Dr Oliver Mathematics
Mathematics: Advanced Higher
2019 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. (a) Differentiate (2)

$$f(x) = x^6 \cot 5x.$$

   (b) Given (3)

$$y = \frac{2x^3 + 1}{x^3 - 4},$$

   find $\frac{dy}{dx}$. Simplify your answer.

   (c) For (3)

$$f(x) = \cos^{-1} 2x,$$

   evaluate $f'(\frac{\sqrt{3}}{4})$.

2. Matrix **A** is defined by

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 \\ -3 & p & 2 \\ -1 & -2 & 5 \end{pmatrix},$$

   where $p \in \mathbb{R}$.

   (a) Given that the determinant of **A** is 3, find the value of p. (3)

   Matrix **B** is defined by

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ q & 3 \\ 4 & 0 \end{pmatrix},$$

   where $q \in \mathbb{R}$.

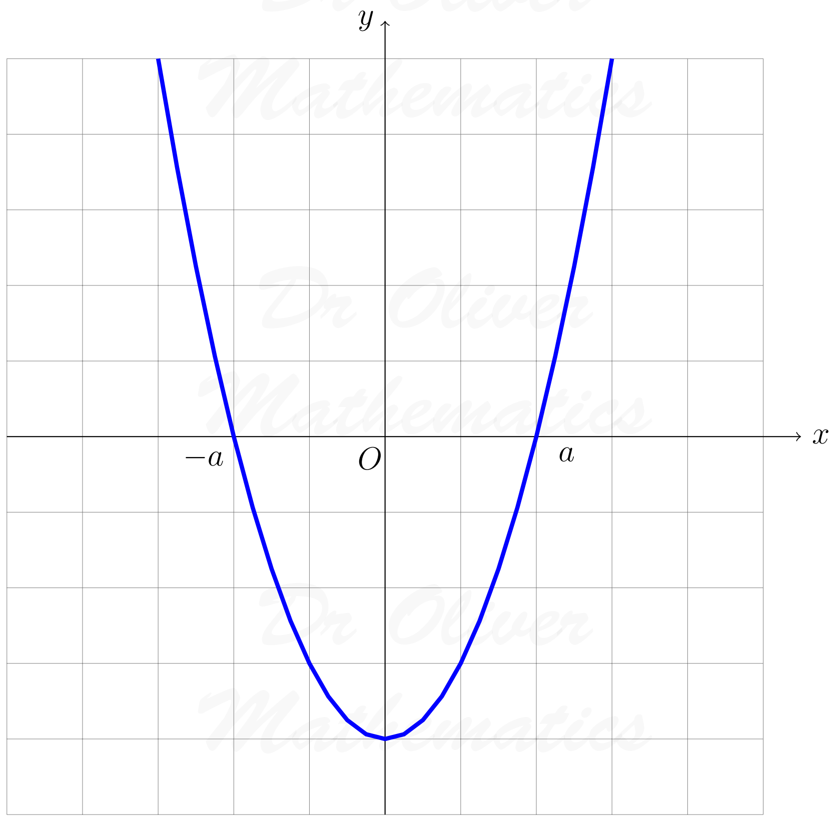
   (b) Find **AB**. (2)

   (c) Explain why **AB** does not have an inverse. (1)

3. The function diagram f(x) is defined by

$$f(x) = x^2 - a^2.$$

   The graph of is shown in the diagram.



(a) State whether $f(x)$ is odd, even or neither. Give a reason for your answer. (1)

(b) Sketch the graph of $y = |f(x)|$. (1)

4. (a) Express (1)

$$\frac{3x^2 + x - 17}{x^2 - x - 12}$$

in the form

$$p + \frac{qx + r}{x^2 - x - 12}$$

where p, q, and r are integers.

(b) Hence express (3)

$$\frac{3x^2 + x - 17}{x^2 - x - 12}$$

with partial fractions.

5. For

$$x = \ln(2t + 7) \text{ and } y = t^2,\ t > 0,$$

find

(a) $\dfrac{dy}{dx}$, (2)

(b) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. (2)

6. A spherical balloon of radius r cm, $r > 0$, deflates at a constant rate of 60 $\mathrm{cm}^3\,\mathrm{s}^{-1}$. (3)

   Calculate the rate of change of the radius with respect to time when $r = 3$.

7. (a) Find an expression of (1)

$$\sum_{r=1}^{n}(6r + 13)$$

   in terms of n.

   (b) Hence, or otherwise, find (2)

$$\sum_{r=p+1}^{20}(6r + 13).$$

8. Find the particular solution of the differential equation (5)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 11\frac{\mathrm{d}y}{\mathrm{d}x} + 28y = 0,$$

   given that $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 9$, when $x = 0$.

9. (a) Write down and simplify the general term in the binomial expansion of (3)

$$\left(2x^2 - \frac{d}{x^3}\right)^7,$$

   where d is a constant.

   (b) Given that the coefficient of $\dfrac{1}{x}$ is $-70\,000$, find the value of d. (2)

10. A curve is defined implicitly by the equation

$$x^2 + y^2 = xy + 12.$$

   (a) Find an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. (3)

   (b) There are two points where the tangent to the curve has equation $x = k$, $k \in \mathbb{R}$. Find the values of k. (2)

11. Let n be a positive integer.

   (a) Find a counterexample to show that the following statement is false. (1)

$$n^2 + n + 1 \text{ is always a prime number.}$$

(b) (i) Write down the contrapositive of: (1)

$$\text{If } n^2 - 2n + 7 \text{ is even, then } n \text{ is odd.}$$

(ii) Use the contrapositive to prove that if $n^2 - 2n + 7$ is even then n is odd. (3)

12. Express 231_{11} in base 7. (3)

13. An electronic device contains a timer circuit that switches off when the voltage, V, reaches a set value. The rate of change of the voltage is given by (5)

$$\frac{\mathrm{d}V}{\mathrm{d}t} = k(12 - V),$$

where k is a constant, t is the time in seconds, and $0 \leqslant V < 12$.

Given that $V = 2$ when $t = 0$, express V in terms of k and t.

14. Prove by induction that (5)

$$\sum_{r=1}^{n} r!\, r = (n+1)! - 1$$

for all positive integers n.

15. The equations of two planes are given below.

$$\begin{aligned} \pi_1 : &\quad 2x - 3y - z = 9 \\ \pi_2 : &\quad x + y - 3z = 2. \end{aligned}$$

(a) Verify that the line of intersection, L_1, of these two planes has parametric equations: (2)

$$x = 2\lambda + 3,\ y = \lambda - 1,\ z = \lambda.$$

Let π_3 be the plane with equation

$$-2x + 4y + 3z = 4.$$

(b) Calculate the acute angle between the line L_1 and the plane π_3. (3)

L_2 is the line perpendicular to π_3 passing through $P(1, 3, -2)$.

(c) Determine whether or not L_1 and L_2 intersect. (4)

16. (a) Use integration by parts to find the exact value of (5)

$$\int_0^1 (x^2 - 2x + 1)\mathrm{e}^{4x}\,\mathrm{d}x.$$

A solid is formed by rotating the curve with equation $y = 4(x-1)\mathrm{e}^{2x}$ between $x = 0$ and $x = 1$ through 2π radians about the x-axis.

(b) Find the exact value of the volume of this solid. (3)

17. The first three terms of a sequence are given by

$$5x + 8,\ -2x + 1,\ x - 4.$$

(a) When $x = 11$, show that the first three terms form the start of a geometric sequence, and state the value of the common ratio. (2)

(b) Given that the entire sequence is geometric for $x = 11$,

(i) state why the associated series has a sum to infinity, and (1)

(ii) calculate this sum to infinity. (2)

There is a second value for x that also gives a geometric sequence.

(c) For this second sequence

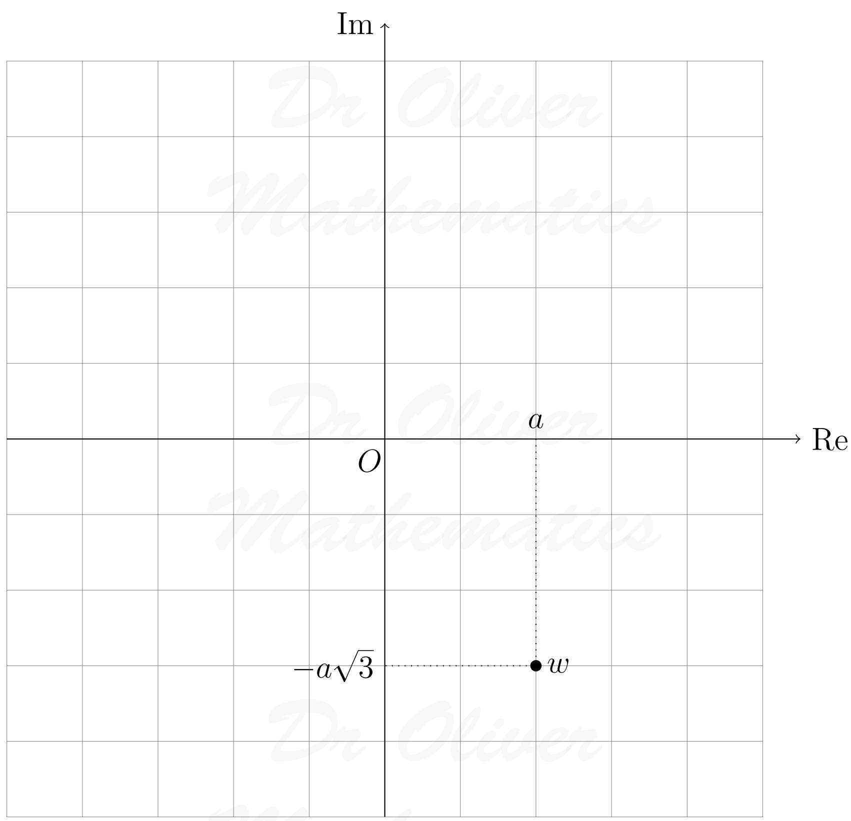
(i) show that (2)

$$x^2 - 8x - 33 = 0,$$

(ii) find the first three terms, and (2)

(iii) state the value of S_{2n} and justify your answer. (1)

18. The complex number w has been plotted on an Argand diagram, as shown below.



(a) Express w in

(i) Cartesian form, (1)

(ii) polar form. (3)

The complex number z_1 is a root of $z^3 = w$, where

$$z_1 = k\left(\cos\tfrac{1}{m}\pi + \sin\tfrac{1}{m}\pi\right),$$

for integers k and m.

(b) Given that $a = 4$,

(i) use de Moivre's theorem to obtain the values of k and m, and (4)

(ii) find the remaining roots. (2)