Further Pure Mathematics 3: Part 1

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Further Mathematics

Hyperbolic functions

$$\sinh x \equiv \frac{e^x - e^{-x}}{2}$$
$$\cosh x \equiv \frac{e^x + e^{-x}}{2}$$
$$\tanh x \equiv \frac{\sinh x}{\cosh x} \equiv \frac{e^x - e^{-x}}{e^x + e^{-x}} \equiv \frac{e^{2x} - 1}{e^{2x} + 1}$$
$$\operatorname{cosech} x \equiv \frac{1}{\sinh x} \equiv \frac{2}{e^x - e^{-x}},\ x \neq 0$$
$$\operatorname{sech} x \equiv \frac{1}{\cosh x} \equiv \frac{2}{e^x + e^{-x}}$$
$$\coth x \equiv \frac{\cosh x}{\sinh x} \equiv \frac{e^x + e^{-x}}{e^x - e^{-x}} \equiv \frac{e^{2x} + 1}{e^{2x} - 1},\ x \neq 0$$

- In order to define the inverse functions, however, we sometimes have to restrict the domain.

Function	Domain	Range
$\sinh x$	$x \in \mathbb{R}$	$x \in \mathbb{R}$
$\operatorname{arsinh} x$	$x \in \mathbb{R}$	$x \in \mathbb{R}$
$\cosh x$	$x \geqslant 0$	$x \geqslant 1$
$\operatorname{arcosh} x$	$x \geqslant 1$	$x \geqslant 0$
$\tanh x$	$x \in \mathbb{R}$	$\lvert x\rvert < 1$
$\operatorname{artanh} x$	$\lvert x\rvert < 1$	$x \in \mathbb{R}$
$\operatorname{cosech} x$	$x \neq 0$	$x \neq 0$
$\operatorname{arcosech} x$	$x \neq 0$	$x \neq 0$
$\operatorname{sech} x$	$x \geqslant 0$	$0 < x \leqslant 1$
$\operatorname{arsech} x$	$0 < x \leqslant 1$	$x \geqslant 0$
$\coth x$	$x \neq 0$	$\lvert x\rvert > 1$
$\operatorname{arcoth} x$	$\lvert x\rvert > 1$	$x \neq 0$

$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right)$$
$$\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right),\ x \geqslant 1$$
$$\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ \lvert x\rvert < 1$$
$$\operatorname{arcosech} x = \ln\left(\frac{1+\sqrt{x^2+1}}{x}\right)$$
$$\operatorname{arsech} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right),\ 0 < x \leqslant 1$$
$$\operatorname{arcoth} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right),\ \lvert x\rvert > 1$$

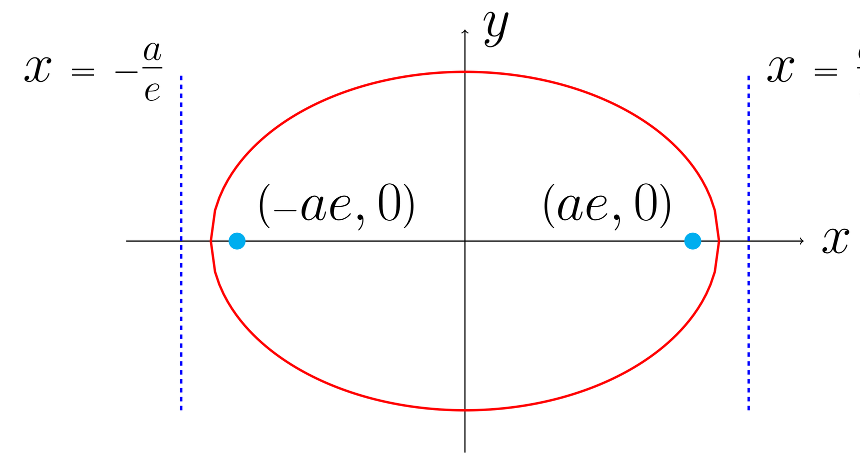
- You need to be able to use **Osborn's Rule** to convert a trigonometric identity into its hyperbolic equivalent. For example,

$$\cos^2 x + \sin^2 x \equiv 1 \to \cosh^2 x - \sinh^2 x \equiv 1$$

Ellipse

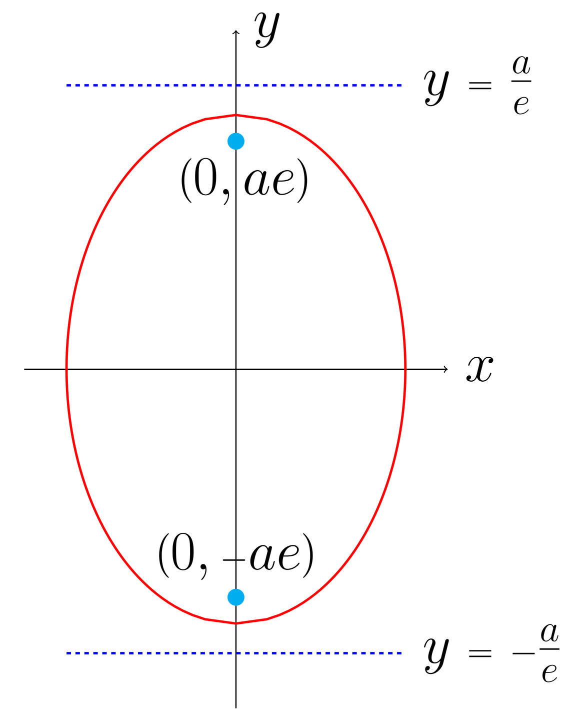
If $a > b$ then
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
is an **ellipse** with

- eccentricity $0 < e < 1$ such that $b^2 = a^2(1 - e^2)$,
- two foci, at $(ae, 0)$ and $(-ae, 0)$,
- two directrices, $x = \frac{a}{e}$ and $x = -\frac{a}{e}$,
- parametric equations $x = a\cos\theta$ and $y = b\sin\theta$.



If $a < b$ then
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
is an **ellipse** with

- eccentricity $0 < e < 1$ such that $a^2 = b^2(1 - e^2)$,
- two foci, at $(0, ae)$ and $(0, -ae)$,
- two directrices, $y = \frac{a}{e}$ and $y = -\frac{a}{e}$,
- parametric equations $x = a\cos\theta$ and $y = b\sin\theta$.

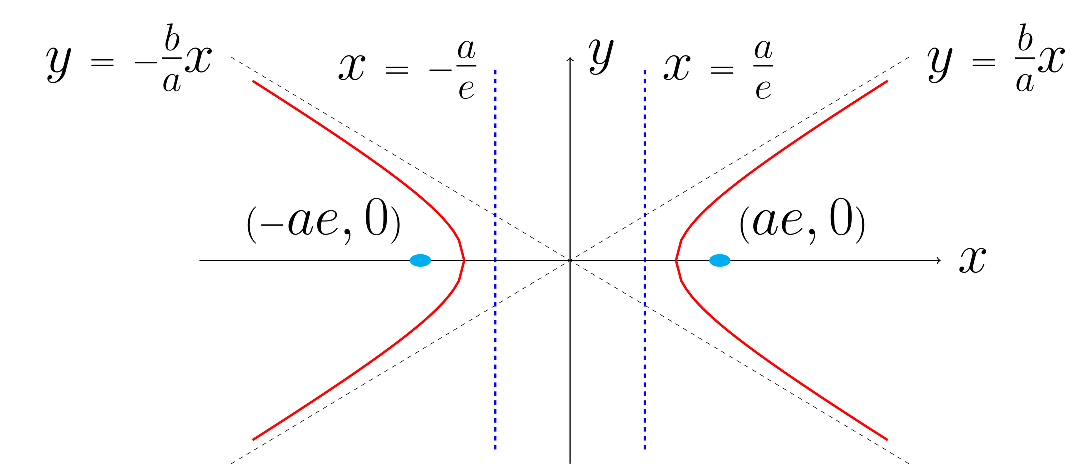


- You should be able to derive the standard results for the equations of the tangents and the normals to the ellipse using parametric equations.
- You should be able to derive the condition that the line $y = mx + c$ is a tangent to the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
if $a^2m^2 + b^2 = c^2$.

Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
is a **hyperbola** with

- eccentricity $e > 1$ such that $b^2 = a^2(e^2 - 1)$,
- two foci, at $(ae, 0)$ and $(-ae, 0)$,
- two directrices, $x = \frac{a}{e}$ and $x = -\frac{a}{e}$,
- parametric equations $x = a\sec\theta$ and $y = b\tan\theta$,
- or $x = a\cosh t$ and $y = b\sinh t$ (although this will only parameterise one branch of the hyperbola),
- asymptotes $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.



- You should be able to derive the standard results for the equations of the tangents and the normals to the hyperbola using parametric equations.
- You should be able to derive the condition that the line $y = mx + c$ is a tangent to the hyperbola
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
if $b^2 + c^2 = a^2m^2$.

Differentiation: Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$
$$\frac{d}{dx}(\cosh x) = \sinh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$
$$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$$
$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$
$$\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$$

Derivatives: Inverse Functions

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}(\operatorname{arsinh} x) = \frac{1}{\sqrt{x^2+1}}$$
$$\frac{d}{dx}(\operatorname{arcosh} x) = \frac{1}{\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\operatorname{artanh} x) = \frac{1}{1-x^2}$$

You should be able to derive all of these results — and this means understanding the domains of the inverse functions. For example,

$$\begin{aligned} & y = \arcsin x \\ \Rightarrow\ & x = \sin y \\ \Rightarrow\ & \frac{dx}{dy} = \cos y \\ \Rightarrow\ & \frac{dx}{dy} = \sqrt{1-\sin^2 y} \\ \Rightarrow\ & \frac{dx}{dy} = \sqrt{1-x^2} \\ \Rightarrow\ & \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

The $\arcsin x$ function has domain $-1 \leqslant x \leqslant 1$ and range $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$. For these values, $\cos y \geqslant 0$ so we can legitimately replace $\cos y$ with the non-negative expression $\sqrt{1-\sin^2 y}$.

Although the derivatives of the $\operatorname{arccosec} x$, $\operatorname{arcsec} x$, and $\operatorname{arccot} x$ (as well as the hyperbolic equivalents) are not required for Further Pure Mathematics 3 you are encouraged to work through their derivatives as a measure of the extent to which you understand the domains and ranges of these functions.