# Dr Oliver Mathematics
# Advanced Level: Pure Mathematics 1
# November 2021: Calculator
# 2 hours

The total number of marks available is 100.
You must write down all the stages in your working.
Inexact answers should be given to three significant figures unless otherwise stated.

1. (3)

$$f(x) = ax^3 + 10x^2 - 3ax - 4.$$

Given that $(x - 1)$ is a factor of $f(x)$, find the value of the constant $a$.

You must make your method clear.

**Solution**

Well,

$$\begin{aligned} f(1) = 0 &\Rightarrow a + 10 - 3a - 4 = 0 \\ &\Rightarrow -2a = -6 \\ &\Rightarrow \underline{\underline{a = 3}}. \end{aligned}$$

2. Given that

$$f(x) = x^2 - 4x + 5, \; x \in \mathbb{R}.$$

(a) express $f(x)$ in the form (2)

$$(x + a)^2 + b,$$

where $a$ and $b$ are integers to be found.

**Solution**

$$\begin{aligned} f(x) &= x^2 - 4x + 5 \\ &= (x^2 - 4x + 4) + 1 \\ &= \underline{\underline{(x - 2)^2 + 1}}; \end{aligned}$$

hence, $\underline{\underline{a = -2}}$ and $\underline{\underline{b = 1}}$.

The curve with equation $y = \mathrm{f}(x)$

- meets the $y$-axis at the point $P$ and
- has a minimum turning point at the point $Q$.

(b) Write down (2)

(i) the coordinates of $P$,

**Solution**
$\underline{\underline{(0, 5)}}$.

(ii) the coordinates of $Q$.

**Solution**
$\underline{\underline{(2, 1)}}$.

3. The sequence $u_1$, $u_2$, $u_3$, ... is defined by

$$u_{n+1} = k - \frac{24}{u_n},\ u_1 = 2,$$

where $k$ is an integer.

Given that

$$u_1 + 2u_2 + u_3 = 0,$$

(a) show that (3)

$$3k^2 - 58k + 240 = 0.$$

**Solution**
So,

$$\begin{aligned} u_1 &= 2 \\ u_2 &= k - \frac{24}{u_1} \\ &= k - \frac{24}{2} \\ &= k - 12 \\ u_3 &= k - \frac{24}{u_2} \\ &= k - \frac{24}{k - 12}. \end{aligned}$$

Now,

$$\begin{aligned} u_1 + 2u_2 + u_3 = 0 &\Rightarrow 2 + 2(k-12) + \left(k - \frac{24}{k-12}\right) = 0 \\ &\Rightarrow 2(k-12) + 2(k-12)^2 + k(k-12) - 24 = 0 \\ &\Rightarrow (2k-24) + 2(k^2 - 24k + 144) + (k^2 - 12k) - 24 = 0 \\ &\Rightarrow \underline{\underline{3k^2 - 58k + 240 = 0}}, \end{aligned}$$

as required.

(b) Find the value of $k$, giving a reason for your answer. (2)

**Solution**

$$\left.\begin{array}{ll} \text{add to:} & -58 \\ \text{multiply to:} & (+3) \times (+240) = +720 \end{array}\right\} -40,\ -18$$

E.g.,

$$\begin{aligned} 3k^2 - 58k + 240 = 0 &\Rightarrow 3k^2 - 40k - 18k + 240 = 0 \\ &\Rightarrow k(3k-40) - 6(3k-40) = 0 \\ &\Rightarrow (k-6)(3k-40) = 0 \\ &\Rightarrow k = 6 \text{ or } k = 13\tfrac{1}{3}; \end{aligned}$$

but $k \in \mathbb{Z}$ so $\underline{\underline{k = 6}}$.

(c) Find the value of $u_3$. (1)

**Solution**

$$u_3 = 6 - \frac{24}{6-12} = 6 - (-4) = \underline{\underline{10}}.$$

4. The curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = x^2 + \ln(2x^2 - 4x + 5)$$

has a single turning point at $x = \alpha$.

(a) Show that $\alpha$ is a solution of the equation (4)

$$2x^3 - 4x^2 + 7x - 2 = 0.$$

**Solution**

$$f(x) = x^2 + \ln(2x^2 - 4x + 5) \Rightarrow f'(x) = 2x + \frac{4x-4}{2x^2-4x+5}$$

and

$$\begin{aligned} f'(0) = 0 &\Rightarrow 2x + \frac{4x-4}{2x^2-4x+5} = 0 \\ &\Rightarrow 2x(2x^2-4x+5) + 4x - 4 = 0 \\ &\Rightarrow (4x^3 - 8x^2 + 10x) + 4x - 4 = 0 \\ &\Rightarrow 4x^3 - 8x^2 + 6x - 4 = 0 \\ &\Rightarrow 2(2x^3 - 4x^2 + 3x - 2) = 0 \\ &\Rightarrow \underline{\underline{2x^3 - 4x^2 + 3x - 2) = 0}}, \end{aligned}$$

as required.

The iterative formula

$$x_{n+1} = \tfrac{1}{7}(2 + 4x_n^2 - 2x_n^3)$$

is used to find an approximate value for $\alpha$.

Starting with $x_1 = 0.3$,

(b) calculate, giving each answer to 4 decimal places, (3)

(i) the value of $x_2$,

**Solution**

$$\begin{aligned} x_2 &= \tfrac{1}{7}[2 + 4((0.3)^2) - 2((0.3)^3)] \\ &= 0.329\,428\,571\,4 \text{ (FCD)} \\ &= \underline{\underline{0.329\,4 \text{ (4 dp)}}}. \end{aligned}$$

(ii) the value of $x_3$.

**Solution**

$$\begin{aligned} x_3 &= \tfrac{1}{7}[2 + 4((0.329\ldots)^2) - 2((0.329\ldots)^3)] \\ &= 0.337\,513\,065\,7 \text{ (FCD)} \\ &= \underline{\underline{0.337\,5 \text{ (4 dp)}}}. \end{aligned}$$

Using a suitable interval and a suitable function that should be stated,

(c) show that $\alpha$ is 0.341 to 3 decimal places. (2)

**Solution**

$$\begin{aligned} f'(0.340\,5) &= -6.74\ldots \times 10^{-4} \\ f'(0.341\,5) &= 1.89\ldots \times 10^{-3} \end{aligned}$$

The function $f'(x)$ is continuous $(x > 0)$ and there is a change of sign and so the root lies $0.340\,5 < \alpha < 0.341\,5$; hence,

$$\underline{\underline{\alpha = 0.341 \text{ (3 dp)}}}.$$

5. A company made a profit of £20 000 in its first year of trading, Year 1.

A model for future trading predicts that the yearly profit will increase by 8% each year, so that the yearly profits will form a geometric sequence.

According to the model,

(a) show that the profit for Year 3 will be £23 328, (1)

**Solution**

$$\begin{aligned} \text{Profit} &= 20\,000 \times (1.08)^2 \\ &= \underline{\underline{£23\,328}}, \end{aligned}$$

as required.

(b) find the first year when the yearly profit will exceed £65 000, (3)

**Solution**

Let $n$ be the first year when the yearly profit will exceed £65 000. Then

$$\begin{aligned} 20\,000 \times (1.08)^{n-1} > 65\,000 &\Rightarrow (1.08)^{n-1} > 3.25 \\ &\Rightarrow \ln(1.08)^{n-1} > \ln 3.25 \\ &\Rightarrow (n-1)\ln 1.08 > \ln 3.25 \\ &\Rightarrow n - 1 > \frac{\ln 3.25}{\ln 1.08} \\ &\Rightarrow n > \frac{\ln 3.25}{\ln 1.08} + 1 \\ &\Rightarrow n > 16.314\ldots; \end{aligned}$$

hence, it will take 17 years.

(c) find the total profit for the first 20 years of trading, giving your answer to the nearest £1 000. (2)

**Solution**

$a = 20\,000$, $n = 20$, and $r = 1.08$:

$$\begin{aligned} S_{20} &= \frac{20\,000(1.08^{20} - 1)}{1.08 - 1} \\ &= 915\,239.286 \text{ (FCD)} \\ &= \text{£}915\,000 \text{ (nearest thousand)}. \end{aligned}$$

6. Figure 1 shows a sketch of triangle $ABC$.

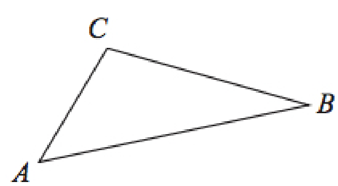


Figure 1: a sketch of triangle $ABC$

Given that

- $\overrightarrow{AB} = -3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$ and
- $\overrightarrow{BC} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}$,

(a) find $\overrightarrow{AC}$, (2)

**Solution**

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AB} + \overrightarrow{BC} \\ &= (-3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}) + (\mathbf{i} + \mathbf{j} + 4\mathbf{k}) \\ &= \underline{\underline{-2\mathbf{i} - 3\mathbf{j} - \mathbf{k}}}.\end{aligned}$$

(b) show that (3)

$$\cos ABC = \tfrac{9}{10}.$$

**Solution**

Well,

$$\begin{aligned}a &= BC \\ &= \sqrt{1^2 + 1^2 + 4^2} \\ &= 3\sqrt{2}, \\ b &= AC \\ &= \sqrt{(-2)^2 + (-3)^2 + (-1)^2} \\ &= \sqrt{14}, \text{ and} \\ c &= AB \\ &= \sqrt{(-3)^2 + (-4)^2 + (-5)^2} \\ &= 5\sqrt{2}.\end{aligned}$$

Now,

$$\begin{aligned}b^2 = a^2 + c^2 - 2bc\cos ABC &\Rightarrow 14 = 18 + 50 - 2 \times 3\sqrt{2} \times 5\sqrt{2} \times \cos ABC \\ &\Rightarrow -54 = -60\cos ABC \\ &\Rightarrow \underline{\underline{\cos ABC = \tfrac{9}{10}}},\end{aligned}$$

as required.

7. The circle $C$ has equation

$$x^2 + y^2 - 10x + 4y + 11 = 0.$$

(a) Find (4)

(i) the coordinates of the centre of $C$,

**Solution**

$$\begin{aligned} & x^2 + y^2 - 10x + 4y + 11 = 0 \\ \Rightarrow\ & (x^2 - 10x) + (y^2 + 4y) = -11 \\ \Rightarrow\ & (x^2 - 10x + 25) + (y^2 + 4y + 4) = -11 + 25 + 4 \\ \Rightarrow\ & (x-5)^2 + (y+2)^2 = 18; \end{aligned}$$

the coordinates of the centre of $C$ are $\underline{\underline{(5, -2)}}$.

(ii) the exact radius of $C$, giving your answer as a simplified surd.

**Solution**
The exact radius of $C$ is

$$\sqrt{18} = \underline{\underline{3\sqrt{2}}}.$$

The line $l$ has equation

$$y = 3x + k,$$

where $k$ is a constant.

Given that $l$ is a tangent to $C$,

(b) find the possible values of $k$, giving your answers as simplified surds. (5)

**Solution**
Simultaneous equations:

$$\begin{aligned} & x^2 + (3x+k)^2 - 10x + 4(3x+k) + 11 = 0 \\ \Rightarrow\ & x^2 + (9x^2 + 6kx + k^2) - 10x + (12x + 4k) + 11 = 0 \\ \Rightarrow\ & 10x^2 + (6k+2)x + (k^2 + 4k + 11) = 0. \end{aligned}$$

Now, given that $l$ is a tangent to $C$:

$$\begin{aligned} b^2 - 4ac = 0 & \Rightarrow (6k+2)^2 - 4(10)(k^2 + 4k + 11) = 0 \\ & \Rightarrow (36k^2 + 24k + 4) - (40k^2 + 160k + 440) = 0 \\ & \Rightarrow -4k^2 - 136k - 436 = 0 \\ & \Rightarrow -4(k^2 + 34k + 109) = 0 \end{aligned}$$

e.g., we now complete the square:

$$\begin{aligned}
&\Rightarrow k^2 + 34k = -109\\
&\Rightarrow k^2 + 34k + 289 = -109 + 289\\
&\Rightarrow (k + 17)^2 = 180\\
&\Rightarrow k + 17 = \pm\sqrt{180}\\
&\Rightarrow \underline{\underline{k = -17 \pm 6\sqrt{5}}}.
\end{aligned}$$

8. A scientist is studying the growth of two different populations of bacteria.

The number of bacteria, $N$, in the **first** population is modelled by the equation

$$N = Ae^{kt},\ t > 0,$$

where $A$ and $k$ are positive constants and $t$ is the time in hours from the start of the study.

Given that

- there were 1 000 bacteria in this population at the start of the study and
- it took exactly 5 hours from the start of the study for this population to double,

(a) find a complete equation for the model. (4)

**Solution**

$$\begin{aligned}
t = 0, N = 1\,000 &\Rightarrow 1\,000 = Ae^0\\
&\Rightarrow A = 1\,000
\end{aligned}$$

and

$$\begin{aligned}
t = 5, N = 2\,000 &\Rightarrow 2\,000 = 1\,000e^{5k}\\
&\Rightarrow e^{5k} = 2\\
&\Rightarrow 5k = \ln 2\\
&\Rightarrow k = \tfrac{1}{5}\ln 2;
\end{aligned}$$

hence,

$$\underline{\underline{N = 1\,000e^{(\frac{1}{5}\ln 2)t}}}.$$

(b) Hence find the rate of increase in the number of bacteria in this population exactly 8 hours from the start of the study. Give your answer to 2 significant figures. (2)

**Solution**

Well,

$$N = 1\,000\mathrm{e}^{(\frac{1}{5}\ln 2)t} \Rightarrow \frac{\mathrm{d}N}{\mathrm{d}t} = 200\ln 2\mathrm{e}^{(\frac{1}{5}\ln 2)t}$$

and

$$\begin{aligned} t = 8 \Rightarrow \frac{\mathrm{d}N}{\mathrm{d}t} &= 200\ln 2\mathrm{e}^{(\frac{1}{5}\ln 2)8} \\ \Rightarrow \frac{\mathrm{d}N}{\mathrm{d}t} &= 420.245\,865\,8 \text{ (FCD)} \\ \Rightarrow \frac{\mathrm{d}N}{\mathrm{d}t} &= 420 \text{ (2 sf)}. \end{aligned}$$

The number of bacteria, $M$, in the **second** population is modelled by the equation

$$M = 500\mathrm{e}^{1.4kt},\ t > 0,$$

where $k$ has the value found in part (a) and $t$ is the time in hours from the start of the study.

Given that $T$ hours after the start of the study, the number of bacteria in the two different populations was the same,

(c) find the value of $T$. (3)

**Solution**

$$\begin{aligned} & 1\,000\mathrm{e}^{(\frac{1}{5}\ln 2)T} = 500\mathrm{e}^{1.4(\frac{1}{5}\ln 2)T} \\ \Rightarrow\ & \frac{\mathrm{e}^{(\frac{1}{5}\ln 2)T}}{\mathrm{e}^{(\frac{7}{25}\ln 2)T}} = \tfrac{1}{2} \\ \Rightarrow\ & \mathrm{e}^{(\frac{1}{5}\ln 2)T-(\frac{7}{25}\ln 2)T} = \tfrac{1}{2} \\ \Rightarrow\ & \mathrm{e}^{(\frac{1}{5}\ln 2-\frac{7}{25}\ln 2)T} = \tfrac{1}{2} \\ \Rightarrow\ & (\tfrac{1}{5}\ln 2 - \tfrac{7}{25}\ln 2)T = \ln\tfrac{1}{2} \\ \Rightarrow\ & T = \frac{\ln\frac{1}{2}}{\frac{1}{5}\ln 2 - \frac{7}{25}\ln 2} \\ \Rightarrow\ & T = 12.5 \text{ hours (exactly!)}. \end{aligned}$$

9.

$$\mathrm{f}(x) = \frac{50x^2+38x+9}{(5x+2)^2(1-2x)},\ x \neq -\tfrac{2}{5},\ x \neq \tfrac{1}{2}.$$

Given that $\mathrm{f}(x)$ can be expressed in the form

$$\frac{A}{5x+2} + \frac{B}{(5x+2)^2} + \frac{C}{1-2x},$$

where $A$, $B$, $C$ are constants.

(a) (i) find the value of $B$ and the value of $C$, (4)

**Solution**

$$\frac{50x^2+38x+9}{(5x+2)^2(1-2x)} \equiv \frac{A}{5x+2} + \frac{B}{(5x+2)^2} + \frac{C}{1-2x}$$
$$\equiv \frac{A(5x+2)+B(1-2x)+C(5x+2)^2}{(5x+2)^2(1-2x)}$$

and hence

$$50x^2+38x+9 \equiv A(5x+2)+B(1-2x)+C(5x+2)^2.$$

$\underline{x=-\frac{2}{5}}$: $1.8 = 1.8B \Rightarrow B = 1$.
$\underline{x=\frac{1}{2}}$: $40.5 = 4.5A + 20.25C \quad (1)$.
$\underline{x=0}$:

$$9 = 2A + 1 + 4C \Rightarrow 4 = A + 2C \Rightarrow 18 = 4.5C + 9C \quad (2).$$

Do (1) − (2):

$$22.5 = 11.25C \Rightarrow \underline{\underline{C=2}}.$$

(ii) show that $A = 0$.

**Solution**

$$4 = A + 4 \Rightarrow \underline{\underline{A=0}}.$$

(b) (i) Use binomial expansions to show that, in ascending powers of $x$, (7)

$$\mathrm{f}(x) = p + qx + rx^2 + \ldots,$$

where $p$, $q$, and $r$ are simplified fractions to be found.

**Solution**

$$\begin{aligned} f(x) &= \frac{1}{(5x+2)^2} + \frac{2}{1-2x} \\ &= (2+5x)^{-2} + 2(1-2x)^{-1} \\ &= \left[2(1+\tfrac{5}{2}x)^{-2}\right] + 2(1-2x)^{-1} \\ &= \tfrac{1}{4}(1+\tfrac{5}{2}x)^{-2} + 2(1-2x)^{-1}. \end{aligned}$$

Now,

$$\begin{aligned} (1+\tfrac{5}{2}x)^{-2} &= (1-(-\tfrac{5}{2}x))^{-2} \\ &= 1 + (-2)(\tfrac{5}{2}x) + \frac{(-2)(-3)}{2!}(\tfrac{5}{2}x)^2 + \dots \\ &= 1 - 5x + \tfrac{75}{4}x^2 + \dots \end{aligned}$$

for

$$|\tfrac{5}{2}x| < 1 \Rightarrow |x| < \tfrac{2}{5}$$

and

$$\begin{aligned} (1-2x)^{-1} &= 1 + (-1)(-2x) + + \frac{(-1)(-2)}{2!}(-2x)^2 + \dots \\ &= 1 + 2x + 4x^2 + \dots \end{aligned}$$

for

$$|2x| < 1 \Rightarrow |x| < \tfrac{1}{2}.$$

Hence,

$$\begin{aligned} f(x) &= \tfrac{1}{4}(1 - 5x + \tfrac{75}{4}x^2 + \dots) + 2(1 + 2x + 4x^2 + \dots) \\ &= \underline{\underline{\tfrac{9}{4} + \tfrac{11}{4}x + \tfrac{203}{16}x^2 + \dots}}. \end{aligned}$$

(ii) Find the range of values of $x$ for which this expansion is valid.

**Solution**

We have to find the most restrictive conditions:

$$\underline{\underline{|x| < \tfrac{2}{5}}}.$$

10. (a) Given that (4)

$$1 + \cos 2\theta + \sin 2\theta \neq 0,$$

prove that

$$\frac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} \equiv \tan\theta.$$

**Solution**

$$\begin{aligned}\frac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} &\equiv \frac{1 - (1 - 2\sin^2\theta) + 2\sin\theta\cos\theta}{1 + (2\cos^2\theta - 1) + 2\sin\theta\cos\theta} \\ &\equiv \frac{2\sin^2\theta + 2\sin\theta\cos\theta}{2\cos^2\theta + 2\sin\theta\cos\theta} \\ &\equiv \frac{2\sin\theta(\sin\theta + \cos\theta)}{2\cos\theta(\sin\theta + \cos\theta)} \\ &\equiv \frac{\sin\theta}{\cos\theta} \\ &\equiv \underline{\underline{\tan\theta}},\end{aligned}$$

as required.

(b) Hence solve, for $0^\circ < x < 180^\circ$, (4)

$$\frac{1 - \cos 4x + \sin 4x}{1 + \cos 4x + \sin 4x} \equiv 3\sin 2x,$$

giving your answers to one decimal place where appropriate.

**Solution**

$$\begin{aligned} & \frac{1 - \cos 4x + \sin 4x}{1 + \cos 4x + \sin 4x} \equiv 3\sin 2x \\ \Rightarrow\ & \frac{1 - \cos 2(2x) + \sin 2(2x)}{1 + \cos 2(2x) + \sin 2(2x)} \equiv 3\sin 2x \\ \Rightarrow\ & \tan 2x \equiv 3\sin 2x \\ \Rightarrow\ & \frac{\sin 2x}{\cos 2x} \equiv 3\sin 2x \\ \Rightarrow\ & \sin 2x \equiv 3\sin 2x\cos 2x \\ \Rightarrow\ & 3\sin 2x\cos 2x - \sin 2x \equiv 0 \\ \Rightarrow\ & \sin 2x(3\cos 2x - 1) \equiv 0 \\ \Rightarrow\ & \sin 2x = 0 \text{ or } \cos 2x = \tfrac{1}{3}.\end{aligned}$$

$\underline{\sin 2x = 0}$:

$$\begin{aligned} \sin 2x = 0 &\Rightarrow 2x = 180 \\ &\Rightarrow \underline{\underline{x = 90}}. \end{aligned}$$

$\underline{\cos 2x = \tfrac{1}{3}}$:

$$\begin{aligned} \cos 2x = \tfrac{1}{3} &\Rightarrow 2x = 75.528\,779\,37,\ 289.471\,220\,6 \text{ (FCD)} \\ &\Rightarrow x = 35.264\,389\,68,\ 144.735\,610\,3\,6 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 35.3,\ 144.7 \text{ (1 dp)}}}. \end{aligned}$$

11. Figure 2 shows a sketch of part of the curve with equation

$$y = (\ln x)^2,\ x > 0.$$

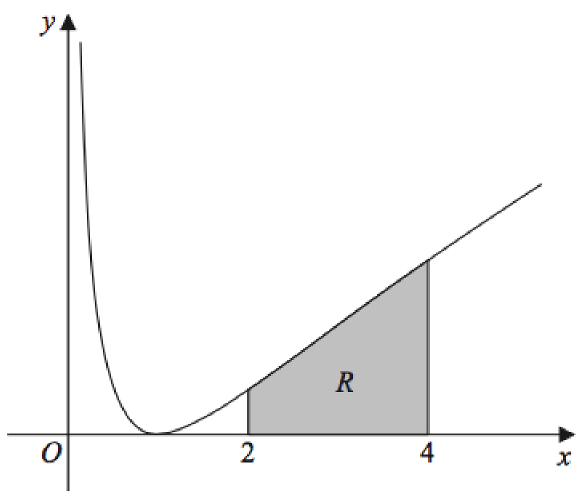


Figure 2: $y = (\ln x)^2$

The finite region $R$, shown shaded in Figure 2, is bounded by the curve, the line with equation $x = 2$, the $x$-axis, and the line with equation $x = 4$.

The table below shows corresponding values of $x$ and $y$, with the values of $y$ given to 4 decimal places.

| $x$ | 2 | 2.5 | 3 | 3.5 | 4 |
|---|---|---|---|---|---|
| $y$ | 0.480 5 | 0.839 6 | 1.206 9 | 1.569 4 | 1.921 8 |

(a) Use the trapezium rule, with all the values of $y$ in the table, to obtain an estimate for the area of $R$, giving your answer to 3 significant figures. (3)

**Solution**

Let $h = 0.5$.

$$\begin{aligned}\text{Area} &\approx \tfrac{1}{2} \times 0.5 \times [0.480\,5 + 2(0.839\,6 + 1.206\,9 + 1.569\,4) + 1.921\,8] \\ &= 2.408\,525 \\ &= \underline{\underline{2.41\ (3\text{ dp})}}.\end{aligned}$$

(b) Use algebraic integration to find the exact area of $R$, giving your answer in the form (5)

$$y = a(\ln 2)^2 + b\ln 2 + c,$$

where $a$, $b$, and $c$ are integers to be found.

**Solution**

We use integration by parts:

$$\begin{aligned}u = (\ln x)^2 &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2\ln x}{x} \\ \frac{\mathrm{d}v}{\mathrm{d}x} = 1 &\Rightarrow v = x.\end{aligned}$$

So

$$\begin{aligned}\int (\ln x)^2\,\mathrm{d}x &= x(\ln x)^2 - \int \left(x \times \frac{2\ln x}{x}\right)\,\mathrm{d}x \\ &= x(\ln x)^2 - \int 2\ln x\,\mathrm{d}x \\ &= x(\ln x)^2 - 2(x\ln x - x) + c \\ &= x(\ln x)^2 - 2x\ln x + 2x + c\end{aligned}$$

and

$$\begin{aligned}\int_2^4 (\ln x)^2\,\mathrm{d}x &= \left[x(\ln x)^2 - 2x\ln x + 2x\right]_{x=2}^4 \\ &= \big(4(\ln 4)^2 - 8\ln 4 + 8\big) - \big(2(\ln 2)^2 - 4\ln 2 + 4\big) \\ &= \big(4(\ln 2^2)^2 - 8\ln 2^2 + 8\big) - \big(2(\ln 2)^2 - 4\ln 2 + 4\big) \\ &= \big(4(2\ln 2)^2 - 16\ln 2 + 8\big) - \big(2(\ln 2)^2 - 4\ln 2 + 4\big) \\ &= \big(16(\ln 2)^2 - 16\ln 2 + 8\big) - \big(2(\ln 2)^2 - 4\ln 2 + 4\big) \\ &= \underline{\underline{12(\ln 2)^2 - 12\ln 2 + 4}};\end{aligned}$$

hence, $\underline{\underline{a = 14}}$, $\underline{\underline{b = -12}}$, and $\underline{\underline{c = 4}}$.

12. Figure 3 is a graph of the trajectory of a golf ball after the ball has been hit until it first hits the ground.

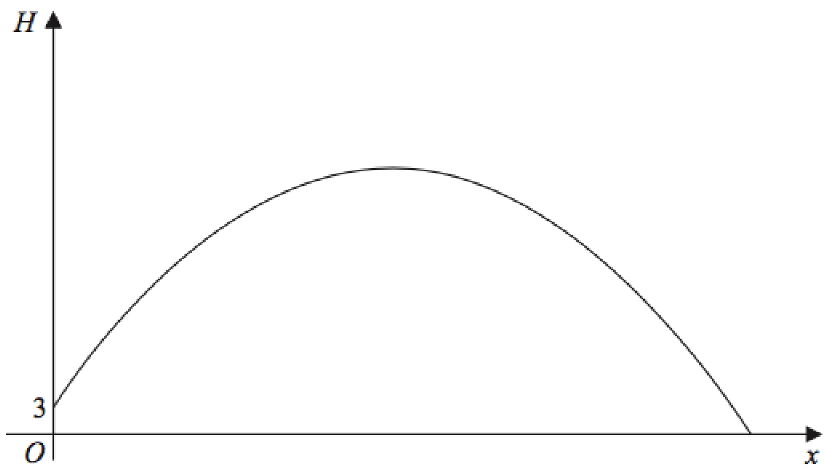


Figure 3: a golf ball

The vertical height, $H$ metres, of the ball above the ground has been plotted against the horizontal distance travelled, $x$ metres, measured from where the ball was hit.

The ball is modelled as a particle travelling in a vertical plane above horizontal ground.

Given that the ball

- is hit from a point on the top of a platform of vertical height 3 m above the ground,
- reaches its maximum vertical height after travelling a horizontal distance of 90 m, and
- is at a vertical height of 27 m above the ground after travelling a horizontal distance of 120 m.

Given also that $H$ is modelled as a **quadratic** function in $x$,

(a) find $H$ in terms of $x$. (5)

**Solution**
Let
$$H(x) = ax^2 + bx + c,$$

where $a$, $b$, and $c$ are all to be determined. Now,

$$\begin{aligned} x = 0, H = 3 &\Rightarrow 3 = 0 + 0 + c \\ &\Rightarrow c = 3. \end{aligned}$$

and

$$\begin{aligned} x = 120, H = 27 &\Rightarrow 27 = 14\,400a + 120b + 3 \\ &\Rightarrow 14\,400a + 120b = 24 \\ &\Rightarrow 600a + 5b = 1 \quad (1). \end{aligned}$$

Next,

$$H(x) = ax^2 + bx + 3 \Rightarrow \frac{\mathrm{d}H}{\mathrm{d}x} = 2ax + b$$

and

$$\begin{aligned} x = 90, \frac{\mathrm{d}H}{\mathrm{d}x} = 0 &\Rightarrow 0 = 180a + b \\ &\Rightarrow 0 = 900a + 5b \quad (2). \end{aligned}$$

Do (2) − (1):

$$\begin{aligned} 300a = -1 &\Rightarrow a = -\tfrac{1}{300} \\ &\Rightarrow b = \tfrac{3}{5}. \end{aligned}$$

Hence,

$$\underline{\underline{H(x) = -\tfrac{1}{300}x^2 + \tfrac{3}{5}x + 3.}}$$

(b) Hence find, according to the model, (3)

(i) the maximum vertical height of the ball above the ground,

**Solution**

$$x = 90 \Rightarrow H(90) = -\tfrac{1}{300}(90^2) + \tfrac{3}{5}(90) + 3 = \underline{\underline{30 \text{ m}}}.$$

(ii) the horizontal distance travelled by the ball, from when it was hit to when it first hits the ground, giving your answer to the nearest metre.

**Solution**

$$
\begin{aligned}
-\tfrac{1}{300}x^2 + \tfrac{3}{5}x + 3 = 0 &\Rightarrow -\tfrac{1}{300}x^2 + \tfrac{3}{5}x = -3 \\
&\Rightarrow -\tfrac{1}{300}[x^2 - 180x] = -3 \\
&\Rightarrow x^2 - 180x = 900 \\
&\Rightarrow x^2 - 180x + 8\,100 = 900 + 8\,100 \\
&\Rightarrow (x-90)^2 = 9\,000 \\
&\Rightarrow x - 90 = \pm 30\sqrt{10} \\
&\Rightarrow x = 90 \pm 30\sqrt{10} \\
&\Rightarrow x = -4.868\,329\,805,\ 184.868\,329\,8 \text{ (FCD)};
\end{aligned}
$$

because $x > 0$, $\underline{\underline{x = 185 \text{ m (nearest metre)}}}$.

(c) The possible effects of wind or air resistance are two limitations of the model. Give one other limitation of this model. (1)

**Solution**

E.g., the ground is unlikely to be horizontal (it will have incline/decline), the spin on the ball, the ball is not a particle.

13. A curve $C$ has parametric equations (3)

$$x = \frac{t^2+5}{t^2+1},\ y = \frac{4t}{t^2+1},\ t \in \mathbb{R}.$$

Show that all points on $C$ satisfy

$$(x-3)^2 + y^2 = 4.$$

**Solution**

$$
\begin{aligned}
(x-3)^2 + y^2 &= \left(\frac{t^2+5}{t^2+1} - 3\right)^2 + \left(\frac{4t}{t^2+1}\right)^2 \\
&= \left(\frac{t^2+5}{t^2+1} - \frac{3(t^2+1)}{t^2+1}\right)^2 + \left(\frac{4t}{t^2+1}\right)^2 \\
&= \frac{(2-2t^2)^2}{(t^2+1)^2} + \frac{(4t)^2}{(t^2+1)^2} \\
&= \frac{(2-2t^2)^2 + (4t)^2}{(t^2+1)^2} \\
&= \frac{(4-8t^2+4t^4) + 16t^2}{(t^2+1)^2} \\
&= \frac{4t^4+8t^2+4}{(t^2+1)^2} \\
&= \frac{4(t^4+2t^2+1}{(t^2+1)^2} \\
&= \frac{4(t^2+1)^2}{(t^2+1)^2} \\
&= \underline{\underline{4}},
\end{aligned}
$$

as required.

14. Given that (4)

$$y = \frac{x-4}{2+\sqrt{x}},\ x > 0,$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{A\sqrt{x}},\ x > 0,$$

where $A$ is a constant to be found.

**Solution**

We use the quotient rule:

$$
\begin{aligned}
u = x - 4 &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \\
v = 2 + \sqrt{x} &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\frac{1}{2}}{\sqrt{x}}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(2+\sqrt{x})\cdot 1 - (x-4)\cdot(\frac{\frac{1}{2}}{\sqrt{x}})}{(2+\sqrt{x})^2} \\
&= \frac{(2+\sqrt{x}) - (\frac{1}{2}\sqrt{x} - \frac{2}{\sqrt{x}})}{(2+\sqrt{x})^2} \\
&= \frac{(2+\frac{1}{2}\sqrt{x} + \frac{2}{\sqrt{x}})}{(2+\sqrt{x})^2} \\
&= \frac{\frac{1}{2\sqrt{x}}(4\sqrt{x}+x+4)}{(2+\sqrt{x})^2} \\
&= \frac{\frac{1}{2\sqrt{x}}(4+4\sqrt{x}+x)}{(2+\sqrt{x})^2} \\
&= \frac{\frac{1}{2\sqrt{x}}(2+\sqrt{x})^2}{(2+\sqrt{x})^2} \\
&= \underline{\underline{\frac{1}{2\sqrt{x}}}};
\end{aligned}
$$

hence, $\underline{\underline{A = 2}}$.

15. (a) Use proof by exhaustion to show that for $n \in \mathbb{N}$, $n \leqslant 4$, (2)

$$(n+1)^3 > 3^n.$$

**Solution**

$$
\begin{aligned}
n = 1: &\quad \text{LHS} = (1+1)^3 = 8, \\
&\quad \text{RHS} = 3^1 = 3, \\
n = 2: &\quad \text{LHS} = (2+1)^3 = 27, \\
&\quad \text{RHS} = 3^2 = 9, \\
n = 3: &\quad \text{LHS} = (3+1)^3 = 64, \\
&\quad \text{RHS} = 3^3 = 27, \\
n = 4: &\quad \text{LHS} = (4+1)^3 = 125, \\
&\quad \text{RHS} = 4^3 = 64.
\end{aligned}
$$

So, we have **proved** by exhaustion the cases for $1 \leqslant n \leqslant 4$.

(b) Given that (4)

$$m^3 + 5$$

is odd, use proof by contradiction to show, using algebra, that $m$ is even.

**Solution**

Let $m$ be odd, e.g.,

$$m = 2n + 1,\ p \in \mathbb{N}.$$

Then

$$\begin{aligned} m^3 + 5 &= (2n+1)^3 + 5 \\ &= (8n^3 + 12n^n + 6n + 1) + 5 \\ &= 8n^3 + 12n^n + 6n + 6 \\ &= 2(n^3 + 6n^n + 3n + 3), \end{aligned}$$

which is an even number. This is a contradiction: if $m^3 + 5$ is odd, then $m$ is even.