Dr Oliver Mathematics
Advanced Level Mathematics
Mechanics: Calculator
2 hours

The total number of marks available is 50.
You must write down all the stages in your working. Note: It goes with the Statistics paper.

1. *In this question, position vectors are given relative to a fixed origin, O.*
   At time t seconds, where $t \geqslant 0$, a particle, P, moves so that its velocity, $\mathbf{v}$ ms^{-1}, is given by
   $$\mathbf{v} = 6t\mathbf{i} - 5t^{\frac{3}{2}}\mathbf{j}.$$
   When $t = 0$, the position vector of P is $(-20\mathbf{i} + 20\mathbf{j})$.

   (a) Find the acceleration of P when $t = 4$. (3)

   Solution
   $$\mathbf{v} = 6t\mathbf{i} - 5t^{\frac{3}{2}}\mathbf{j} \Rightarrow \mathbf{a} = 6\mathbf{i} - \tfrac{15}{2}t^{\frac{1}{2}}\mathbf{j}$$
   and
   $$t = 4 \Rightarrow \mathbf{a} = \underline{\underline{(6\mathbf{i} - 15\mathbf{j}) \text{ ms}^{-2}}}.$$

   (b) Find the position vector of P when $t = 4$. (3)

   Solution
   $$\mathbf{v} = 6t\mathbf{i} - 5t^{\frac{3}{2}}\mathbf{j} \Rightarrow \mathbf{s} = 3t^2\mathbf{i} - 2t^{\frac{5}{2}}\mathbf{j} + \mathbf{c}.$$
   Now,
   $$t = 0 \Rightarrow -20\mathbf{i} + 20\mathbf{j} = \mathbf{c}$$
   which means
   $$\mathbf{s} = (3t^2 - 20)\mathbf{i} + (20 - 2t^{\frac{5}{2}})\mathbf{j}.$$
   Finally,
   $$t = 4 \Rightarrow \mathbf{s} = \underline{\underline{(28\mathbf{i} - 44\mathbf{j}) \text{ m}}}.$$

2. A particle, P, moves with constant acceleration $(2\mathbf{i} - 3\mathbf{j})$ ms^{-2}.
   At time $t = 0$, the particle is at the point A and is moving with velocity $(-\mathbf{i} + 4\mathbf{j})$ ms^{-1}.
   At time $t = T$ seconds, P is moving in the direction of vector $(3\mathbf{i} - 4\mathbf{j})$.

   (a) Find the value of T. (4)

Solution

$$\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} \Rightarrow \mathbf{v} = 2t\mathbf{i} - 3t\mathbf{j} + \mathbf{c}.$$

Now,

$$t = 0 \Rightarrow -\mathbf{i} + 4\mathbf{j} = \mathbf{c}$$

and so

$$\mathbf{v} = (2t - 1)\mathbf{i} + (4 - 3t)\mathbf{j}.$$

Next,

$$t = T \Rightarrow \mathbf{v} = (2T - 1)\mathbf{i} + (4 - 3T)\mathbf{j}$$

is parallel to $(3\mathbf{i} - 4\mathbf{j})$:

$$\begin{aligned} 2T - 1 = 3\mu \Rightarrow 2T &= 3\mu + 1 \\ \Rightarrow T &= \tfrac{3}{2}\mu + \tfrac{1}{2} \end{aligned}$$

and

$$\begin{aligned} 4 - 3T = -4\mu \Rightarrow 3T &= 4\mu + 4 \\ \Rightarrow T &= \tfrac{4}{3}\mu + \tfrac{4}{3}. \end{aligned}$$

So

$$\begin{aligned} \tfrac{3}{2}\mu + \tfrac{1}{2} = \tfrac{4}{3}\mu + \tfrac{4}{3} &\Rightarrow \tfrac{1}{6}\mu = \tfrac{5}{6} \\ &\Rightarrow \mu = 5 \\ &\Rightarrow \underline{\underline{T = 8}}. \end{aligned}$$

At time $t = 4$ seconds, P is at the point B.

(b) Find the distance AB (4)

Solution

$$\mathbf{v} = (2t - 1)\mathbf{i} + (4 - 3t)\mathbf{j} \Rightarrow \mathbf{s} = (t^2 - t)\mathbf{i} + (4t - \tfrac{3}{2}t^2)\mathbf{j} + \mathbf{d}.$$

Now,

$$t = 0 \Rightarrow \mathbf{s} = \mathbf{d}$$

and

$$t = 4 \Rightarrow \mathbf{s} = 12\mathbf{i} - 8\mathbf{j} + \mathbf{d}.$$

Finally,

$$\begin{aligned} AB &= \sqrt{12^2 + 8^2} \\ &= 4\sqrt{13}. \end{aligned}$$

3. Two blocks, A and B, of masses $2m$ and $3m$ respectively, are attached to the ends of a light string.
Initially, A is held at rest on a fixed rough plane.
The plane is inclined at an angle α to the horizontal ground, where $\tan\alpha = \frac{5}{12}$.
The string passes over a small smooth pulley, P, fixed at the top of the plane.
The part of the string from A to P is parallel to a line of greatest slope of the plane.
Block B hangs freely below P, as shown in Figure 1.

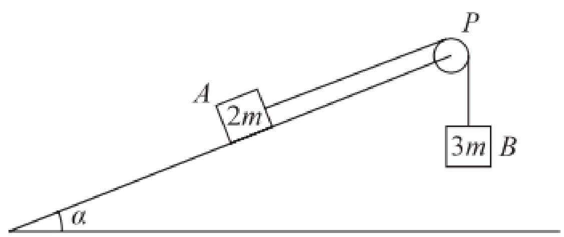

Figure 1: two blocks, A and B, of masses $2m$ and $3m$ respectively

The coefficient of friction between A and the plane is $\frac{2}{3}$.
The blocks are released from rest with the string taut and A moves up the plane.
The blocks are modelled as particles and the string is modelled as being inextensible.

(a) Show the (8)

$$T = \tfrac{12}{5}mg.$$

Solution

Let T N be the tension, R N be the normal reaction, and let a ms^{-2} be the acceleration. Now, Newton's Second Law:

$$\begin{aligned} \text{Parallel:}&\quad T - F - 2mg\sin\alpha = 2ma \\ \text{Perpendicular:}&\quad R = 2mg\cos\alpha \\ F = \mu R:&\quad F = \tfrac{2}{3}R \\ B:&\quad 3mg - T = 3ma. \end{aligned}$$

Next,

$$3mg - T = 3ma \Rightarrow a = \frac{3mg - T}{3m}.$$

Finally,

$$\begin{aligned} F = T - 2mg\sin\alpha - 2ma \Rightarrow T - 2mg\sin\alpha - 2ma &= \tfrac{2}{3}(2mg\cos\alpha) \\ \Rightarrow T &= \tfrac{4}{3}mg\cos\alpha + 2mg\sin\alpha + 2ma \\ \Rightarrow T &= \tfrac{4}{3}mg(\tfrac{12}{13}) + 2mg(\tfrac{5}{13}) + 2ma \\ \Rightarrow T &= 2mg + 2m\left(\frac{3mg - T}{3m}\right) \\ \Rightarrow T &= 2mg + \tfrac{2}{3}(3mg - T) \\ \Rightarrow T &= 2mg + 2mg - \tfrac{2}{3}T \\ \Rightarrow \tfrac{5}{3}T &= 4mg \\ \Rightarrow \underline{\underline{T = \tfrac{12}{5}mg}}, \end{aligned}$$

as required.

After B reaches the ground, A continues to move up the plane until it comes to rest before reaching P.

(b) Determine whether A will remain at rest, carefully justifying your answer. (2)

Solution

Well,

$$F_{\text{max}} = \tfrac{4}{3}mg\cos\alpha = \tfrac{16}{3}mg$$

whereas

$$F_{\text{parallal}} = 2mg\sin\alpha = \tfrac{10}{13}mg;$$

hence, A does not move.

(c) Suggest two refinements to the model that would make it more realistic. (2)

Solution

E.g., extensible string, mass of the string, friction at pulley, do not model the blocks as particles, air resistance.

4. A ramp, AB, of length 8 m and mass 20 kg, rests in equilibrium with end A on rough horizontal ground.
The ramp rests on a smooth solid cylindrical drum which is party under the ground.
The drum is fixed with its axis at the same horizontal level as A.

The point of contact between the ramp and the drum is C, where $AC = 5$ m, as shown in Figure 2.

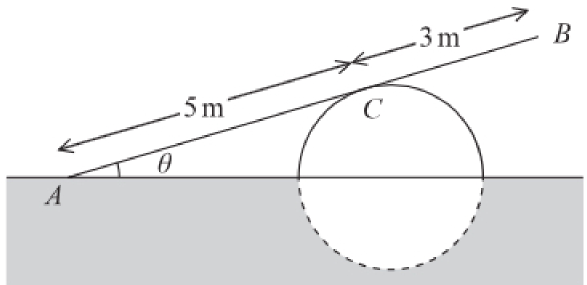

Figure 2: a ramp, AB, of length 8 m and mass 20 kg

The ramp is resting in a vertical plane which is perpendicular to the axis of the drum, at an angle θ to the horizontal, where $\tan\theta = \frac{7}{24}$.
The ramp is modelled as a uniform rod.

(a) Explain why the reaction from the drum on the ramp at point C acts in a direction which is perpendicular to the ramp. (1)

Solution

The drum is smooth.

(b) Find the magnitude of the resultant force acting on the ramp at A. (9)

Solution

C: N N be the normal reaction.
A: F N be the frictional force and R N be the normal reaction.

$$\begin{aligned} R(\updownarrow): &\quad R + N\cos\alpha = 20g \\ R(\leftrightarrow): &\quad F = N\sin\alpha \\ \text{Limiting equilibrium}: &\quad F = \mu R \\ \text{Moments about } A: &\quad (4\cos\alpha)(20g) = 5N. \end{aligned}$$

Now,

$$\begin{aligned} N &= 16g\cos\alpha = 150.528 \\ R &= 20g - N\cos\alpha = 51.493\,12 \\ F &= N\sin\alpha = 42.147\,84. \end{aligned}$$

Finally,

$$\begin{aligned}\text{force} &= \sqrt{F^2 + R^2}\\ &= 66.543\,082\,47 \text{ (FCD)}\\ &= \underline{\underline{67 \text{ N (2 sf)}}}.\end{aligned}$$

The ramp is still in equilibrium in the position shown in Figure 2 but the ramp is not now modelled as being uniform.
Given that the centre of mass of the ramp is assumed to be closer to A than to B,

(c) state how this would affect the magnitude of the normal reaction between the ramp and the drum at C. (1)

Solution
The magnitude of the normal reaction at C will decrease.

5. The points A and B lie 50 m apart on level ground.
At time $t = 0$, two small balls, P and Q, are projected in the vertical plane containing AB.
Ball P is projected from A with speed 20 ms^{-1} at 30° to AB.
Ball Q is projected from B with speed u ms^{-1} at angle θ to BA, as shown in Figure 3.

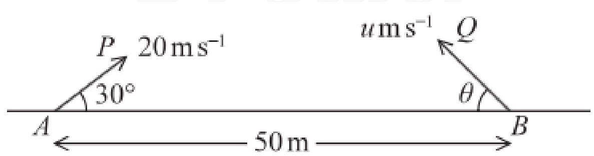

Figure 3: two small balls, P and Q

At time $t = 2$ seconds, P and Q collide.
Until they collide, the balls are modelled as particles moving freely under gravity.

(a) Find the magnitude and direction of P at the instant before it collides with Q. (6)

Solution
At $t = 2$, P's horizontal velocity is $20\cos 30°$ and its vertical velocity is

$$\begin{aligned}v &= 20\sin 30° + 2(-9.8)\\ &= 20\sin 30° - 19.6.\end{aligned}$$

Finally, the velocity of P is

$$\begin{aligned} \text{velocity} &= \sqrt{(20\cos 30^\circ)^2 + (20\sin 30^\circ - 19.6)^2} \\ &= 19.803\,030\,07 \text{ (FCD)} \\ &= \underline{\underline{20 \text{ ms}^{-1} \text{ (2 sf)}}} \end{aligned}$$

and

$$\begin{aligned} \text{direction} &= \tan^{-1}\left(\frac{20\sin 30^\circ - 19.6}{20\cos 30^\circ}\right) \\ &= -28.997\,686\,08 \text{ (FCD)} \\ &= -29^\circ \text{ (2 sf)} \end{aligned}$$

hence, it is $\underline{\underline{29^\circ \text{ (2 sf)}}}$ below the x-axis.

(b) Find (6)

(i) the size of angle θ,

Solution
Horizontally:

$$\begin{aligned} 2(20\cos 20^\circ) + 2(u\cos\theta^\circ) = 50 &\Rightarrow 20\cos 20^\circ + u\cos\theta^\circ = 25 \\ &\Rightarrow u\cos\theta^\circ = 25 - 20\cos 30^\circ \quad (1). \end{aligned}$$

Vertically: $s = ut + \frac{1}{2}a^2t$:

$$\begin{aligned} & 2(20\sin 30^\circ) + \tfrac{1}{2}(-9.8)^2(2) = 2(u\sin\theta^\circ) + \tfrac{1}{2}(-9.8)^2(2) \\ \Rightarrow\quad & 20\sin 30^\circ = u\sin\theta^\circ \quad (2). \end{aligned}$$

Now, divide (2) by (1):

$$\begin{aligned} \tan\theta^\circ = \frac{u\sin\theta^\circ}{u\cos\theta^\circ} &\Rightarrow \tan\theta^\circ = \frac{20\sin 30^\circ}{25 - 20\cos 30^\circ} \\ &\Rightarrow \theta = 52.477\,568\,49 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\theta = 52 \text{ (2 sf)}}}. \end{aligned}$$

(ii) the value of u.

Solution

$$\begin{aligned} u &= \frac{20 \sin 30^\circ}{\sin \theta^\circ} \\ &= 12.608\,512\,85 \text{ (FCD)} \\ &= \underline{\underline{13 \text{ ms}^{-1} \text{ (2 sf)}}}. \end{aligned}$$

(c) State one limitation of the model, other than air resistance, that could affect the accuracy of your answers. (1)

Solution

E.g., the model does not take account of the fact that they are not particles, the model does not take account of the size(s) of the balls, the model does not take account of the spin of the balls, the model does not take account of the wind, the acceleration due to gravity is not exactly 9.8.