Dr Oliver Mathematics
Advanced Subsidiary Paper 1: Pure Mathematics
June 2022: Calculator
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.
Inexact answers should be given to three significant figures unless otherwise stated.

1. Find (4)

$$\int \left(8x^3 - \frac{3}{2\sqrt{x}} + 5\right) \, \mathrm{d}x,$$

giving your answer in simplest form.

Solution

$$\begin{aligned}\int \left(8x^3 - \frac{3}{2\sqrt{x}} + 5\right) \, \mathrm{d}x &= \int \left(8x^3 - \tfrac{3}{2}x^{-\frac{1}{2}} + 5\right) \, \mathrm{d}x \\ &= \underline{\underline{2x^4 - 3x^{\frac{1}{2}} + 5x + c}}.\end{aligned}$$

2.

$$\mathrm{f}(x) = 2x^3 + 5x^2 + 2x + 15.$$

(a) Use the factor theorem to show that $(x + 3)$ is a factor of $\mathrm{f}(x)$. (2)

Solution

Well,

$$\begin{aligned}\mathrm{f}(3) &= 2[(-3)^3] + 5[(-3)^2] + 2(-3) + 15 \\ &= -54 + 45 - 6 + 15 \\ &= 0;\end{aligned}$$

as their is no remainder, we conclude that $(x + 3)$ is a $\underline{\underline{\text{factor}}}$ of $\mathrm{f}(x)$.

(b) Find the constants a, b, and c such that (2)

$$\mathrm{f}(x) = (x + 3)(ax^2 + bx + c).$$

Solution

We use synthetic division:

-3	2	5	2	15
	$\downarrow$	-6	3	-15
	2	-1	5	0

So,

$$2x^3 + 5x^2 + 2x + 15 = \underline{\underline{(x+3)(2x^2 - x + 5)}};$$

hence, $\underline{\underline{a = 2}}$, $\underline{\underline{b = -1}}$, and $\underline{\underline{c = 5}}$.

(c) Hence show that $\mathrm{f}(x) = 0$ has only one real root. (2)

Solution

Well,

$$\begin{aligned} b^2 - 4ac &= (-1)^2 - 4 \times 2 \times 5 \\ &= 1 - 40 \\ &= -39 \\ &< 0; \end{aligned}$$

no real roots! Hence, $\mathrm{f}(x) = 0$ has $\underline{\underline{\text{only one real root}}}$.

(d) Write down the real root of the equation (1)

$$\mathrm{f}(x - 5) = 0.$$

Solution

$-3 + 5 = \underline{\underline{2}}$.

3. The triangle PQR is such that $\overrightarrow{PQ} = 3\mathbf{i} + 5\mathbf{j}$ and $\overrightarrow{PR} = 13\mathbf{i} - 15\mathbf{j}$.

(a) Find $\overrightarrow{QR}$. (2)

Solution

$$
\begin{aligned}
\overrightarrow{QR} &= \overrightarrow{QP} + \overrightarrow{PR} \\
&= -\overrightarrow{PQ} + \overrightarrow{PR} \\
&= -(3\mathbf{i} + 5\mathbf{j}) + (13\mathbf{i} - 15\mathbf{j}) \\
&= -3\mathbf{i} - 5\mathbf{j} + 13\mathbf{i} - 15\mathbf{j} \\
&= \underline{\underline{10\mathbf{i} - 20\mathbf{j}}}.
\end{aligned}
$$

(b) Hence find $|\overrightarrow{QR}|$ giving your answer as a simplified surd. (2)

Solution

$$
\begin{aligned}
|\overrightarrow{QR}| &= \sqrt{10^2 + (-20)^2} \\
&= \sqrt{100 + 400} \\
&= \sqrt{500} \\
&= \sqrt{100 \times 5} \\
&= \sqrt{100} \times \sqrt{5} \\
&= \underline{\underline{10\sqrt{5}}}.
\end{aligned}
$$

The point S lies on the line segment QR so that

$$QS : SR = 3 : 2.$$

(c) Find $\overrightarrow{PS}$. (2)

Solution

$$
\begin{aligned}
\overrightarrow{PS} &= \overrightarrow{PQ} + \overrightarrow{QS} \\
&= \overrightarrow{PQ} + \tfrac{3}{5}\overrightarrow{QR} \\
&= (3\mathbf{i} + 5\mathbf{j}) + \tfrac{3}{5}(10\mathbf{i} - 20\mathbf{j}) \\
&= 3\mathbf{i} + 5\mathbf{j} + 6\mathbf{i} - 12\mathbf{j} \\
&= \underline{\underline{9\mathbf{i} - 7\mathbf{j}}}.
\end{aligned}
$$

4. Figure 1 shows a sketch of triangle ABC with

- $AB = (x + 2)$ cm,

- $BC = (3x + 10)$ cm,
- $AC = 7x$ cm,
- angle $BAC = 60°$, and
- angle $ACB = \theta°$.

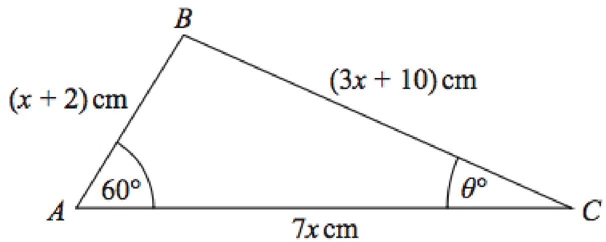


Figure 1: a sketch of triangle ABC

(a) (i) Show that (3)

$$17x^2 - 35x - 48 = 0.$$

Solution
We use the cosine rule:

$$\begin{aligned} & BC^2 = AB^2 + AC^2 - 2 \times AB \times AC \times \cos A \\ \Rightarrow \quad & (3x + 10)^2 = (x + 2)^2 + (7x)^2 - 2 \times (x + 2) \times (7x) \times \cos 60° \end{aligned}$$

$\times$	$3x$	$+10$
$3x$	$9x^2$	$+30x$
$+10$	$+30x$	$+100$

×	x	$+2$
x	x^2	$+2x$
$+2$	$+2x$	$+4$

$$\begin{aligned}
&\Rightarrow \quad 9x^2 + 60x + 100 = x^2 + 4x + 4 + 49x^2 - 7x(x+2) \\
&\Rightarrow \quad 9x^2 + 60x + 100 = x^2 + 4x + 4 + 49x^2 - 7x^2x - 14x \\
&\Rightarrow \quad 0 = 34x^2 - 70x - 96 \\
&\Rightarrow \quad 2(17x^2 - 35x - 48) = 0 \\
&\Rightarrow \quad \underline{\underline{17x^2 - 35x - 48 = 0}},
\end{aligned}$$

as required.

(ii) Hence find the value of x. (1)

Solution

add to: -35
multiply to: $(+17) \times (-48) = -816$ $\Big\}$ -51, $+16$

E.g.,

$$\begin{aligned}
17x^2 - 35x - 48 = 0 &\Rightarrow 17x^2 - 51x + 16x - 48 = 0 \\
&\Rightarrow 17x(x-3) + 16(x-3) = 0 \\
&\Rightarrow (17x + 16)(x - 3) = 0 \\
&\Rightarrow 17x + 16 = 0 \text{ or } x - 3 = 0 \\
&\Rightarrow x = -\tfrac{16}{17} \text{ or } x = 3;
\end{aligned}$$

now, $x \neq -\frac{16}{17}$ because $7x$ is a *length*. So, $\underline{\underline{x = 3}}$.

(b) Hence find the value of θ giving your answer to one decimal place. (2)

Solution

We use the sine rule:

$$\frac{\sin C}{AB} = \frac{\sin A}{BC} \Rightarrow \frac{\sin\theta}{5} = \frac{\sin 60^\circ}{19}$$
$$\Rightarrow \sin\theta = \frac{5\sin 60^\circ}{19}$$
$$\Rightarrow \theta = 13.173\,551\,111 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\theta = 13.2^\circ \text{ (3 sf)}}}.$$

5. The mass, A kg, of algae in a small pond, is modelled by the equation

$$A = pq^t,$$

where p and q are constants and t is the number of weeks after the mass of algae was first recorded.

Data recorded indicates that there is a linear relationship between t and $\log_{10} A$ given by the equation

$$\log_{10} A = 0.03t + 0.5.$$

(a) Use this relationship to find a complete equation for the model in the form (4)

$$A = pq^t,$$

giving the value of p and the value of q each to 4 significant figures.

Solution

$$\log_{10} A = 0.03t + 0.5 \Rightarrow A = 10^{0.03t+0.5}$$
$$\Rightarrow A = 10^{0.5} \cdot (10^{0.03})^t$$
$$\Rightarrow A = 3.162\,277\,66 \cdot (1.071\,519\,305)^t \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{A = 3.162 \cdot (1.072)^t \text{ (4 sf)}}};$$

hence, $\underline{\underline{p = 3.162 \text{ (4 sf)}}}$ and $\underline{\underline{q = 1.072 \text{ (4 sf)}}}$.

(b) With reference to the model, interpret (2)

(i) the value of the constant p,

Solution
E.g., p is the initial mass of algae.

(ii) the value of the constant q.

Solution
E.g., q is the weekly rate, the ratio of algae from one week to the next.

(c) Find, according to the model, (3)

(i) the mass of algae in the pond when $t = 8$, giving your answer to the nearest 0.5 kg,

Solution

$$\begin{aligned} A &= 3.162\ldots \cdot (1.071\ldots)^8 \\ &= 5.495\,408\,739 \text{ (FCD)} \\ &= \underline{\underline{5.5 \text{ kg (nearest 0.5)}}}. \end{aligned}$$

(ii) the number of weeks it takes for the mass of algae in the pond to reach 4 kg.

Solution
Let t be the number of weeks. Then

$$\begin{aligned} 4 = 3.162\ldots \cdot (1.071\ldots)^t &\Rightarrow (1.071\ldots)^t = \frac{4}{3.162\ldots} \\ &\Rightarrow \log_{10}(1.071\ldots)^t = \log_{10}\left(\frac{4}{3.162\ldots}\right) \\ &\Rightarrow t\log_{10}(1.071\ldots) = \log_{10}\left(\frac{4}{3.162\ldots}\right) \\ &\Rightarrow t = \frac{\log_{10}\left(\frac{4}{3.162\ldots}\right)}{\log_{10} 1.071\ldots} \\ &\Rightarrow t = 3.401\,999\,711 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{t = 3.4 \text{ weeks (1 dp)}}}. \end{aligned}$$

(d) State one reason why this may not be a realistic model in the long term. (1)

Solution
E.g., as $t \to \infty$, $\underline{\underline{A \to \infty}}$, the weather may affect the rate of growth.

6. (a) Find the first 4 terms, in ascending powers of x, of the binomial expansion of (4)

$$\left(3 - \frac{2x}{9}\right)^8,$$

giving each term in simplest form.

Solution

$$\left(3 - \frac{2x}{9}\right)^8$$

$$\Rightarrow \quad 3^8 + \binom{8}{1}(3^7)\left(-\frac{2x}{9}\right) + \binom{8}{2}(3^6)\left(-\frac{2x}{9}\right)^2 + \binom{8}{3}(3^5)\left(-\frac{2x}{9}\right)^3 + \ldots$$

$$\Rightarrow \quad \underline{\underline{6\,561 - 3\,888x + 1\,008x^2 - \tfrac{448}{3}x^3 + \ldots}}.$$

$$\mathrm{f}(x) = \left(\frac{x-1}{2x}\right)\left(3 - \frac{2x}{9}\right)^8.$$

(b) Find the coefficient of x^2 in the series expansion of $\mathrm{f}(x)$, giving your answer as a simplified fraction. (2)

Solution

$$\begin{aligned}\mathrm{f}(x) &= \left(\frac{x-1}{2x}\right)\left(3 - \frac{2x}{9}\right)^8\\ &= \left(\tfrac{1}{2} - \frac{1}{2x}\right)\left[6\,561 - 3\,888x + 1\,008x^2 - \tfrac{448}{3}x^3 + \ldots\right]\\ &= \ldots + \left[\tfrac{1}{2}x^2(1\,008x) - \frac{1}{2x}(-\tfrac{448}{3}x^3)\right] + \ldots\\ &= \ldots + \tfrac{1\,736}{3}x^2 + \ldots;\end{aligned}$$

hence, the coefficient of x^2 in the series expansion of $\mathrm{f}(x)$ is $\underline{\underline{\tfrac{1\,736}{3}}}$.

7. (a) Factorise completely (2)

$$9x - x^3.$$

Solution

$$9x - x^3 = x(9 - x^2)$$

difference of two squares:

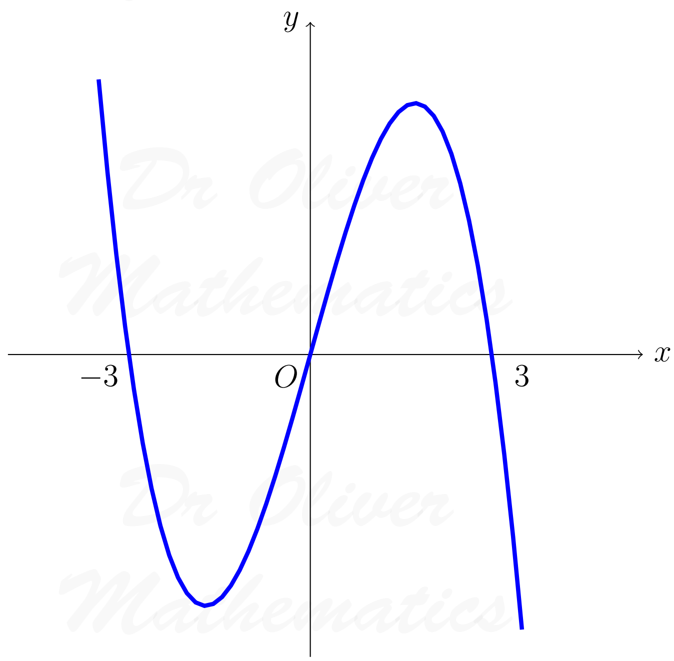
$$= x(3^2 - x^2)$$
$$= \underline{\underline{x(3 - x)(3 + x)}}.$$

The curve C has equation

$$y = 9x - x^3.$$

(b) Sketch C showing the coordinates of the points at which the curve cuts the x-axis. (2)

Solution



The line l has equation $y = k$, where k is a constant.

Given that C and l intersect at 3 distinct points,

(c) find the range of values for k, writing your answer in set notation. (3)
Solutions relying on calculator technology are not acceptable.

Solution

Well,

$$y = 9x - x^3 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 9 - 3x^2$$

and

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 9 - 3x^2 &= 0 \\ \Rightarrow 3x^2 &= 9 \\ \Rightarrow x^2 &= 3 \\ \Rightarrow x &= \pm\sqrt{3}. \end{aligned}$$

Now,

$$x = 3 \Rightarrow y = 9(\sqrt{3}) - (\sqrt{3})^3 = 6\sqrt{3}$$

and

$$x = -3 \Rightarrow y = 9(-\sqrt{3}) - (-\sqrt{3})^3 = -6\sqrt{3}.$$

Hence,

$$\underline{\underline{\left\{x \in \mathbb{R} : -6\sqrt{3} < x < 6\sqrt{3}\right\}}}.$$

8. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

The air pressure, P kg/cm^2, inside a car tyre, t minutes from the instant when the tyre developed a puncture is given by the equation

$$P = k + 1.4\mathrm{e}^{-0.5t},\ t \in \mathbb{R},\ t \geqslant 0,$$

where k is a constant.

Given that the initial air pressure inside the tyre was 2.2 kg/cm^2,

(a) state the value of k. (1)

Solution

$$\begin{aligned} 2.2 = k + 1.4\mathrm{e}^{-0.5\times 0} &\Rightarrow 2.2 = k + 1.4 \\ &\Rightarrow \underline{\underline{k = 0.8}}. \end{aligned}$$

From the instant when the tyre developed the puncture,

(b) find the time taken for the air pressure to fall to 1 kg/cm^2. (3)
Give your answer in minutes to one decimal place.

Solution

$$\begin{aligned} 1 = 0.8 + 1.4\mathrm{e}^{-0.5t} &\Rightarrow 1.4\mathrm{e}^{-0.5t} = 0.2 \\ &\Rightarrow \mathrm{e}^{-0.5t} = \tfrac{1}{7} \\ &\Rightarrow \mathrm{e}^{0.5t} = 7 \\ &\Rightarrow 0.5t = \ln 7 \\ &\Rightarrow t = 2\ln 7 \text{ (exact!)} \\ &\Rightarrow \underline{\underline{t = 3.89 \text{ mins (3 sf)}}}. \end{aligned}$$

(c) Find the rate at which the air pressure in the tyre is decreasing exactly 2 minutes from the instant when the tyre developed the puncture. (2)
Give your answer in kg/cm^2 per minute to 3 significant figures.

Solution

$$P = 0.8 + 1.4\mathrm{e}^{-0.5t} \Rightarrow \frac{\mathrm{d}P}{\mathrm{d}t} = -0.7\mathrm{e}^{-0.5t}$$

and

$$\begin{aligned} t = 2 \Rightarrow \frac{\mathrm{d}P}{\mathrm{d}t} &= -0.7\mathrm{e}^{-1} \\ \Rightarrow \frac{\mathrm{d}P}{\mathrm{d}t} &= -0.257\,515\,608\,8 \text{ (FCD)}; \end{aligned}$$

hence, it is going down at $\underline{\underline{0.258 \text{ kg/cm}^2 \text{ (3 sf)}}}$.

9. (a) Given that $p = \log_3 x$, where $x > 0$, find in simplest form in terms of p, (2)

(i) $\log_3\left(\frac{x}{9}\right)$

Solution

$$\begin{aligned} \log_3\left(\frac{x}{9}\right) &= \log_3(x) - \log_3(9) \\ &= \log_3(x) - \log_3\left(3^2\right) \\ &= \log_3 x - 2 \\ &= \underline{\underline{p-2}}. \end{aligned}$$

(ii) $\log_3(\sqrt{x})$.

Solution

$$\begin{aligned}\log_3(\sqrt{x}) &= \log_3\left(x^{\frac{1}{2}}\right)\\ &= \tfrac{1}{2}\log_3 x\\ &= \underline{\underline{\tfrac{1}{2}p}}.\end{aligned}$$

(b) Hence, or otherwise, solve (4)

$$2\log_3\left(\frac{x}{9}\right) + 3\log_3\left(\sqrt{x}\right) = -11,$$

giving your answer as a simplified fraction.
Solutions relying on calculator technology are not acceptable.

Solution

$$\begin{aligned}& 2\log_3\left(\frac{x}{9}\right) + 3\log_3\left(\sqrt{x}\right) = -11\\ \Rightarrow\quad & 2(p-2) + 3(\tfrac{1}{2}p) = -11\\ \Rightarrow\quad & 2p - 4 + \tfrac{3}{2}p = -11\\ \Rightarrow\quad & \tfrac{7}{2}p = -7\\ \Rightarrow\quad & p = -2\\ \Rightarrow\quad & \log_3 x = -2\\ \Rightarrow\quad & x = 3^{-2}\\ \Rightarrow\quad & \underline{\underline{x = \tfrac{1}{9}}}.\end{aligned}$$

10. **In this question you must show all stages of your working.**
Solutions relying on calculator technology are not acceptable.

Figure 2 shows a sketch of part of the curve C with equation

$$y = \tfrac{1}{3}x^2 - 2\sqrt{x} + 3,\ x \geqslant 0.$$

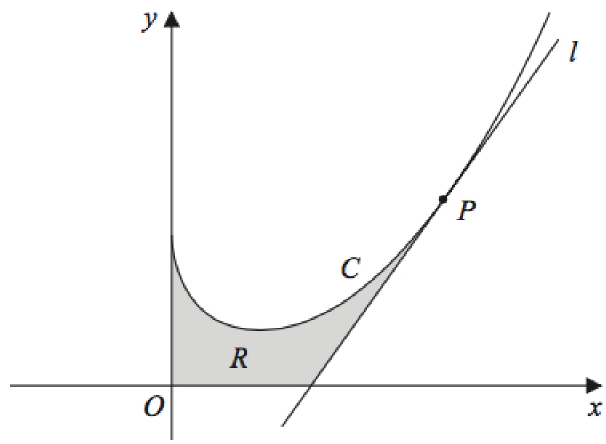


Figure 2: a sketch of part of the curve C

The point P lies on C and has x-coordinate 4.

The line l is the tangent to C at P.

(a) Show that l has equation (5)

$$13x - 6y - 26 = 0.$$

Solution

$$x = 4 \Rightarrow \tfrac{1}{3}(4^2) - 2\sqrt{4} + 3 = \tfrac{13}{3}$$

so $P(4, \frac{13}{3})$. Now,

$$y = \tfrac{1}{3}x^2 - 2\sqrt{x} + 3 \Rightarrow y = \tfrac{1}{3}x^2 - 2x^{\frac{1}{2}} + 3$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{2}{3}x - x^{-\frac{1}{2}}$$

and

$$x = 4 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{2}{3}(4) - 4^{-\frac{1}{2}}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{13}{6}.$$

Hence, l has equation

$$y - \tfrac{13}{3} = \tfrac{13}{6}(x - 4) \Rightarrow 6y - 26 = 13(x - 4)$$
$$\Rightarrow 6y - 26 = 13x - 52$$
$$\Rightarrow \underline{\underline{13x - 6y - 26 = 0}},$$

as required.

The region R, shown shaded in Figure 2, is bounded by the y-axis, the curve C, the line l, and the x-axis.

(b) Find the exact area of R. (5)

Solution

We need to find where l crosses the x-axis:

$$\begin{aligned} y = 0 &\Rightarrow 13x = 26 \\ &\Rightarrow x = 2. \end{aligned}$$

Now,

$$\begin{aligned} \text{area of } R &= \text{area under the curve} - \text{area under } l \\ &= \int_0^4 \left(\tfrac{1}{3}x^2 - 2x^{\frac{1}{2}} + 3\right) \mathrm{d}x - \tfrac{1}{2} \times 2 \times \tfrac{13}{3} \\ &= \left[\tfrac{1}{9}x^3 - \tfrac{4}{3}x^{\frac{3}{2}} + 3x\right]_{x=0}^{4} - \tfrac{13}{3} \\ &= \left\{\left(\tfrac{64}{9} - \tfrac{32}{3} + 12\right) - (0 - 0 + 0)\right\} - \tfrac{13}{3} \\ &= \tfrac{76}{3} - \tfrac{13}{3} \\ &= \underline{\underline{4\tfrac{1}{9}}}. \end{aligned}$$

11. Figure 3 shows the circle C with equation

$$x^2 + y^2 - 10x - 8y + 32 = 0$$

and the line l with equation

$$2y + x + 6 = 0.$$

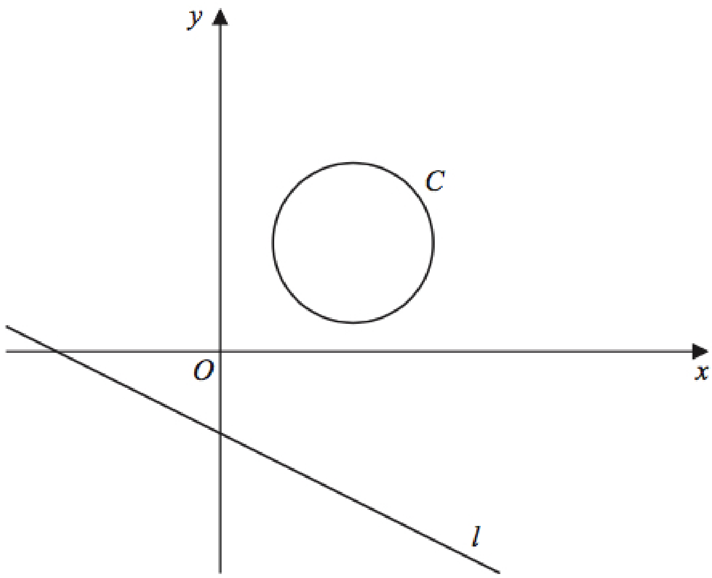


Figure 3: $x^2 + y^2 - 10x - 8y + 32 = 0$

(a) Find (3)

(i) the coordinates of the centre of C,

Solution

We have to complete the square, *twice*:

$$\begin{aligned} & x^2 + y^2 - 10x - 8y + 32 = 0 \\ \Rightarrow\ & x^2 - 10x + y^2 - 8y = -32 \\ \Rightarrow\ & (x^2 - 10x + 25) + (y^2 - 8y + 16) = -32 + 25 + 16 \\ \Rightarrow\ & (x-5)^2 + (y-4)^2 = 9 \\ \Rightarrow\ & (x-5)^2 + (y-4)^2 = 3^2; \end{aligned}$$

hence, the centre is $\underline{\underline{(5, 4)}}$ ...

(ii) the radius of C.

Solution

...and the radius is $\underline{\underline{3}}$.

Solution

(b) Find the shortest distance between C and l. (5)

Solution
Well,

$$\begin{aligned} 2y + x + 6 = 0 &\Rightarrow 2y = -x - 6 \\ &\Rightarrow y = -\tfrac{1}{2}x - 3 \end{aligned}$$

and

$$m_{\text{normal}} = -\frac{1}{-\frac{1}{2}} = 2.$$

Now, it goes through the point $(5, 4)$ and the equation of the normal is

$$\begin{aligned} y - 4 = 2(x - 5) &\Rightarrow y - 4 = 2x - 10 \\ &\Rightarrow y = 2x - 6. \end{aligned}$$

We need to find the point at which these two lines intersect:

$$\begin{aligned} 2y + x + 6 = 0 &\Rightarrow 2(2x - 6) + x + 6 = 0 \\ &\Rightarrow 4x - 12 + x + 6 = 0 \\ &\Rightarrow 5x = 6 \\ &\Rightarrow x = \tfrac{6}{5} \\ &\Rightarrow y = 2(\tfrac{6}{5}) - 6 = -\tfrac{18}{5}; \end{aligned}$$

so, the point is $(\frac{6}{5}, -\frac{18}{5})$. Finally,

$$\begin{aligned} \text{shortest distance} &= \sqrt{(5 - \tfrac{6}{5})^2 + [4 - (-\tfrac{18}{5})]^2} - 3 \\ &= \sqrt{14.44 + 57.76} - 3 \\ &= \sqrt{72.2} - 3 \\ &= \underline{\underline{\tfrac{19}{5}\sqrt{5} - 3}}. \end{aligned}$$

12. A company makes drinks containers out of metal.

The containers are modelled as closed cylinders with base radius r cm and height h cm and the capacity of each container is 355 cm^3.

The metal used

- for the circular base and the curved side costs 0.04 pence/cm^2 and
- for the circular top costs 0.09 pence/cm^2

Both metals used are of negligible thickness.

(a) Show that the total cost, C pence, of the metal for one container is given by (4)

$$C = 0.13\pi r^2 + \frac{28.4}{r}.$$

Solution

Let r cm be the radius and let h cm be the height. Now,

$$\pi r^2 h = 355 \Rightarrow h = \frac{355}{\pi r^2}$$

and

$$\begin{aligned} C &= \text{bottom} + \text{curved part} + \text{top} \\ &= (0.04 \times \pi r^2) + (0.04 \times 2\pi r h) + (0.09 \times \pi r^2) \\ &= 0.13\pi r^2 + 0.08\pi r \left(\frac{355}{\pi r^2}\right) \\ &= \underline{\underline{0.13\pi r^2 + \frac{28.4}{r}}}, \end{aligned}$$

as required.

(b) Use calculus to find the value of r for which C is a minimum, giving your answer to 3 significant figures. (4)

Solution

$$\begin{aligned} C = 0.13\pi r^2 + \frac{28.4}{r} &\Rightarrow C = 0.13\pi r^2 + 28.4 r^{-1} \\ &\Rightarrow \frac{\mathrm{d}C}{\mathrm{d}r} = 0.26\pi r - 28.4 r^{-2} \end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}C}{\mathrm{d}r} = 0 &\Rightarrow 0.26\pi r - 28.4r^{-2} = 0\\ &\Rightarrow 0.26\pi r = 28.4r^{-2}\\ &\Rightarrow r^3 = \frac{1\,420}{13\pi}\\ &\Rightarrow r = \sqrt[3]{\frac{1\,420}{13\pi}}\\ &\Rightarrow r = 3.263\,861\,387 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{r = 3.26 \text{ cm (3 sf)}}}.\end{aligned}$$

(c) Using $\frac{\mathrm{d}^2C}{\mathrm{d}r^2}$ prove that the cost is minimised for the value of r found in part (b). (2)

Solution

$$\frac{\mathrm{d}C}{\mathrm{d}r} = 0.26\pi r - 28.4r^{-2} \Rightarrow \frac{\mathrm{d}^2C}{\mathrm{d}r^2} = 0.26\pi + 56.8r^{-3}$$

and

$$r = 3.263\ldots \Rightarrow \frac{\mathrm{d}^2C}{\mathrm{d}r^2} = 0.26\pi + 56.8(3.263\ldots^{-3}) > 0.$$

Hence, the cost is $\underline{\underline{\text{minimised}}}$ for the value of r

(d) Hence find the minimum value of C, giving your answer to the nearest integer. (2)

Solution

$$\begin{aligned}C &= 0.13\pi(3.263\ldots^2) + \frac{28.4}{3.263\ldots}\\ &= 13.052\,024\,87 \text{ (FCD)}\\ &= \underline{\underline{13 \text{ (nearest integer)}}}.\end{aligned}$$

13. **In this question you must show all stages of your working.**
Solutions relying entirely on calculator technology are not acceptable.

(a) Show that (3)

$$\frac{1}{\cos\theta} + \tan\theta \equiv \frac{\cos\theta}{1-\sin\theta},\ \theta \neq (2n+1)90^\circ,\ n \in \mathbb{Z}.$$

Solution

$$\begin{aligned}\frac{1}{\cos\theta}+\tan\theta &\equiv \frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}\\ &\equiv \frac{1+\sin\theta}{\cos\theta}\\ &\equiv \frac{1+\sin\theta}{\cos\theta}\times\frac{1-\sin\theta}{1-\sin\theta}\\ &\equiv \frac{(1+\sin\theta)(1-\sin\theta)}{\cos\theta(1-\sin\theta)}\\ &\equiv \frac{1-\sin^2\theta}{\cos\theta(1-\sin\theta)}\\ &\equiv \frac{\cos^2\theta}{\cos\theta(1-\sin\theta)}\\ &\equiv \underline{\underline{\frac{\cos\theta}{1-\sin\theta}}},\end{aligned}$$

as required.

Given that $\cos 2x \neq 0$,

(b) solve for $0^\circ < x < 90^\circ$, (5)

$$\frac{1}{\cos 2x}+\tan 2x = 3\cos 2x,$$

giving your answers to one decimal place.

Solution

$$\begin{aligned}\frac{1}{\cos 2x}+\tan 2x = 3\cos 2x &\Rightarrow \frac{\cos 2x}{1-\sin 2x} = 3\cos 2x\\ &\Rightarrow \frac{\cos 2x}{1-\sin 2x} - 3\cos 2x = 0\\ &\Rightarrow \cos 2x\left(\frac{1}{1-\sin 2x}-3\right) = 0\\ &\Rightarrow \frac{1}{1-\sin 2x} - 3 = 0\\ &\Rightarrow \frac{1}{1-\sin 2x} = 3\\ &\Rightarrow 1-\sin 2x = \tfrac{1}{3}\\ &\Rightarrow \sin 2x = \tfrac{2}{3},\end{aligned}$$

as $\cos 2x \neq 0$. Finally,

$$\begin{aligned} \sin 2x = \tfrac{2}{3} &\Rightarrow 2x = 41.810\,314\,9,\ 138.189\,685\,1 \text{ (FCD)} \\ &\Rightarrow x = 20.905\,157\,45,\ 69.094\,842\,55 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 20.9,\ 69.1 \text{ (1 dp)}}}. \end{aligned}$$

14. (a) A student states: (1)

"if x^2 is greater than 9, then x must be greater than 3."

Determine whether or not this statement is true, giving a reason for your answer.

Solution
False: consider $\underline{\underline{x = -4}}$. We have $(-4)^2 = 16$ but $x \ngtr 3$.

(b) Prove that for all positive integers n, (3)

$$n^3 + 3n^2 + 2n$$

is divisible by 6.

Solution
We factorise:

$$n^3 + 3n^2 + 2n = n(n^2 + 3n + 2)$$

$$\left.\begin{array}{ll} \text{add to:} & +3 \\ \text{multiply to:} & +2 \end{array}\right\} + 1,\ +2$$

$$= n(n+1)(n+2),$$

three consecutive numbers!
So, it is a multiple of 2 and it is a multiple of 3.
Hence, $n^3 + 3n^2 + 2n$ is divisible by $\underline{\underline{6}}$.