch of a curve with equation $y = \mathrm{f}(x), x \in \mathbb{R}$.

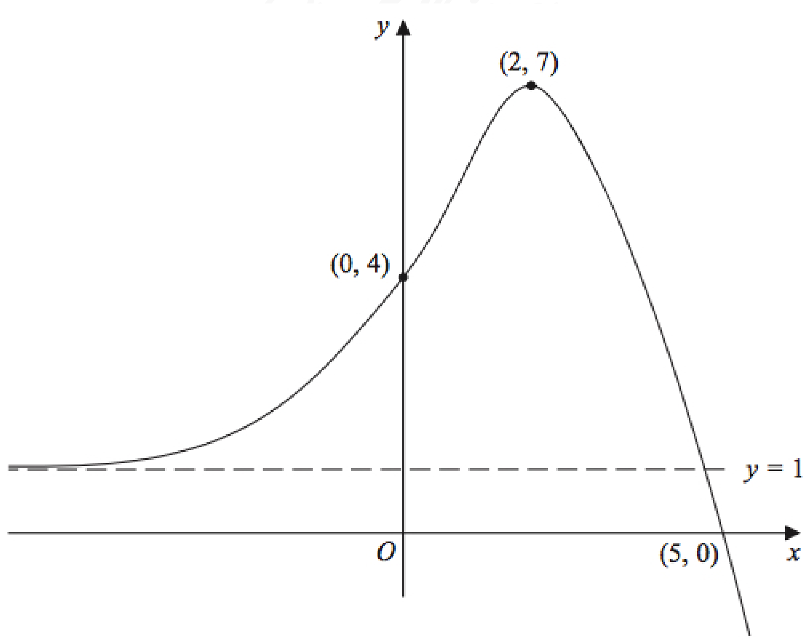



Figure 1: $y = \mathrm{f}(x), x \in \mathbb{R}$

The curve crosses the $y$-axis at $(0, 4)$ and crosses the $x$-axis at $(5, 0)$.
The curve has a single turning point, a maximum, at $(2, 7)$.
The line with equation $y = 1$ is the only asymptote to the curve.

(a) State the coordinates of the turning point on the curve with equation $y = \mathrm{f}(x - 2)$. (1)

**Solution**
$\underline{\underline{(4, 7)}}$.

(b) State the solution of the equation $\mathrm{f}(2x) = 0$. (1)

**Solution**
$\underline{\underline{x = \frac{5}{2}}}$.

(c) State the equation of the asymptote to the curve with equation $y = \mathrm{f}(-x)$. (1)

**Solution**
Rotate it in the $y$-direction: $\underline{\underline{y = 1}}$.

Given that the line with equation $y = k$, where $k$ is a constant, meets the curve $y = \mathrm{f}(x)$ at only one point,

(d) state the set of possible values for $k$. (2)

**Solution**
$\underline{\underline{k \leqslant 1}}$ or $\underline{\underline{k = 7}}$.

6. A sequence $a_1$, $a_2$, $a_3$, ... is defined by

$$a_1 = 4,$$
$$a_{n+1} = \frac{a_n}{a_n + 1}, \; n \geqslant 1, \; n \in \mathbb{N}.$$

(a) Find the values of $a_2$, $a_3$, and $a_4$.
Write your answers as simplified fractions. (3)

**Solution**

$$
\begin{aligned}
a_2 &= \frac{4}{4+1} = \underline{\underline{\tfrac{4}{5}}}. \\
a_3 &= \frac{\frac{4}{5}}{\frac{4}{5}+1} \\
&= \frac{\frac{4}{5}}{\frac{9}{5}} \\
&= \underline{\underline{\tfrac{4}{9}}}. \\
a_3 &= \frac{\frac{4}{9}}{\frac{4}{9}+1} \\
&= \frac{\frac{4}{9}}{\frac{13}{9}} \\
&= \underline{\underline{\tfrac{4}{13}}}.
\end{aligned}
$$

Given that

$$a_n = \frac{4}{pn+q},$$

where $p$ and $q$ are constants,

(b) state the value of $p$ and the value of $q$. (2)

**Solution**

Compare the bottom lines of $a_2$ and $a_3$:

$$2p + q = 5 \text{ and } 3p + q = 9.$$

Subtract:

$$\underline{\underline{p = 4 \text{ and } q = -3}}.$$

(c) Hence calculate the value of $N$ such that $a_N = \frac{4}{321}$. (2)

**Solution**

$$
\begin{aligned}
4N - 3 = 321 &\Rightarrow 4N = 324 \\
&\Rightarrow \underline{\underline{N = 81}}.
\end{aligned}
$$

7. The equation

$$20x^2 = 4kx - 13kx^2 + 2,$$

where $k$ is a constant, has no real roots.

(a) Show that $k$ satisfies the inequality (4)

$$2k^2 + 13k + 20 < 0.$$

**Solution**

$$20x^2 = 4kx - 13kx^2 + 2 \Rightarrow (20 + 13k)x^2 - 4kx - 2 = 0.$$

Now, '$b^2 - 4ac < 0$':

$$\begin{aligned} & (-4k)^2 - 4 \times (20 + 13k) \times (-2) < 0 \\ \Rightarrow \quad & 16k^2 + 8(20 + 13k) < 0 \\ \Rightarrow \quad & 16k^2 + 104k + 160 < 0 \\ \Rightarrow \quad & \underline{\underline{2k^2 + 13k + 20 < 0}}, \end{aligned}$$

as required.

(b) Find the set of possible values for $k$. (4)

**Solution**

$$\left.\begin{array}{ll} \text{add to:} & +13 \\ \text{multiply to:} & (+2) \times (+20) = +40 \end{array}\right\} +5, +8$$

$$\begin{aligned} 2k^2 + 13k + 20 < 0 & \Rightarrow 2k^2 + 5k + 8k + 20 < 0 \\ & \Rightarrow k(2k + 5) + 4(2k + 5) < 0 \\ & \Rightarrow (2k + 5)(k + 4) < 0 \\ & \Rightarrow \underline{\underline{-4 < k < -\tfrac{5}{2}}}. \end{aligned}$$

8. Figure 2 shows the straight line $l_1$ with equation $4y = 5x + 12$.

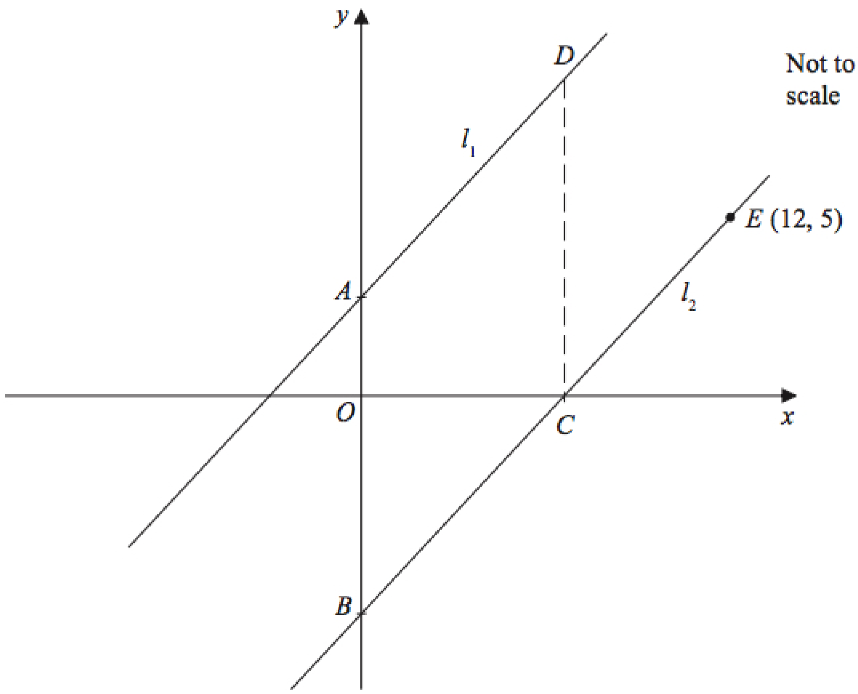



Figure 2: two straight lines

(a) State the gradient of $l_1$. (1)

**Solution**

$$4y = 5x + 12 \Rightarrow y = \tfrac{5}{4}x + 3$$

so the gradient is $\underline{\underline{\tfrac{5}{4}}}$.

The line $l_2$ is parallel to $l_1$ and passes through the point $E(12, 5)$, as shown in Figure 2.

(b) Find the equation of $l_2$. Write your answer in the form $y = mx + c$, where $m$ and $c$ are constants to be determined. (3)

**Solution**

$$\begin{aligned} y - 5 = \tfrac{5}{4}(x - 12) &\Rightarrow y - 5 = \tfrac{5}{4}x - 15 \\ &\Rightarrow \underline{\underline{y = \tfrac{5}{4}x - 10}}. \end{aligned}$$

The line $l_2$ cuts the $x$-axis at the point $C$ and the $y$-axis at the point $B$.

(c) Find the coordinates of (2)

(i) the point $B$,

**Solution**
$\underline{\underline{B(0,-10)}}$.

(ii) the point $C$.

**Solution**

$$\begin{aligned} y = 0 &\Rightarrow \tfrac{5}{4}x - 10 = 0 \\ &\Rightarrow \tfrac{5}{4}x = 10 \\ &\Rightarrow \tfrac{1}{4}x = 2 \\ &\Rightarrow x = 8, \end{aligned}$$

and $\underline{\underline{C(8,0)}}$.

The line $l_1$ cuts the $y$-axis at the point $A$.

The point $D$ lies on $l_1$ such that $ABCD$ is a parallelogram, as shown in Figure 2.

(d) Find the area of $ABCD$. (2)

**Solution**

$$\begin{aligned} \text{Area} &= 8 \times (\tfrac{5}{4} \times 8 + 3) \\ &= 8 \times (10 + 3) \\ &= 8 \times 13 \\ &= \underline{\underline{104 \text{ units}^2}}. \end{aligned}$$

9. The curve $C$ has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}'(x) = (x-3)(3x+5).$$

Given that the point $P(1,20)$ lies on $C$,

(a) find $\mathrm{f}(x)$, simplifying each term. (5)

**Solution**

$$\begin{aligned} \mathrm{f}'(x) = (x-3)(3x+5) &\Rightarrow \mathrm{f}'(x) = 3x^2 - 4x - 15 \\ &\Rightarrow \underline{\underline{\mathrm{f}(x) = x^3 - 2x^2 - 15x + c}}. \end{aligned}$$

(b) Show that (3)

$$f(x) = (x-3)^2(x+A),$$

where $A$ is a constant to be found.

**Solution**

Now, $P(1, 20)$ lies on $C$ so

$$20 = 1 - 2 - 15 + c \Rightarrow c = 36$$

and

$$f(x) = x^3 - 2x^2 - 15x + 36.$$

Well, we can do a bit synthetic division (twice):

$$\begin{array}{c|cccc} 3 & 1 & -2 & -15 & 36 \\ & \downarrow & 3 & 3 & -36 \\ \hline & 1 & 1 & -12 & 0 \end{array}$$

$$\begin{array}{c|ccc} 3 & 1 & 1 & -12 \\ & \downarrow & 3 & 12 \\ \hline & 1 & 4 & 0 \end{array}$$

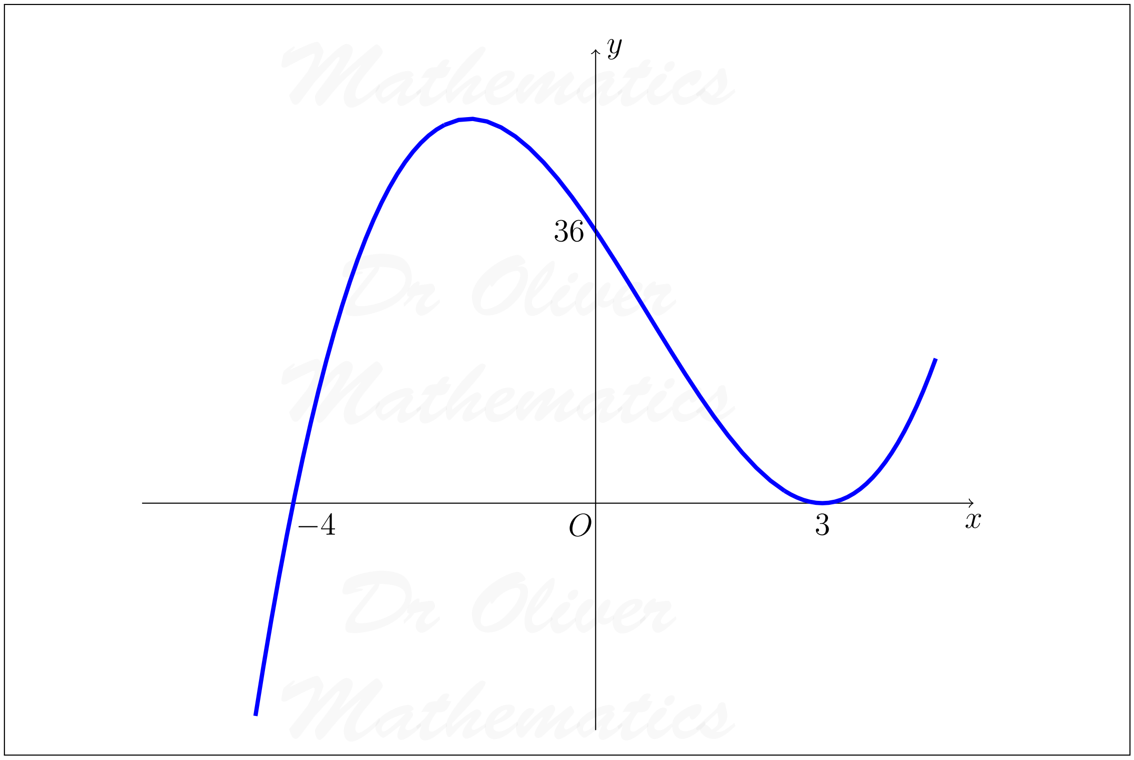
So,

$$\begin{aligned} f(x) &= (x-3)(x-3)(x+4) \\ &= \underline{\underline{(x-3)^2(x+4)}}; \end{aligned}$$

hence, $\underline{\underline{A = 4}}$.

(c) Sketch the graph of $C$. Show clearly the coordinates of the points where $C$ cuts or meets the $x$-axis and where $C$ cuts the $y$-axis. (4)

**Solution**




10. Figure 3 shows a sketch of part of the curve $C$ with equation

$$y = \tfrac{1}{2}x + \frac{27}{x} - 12,\ x > 0.$$

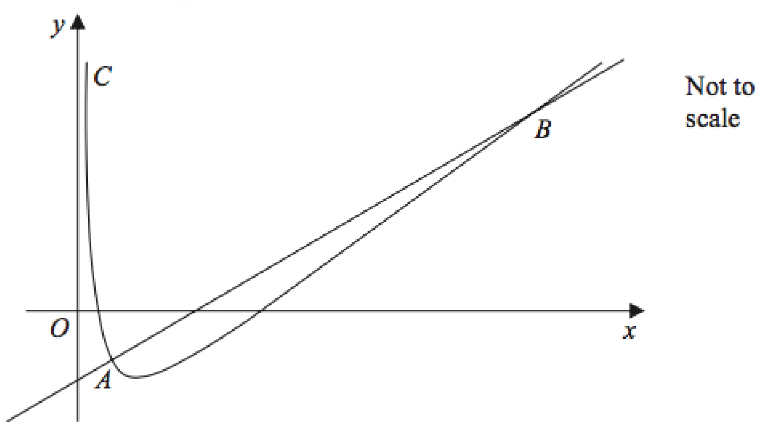



Figure 3: $y = \frac{1}{2}x + \frac{27}{x} - 12,\ x > 0$

The point $A$ lies on $C$ and has coordinates $(3, -\frac{3}{2})$.

(a) Show that the equation of the normal to $C$ at $A$ can be written as (5)

$$10y = 4x - 27.$$

**Solution**

$$y = \tfrac{1}{2}x + \frac{27}{x} - 12 \Rightarrow y = \tfrac{1}{2}x + 27x^{-1} - 12$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{2} - 27x^{-2}.$$

At $x = 3$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{2} - 3 = -\tfrac{5}{2}$$

and the gradient of the normal is $\frac{2}{5}$. Finally, the equation of the normal is

$$y + \tfrac{3}{2} = \tfrac{2}{5}(x - 3) \Rightarrow 10y + 15 = 4(x - 3)$$
$$\Rightarrow 10y + 15 = 4x - 12$$
$$\Rightarrow \underline{\underline{10y = 4x - 27}},$$

as required.

The normal to $C$ at $A$ meets $C$ again at the point $B$, as shown in Figure 3.

(b) Use algebra to find the coordinates of $B$. (5)

**Solution**

$$10y = 4x - 27 \Rightarrow y = \frac{4x - 27}{10}$$

and

$$\tfrac{1}{2}x + \frac{27}{x} - 12 = \frac{4x - 27}{10}$$
$$\times(10x) \Rightarrow \quad 5x^2 + 270 - 120x = 4x^2 - 27x$$
$$\Rightarrow \quad x^2 - 93x + 270 = 0$$

$$\left.\begin{array}{ll}\text{add to:} & -93 \\ \text{multiply to:} & (+1) \times (+270) = +270\end{array}\right\} -90,\ -3$$

$$\Rightarrow \quad (x - 3)(x - 90) = 0$$
$$\Rightarrow \quad x = 3 \text{ or } x = 90.$$

Finally,

$$x = 90 \Rightarrow y = \tfrac{2}{5} \times 90 - \tfrac{27}{10} = 33.3;$$

hence, $\underline{\underline{B(90, 33.3)}}$.