Dr Oliver Mathematics
OCR FMSQ Additional Mathematics
2023 Paper
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.
Final answers should be given correct to three significant figures where appropriate.

1. Find the term independent of x in the expansion of (3)

$$\left(x + \frac{2}{x}\right)^6.$$

Solution

The general term is

$$\begin{aligned}\binom{6}{n} x^n \left(\frac{2}{x}\right)^{6-n} &= \binom{6}{n} x^n (2^{6-n}) x^{-(6-n)} \\ &= \binom{6}{n} x^n 2^{6-n} x^{n-6} \\ &= \binom{6}{n} 2^{6-n} x^{2n-6}.\end{aligned}$$

So, the he term independent of x is

$$2n - 6 = 0 \Rightarrow n = 3$$

and the term is

$$\binom{6}{3} 2^3 = \underline{\underline{160}}.$$

2. Jack throws 4 ordinary six-sided dice numbered 1 to 6. (3)

   Find the probability that he throws at least one 3.

Solution

Well,

$$\begin{aligned} \text{P(at least one 3)} &= 1 - \text{P(no 3s)} \\ &= 1 - (\tfrac{5}{6})^4 \\ &= \underline{\underline{\tfrac{671}{1\,296}}}. \end{aligned}$$

3. Use long division to find the quotient and the remainder when (3)

$$x^3 + 3x^2 + 5x - 3$$

is divided by $(x + 1)$.

Solution

$$\begin{array}{r|rrrr} & & x^2 & +2x & +3 \\ \hline x+1 & x^3 & +3x^2 & +5x & -3 \\ & x^3 & +x^2 & & \\ \hline & & 2x^2 & +5x & -3 \\ & & 2x^2 & +2x & \\ \hline & & & 3x & -3 \\ & & & 3x & +3 \\ \hline & & & & -6 \end{array}$$

So, the quotient is

$$\underline{\underline{x^2 + 2x + 3}}$$

and the remainder is

$$\underline{\underline{-6}}.$$

4. Simplify the following.

(a) $\dfrac{1}{x-2} - \dfrac{2}{x+1}$. (2)

Solution

$\times$	x	-2
x	x^2	$-2x$
$+1$	$+x$	-2

so

$$\begin{aligned}\frac{1}{x-2}-\frac{2}{x+1} &= \frac{(x+1)-2(x-2)}{(x-2)(x+1)} \\ &= \frac{x+1-2x+4}{(x-2)(x+1)} \\ &= \underline{\underline{\frac{-x+5}{(x-2)(x+1)}}}.\end{aligned}$$

(b) **In this question you must show detailed reasoning.** (3)

$\dfrac{2}{5-\sqrt{2}}+\dfrac{1}{5+\sqrt{2}}$.

Solution

$\times$	5	$-\sqrt{2}$
5	25	$-5\sqrt{2}$
$+\sqrt{2}$	$+5\sqrt{2}$	-2

so

$$\begin{aligned}\frac{2}{5-\sqrt{2}}+\frac{1}{5+\sqrt{2}} &= \frac{2(5+\sqrt{2})+(5-\sqrt{2})}{23} \\ &= \frac{10+2\sqrt{2}+5-\sqrt{2}}{23} \\ &= \underline{\underline{\frac{15+\sqrt{2}}{23}}}.\end{aligned}$$

5. You are given that

$$\sin\theta = -0.6 \text{ for } 270^\circ \leqslant \theta \leqslant 360^\circ.$$

(a) Find the value of θ. (2)

Solution

$$\sin\theta = -0.6 \Rightarrow \theta = 323.130\,102\,4 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\theta = 323^\circ \text{ (3 sf)}}}.$$

(b) Using Pythagoras' theorem, determine the **exact** value of $\tan\theta$. (4)

Solution

Well,

$$\begin{aligned} \text{opp}^2 + \text{adj}^2 = \text{hyp}^2 &\Rightarrow 0.6^2 + \text{adj}^2 = 1^2 \\ &\Rightarrow 0.36 + \text{adj}^2 = 1 \\ &\Rightarrow \text{adj}^2 = 0.64 \\ &\Rightarrow \text{adj} = 0.8. \end{aligned}$$

So,

$$\tan\theta = \frac{-0.6}{0.8} = \underline{\underline{-\tfrac{3}{4}}}.$$

6. A car accelerates from rest in a straight line.

At time t seconds its velocity, v m s^{-1}, is given by the equation

$$v = 20\left(1 - 2^{-\frac{1}{2}t}\right).$$

(a) Calculate the velocity of the car when $t = 4$, 6, and 8 seconds. (2)

Solution

Well,

$$t = 4 \Rightarrow v = 20\left(1 - 2^{-2}\right)$$
$$\Rightarrow \underline{\underline{v = 15 \text{ m s}^{-1}}},$$
$$t = 6 \Rightarrow v = 20\left(1 - 2^{-3}\right)$$
$$\Rightarrow \underline{\underline{v = 17\tfrac{1}{2} \text{ m s}^{-1}}},$$
$$t = 8 \Rightarrow v = 20\left(1 - 2^{-4}\right)$$
$$\Rightarrow \underline{\underline{v = 18\tfrac{3}{4} \text{ m s}^{-1}}}.$$

(b) Hence calculate an estimate of the acceleration of the car at $t = 6$ seconds. Give your answer correct to **2** significant figures. (2)

Solution

$$\frac{18.75 - 15}{4} = \frac{3.75}{4}$$
$$= \underline{\underline{0.9375 \text{ m s}^{-1}}}.$$

(c) Explain how this estimate could be improved. (1)

Solution

E.g., smaller intervals.

7. You are given that the equation

$$3^x - 4x^2 = 0$$

has three roots, α, β, and γ where $\alpha < 0$ and $\gamma > 3$.

(a) By considering the value of (2)

$$3^x - 4x^2 = 0$$

when $x = 0$ and $x = 1$, show that β lies between 0 and 1.

Solution

Well,

$$x = 0 \Rightarrow y = 1,$$
$$x = 1 \Rightarrow y = -1;$$

because the function is continuous, it must have a solution between $x = 0$ and $x = 1$.

(b) By considering appropriate values of x, determine the value of β correct to 1 decimal place. (3)

Solution

Number	$3^x - 4x^2$	Value
0.75	0.029 5	Too big
0.80	−0.151	Too small

So,

$$0.75 < x < 0.80,$$

and

$$x = 0.8 \text{ (1 dp)}.$$

8. The triangle ABC is such that $AB = 12$ cm and angle $BAC = 50°$.

(a) Given that $BC = 10$ cm, determine the two possible values of angle ACB. (4)

Solution
Sine rule:

$$\frac{\sin ACB}{AB} = \frac{\sin BAC}{BC} \Rightarrow \frac{\sin ACB}{12} = \frac{\sin 50^\circ}{10}$$
$$\Rightarrow \sin ACB = \frac{12 \sin 50^\circ}{10}$$
$$\Rightarrow \angle ACB = 66.817\,167\,09,\ 113.182\,832\,9 \text{ (FCD)}$$
$$\Rightarrow \angle ACB = 66.8^\circ,\ 113^\circ \text{ (3 sf)}.$$

(b) State **two** conditions for BC such that if **either** of them is satisfied there will be only **one** value for the angle ACB. (2)

Solution
E.g., $BC > 12$ cm
BC is perpendicular to AC.

9. The point A has the coordinate $(3, 7)$ and the point B has the coordinate $(7, 1)$. (5)

Find the equation of the perpendicular bisector of AB.

Solution

Let C be the midpoint. Then C is the point

$$\left(\frac{3+7}{2}, \frac{7+1}{2}\right) = (5, 4).$$

Now,

$$\begin{aligned} m_{AB} &= \frac{1-7}{7-3} \\ &= \frac{-6}{4} \\ &= -\frac{3}{2} \end{aligned}$$

and

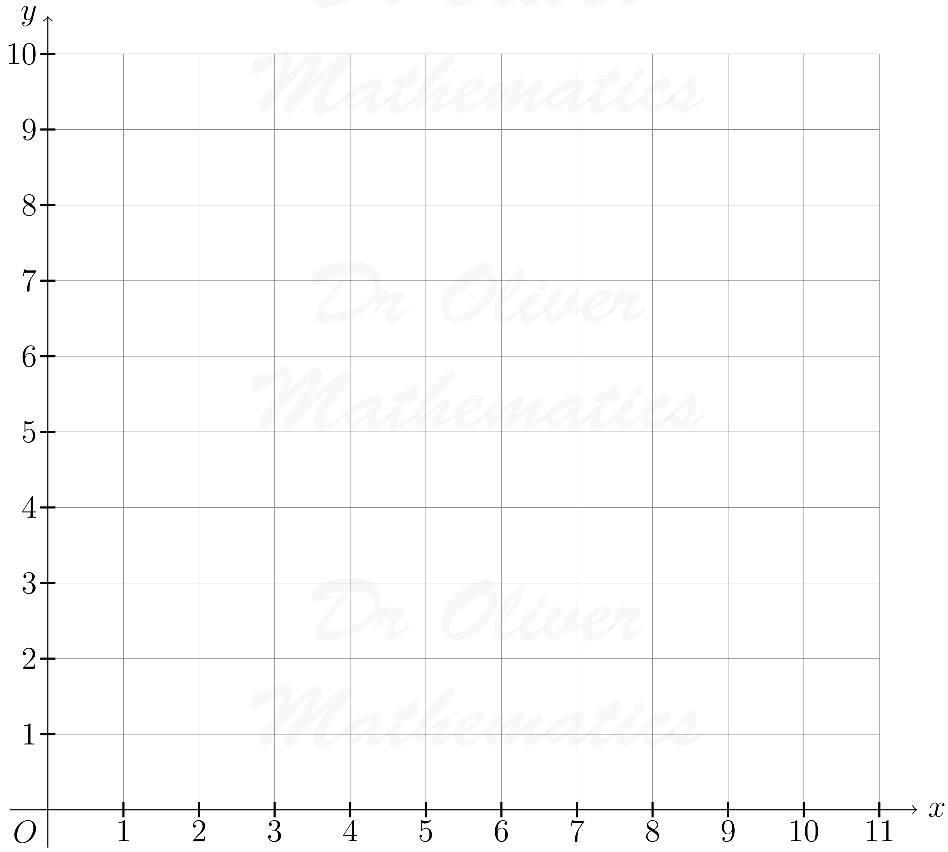
$$m_{\text{normal}} = \tfrac{2}{3}.$$

Finally, the equation of the perpendicular bisector of AB is

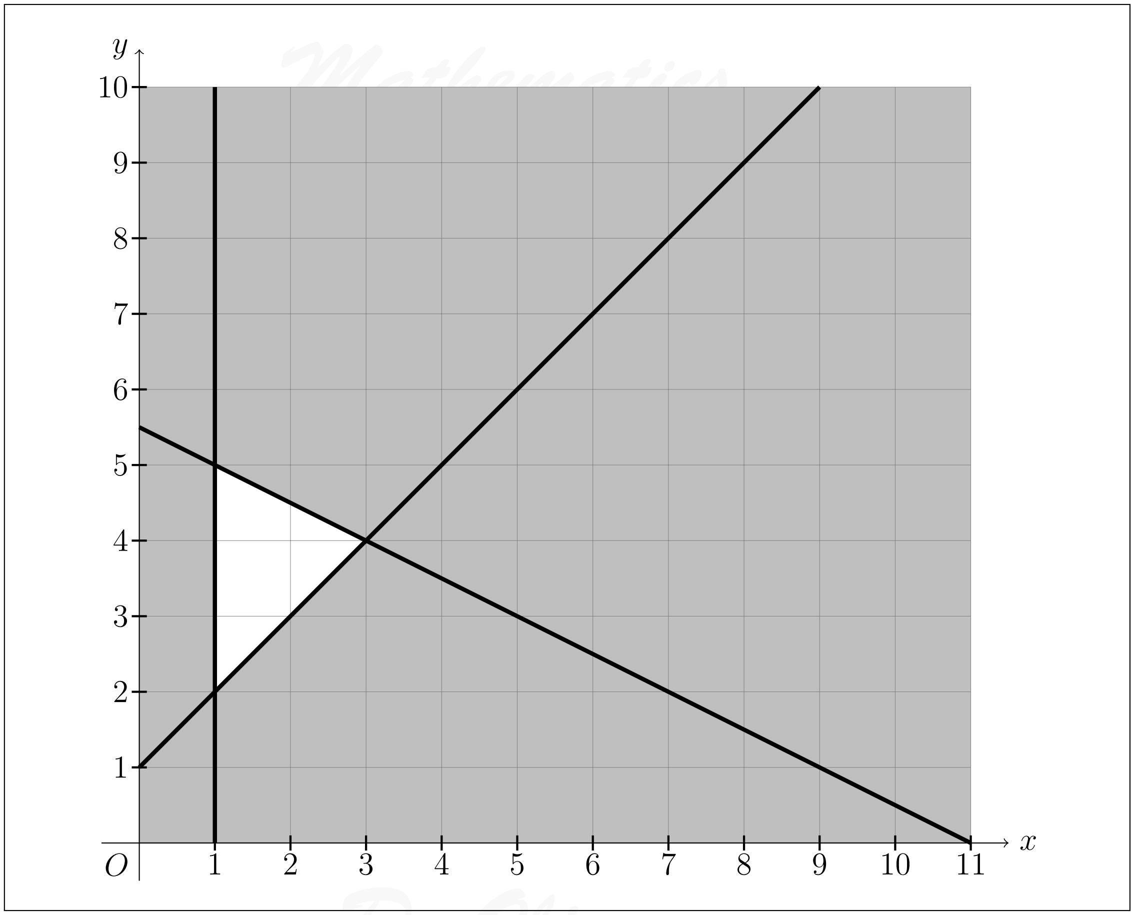
$$\begin{aligned} y - 4 = \tfrac{2}{3}(x-5) &\Rightarrow y - 4 = \tfrac{2}{3}x - \tfrac{10}{3} \\ &\Rightarrow \underline{\underline{y = \tfrac{2}{3}x + \tfrac{2}{3}}}. \end{aligned}$$

10. (a) On the grid below, indicate the region for which the following inequalities hold. You should shade the region that is **not** satisfied by the inequalities. (4)

- $y \geqslant x + 1$,
- $x \geqslant 1$, and
- $x + 2y \leqslant 11$.



Solution



(b) Find the maximum value of (2)

$$x + y$$

subject to these conditions.

Solution

Well, all have to do is evaluate $x + y$ at each of the critical points:

$$x = 1, y = 2 \Rightarrow x + y = 3$$
$$x = 3, y = 4 \Rightarrow x + y = 7$$
$$x = 5, y = 1 \Rightarrow x + y = 6;$$

hence, subject to these conditions,

$$x + y = \underline{\underline{7}}.$$

11. Amir asked 80 people about their preferences for the drinks tea, coffee or hot chocolate.

The results of his investigation were as follows.

- 25 liked all three drinks.
- 3 liked tea and coffee but not hot chocolate.
- 4 liked hot chocolate and coffee but not tea.
- 5 liked tea but neither of the other two drinks.
- 35 liked tea and hot chocolate.
- 48 liked hot chocolate.
- 47 liked coffee.

(a) Draw a Venn diagram to represent these data. (3)

Solution

35 liked tea and hot chocolate:

$$35 - 25 = 10.$$

48 liked hot chocolate:

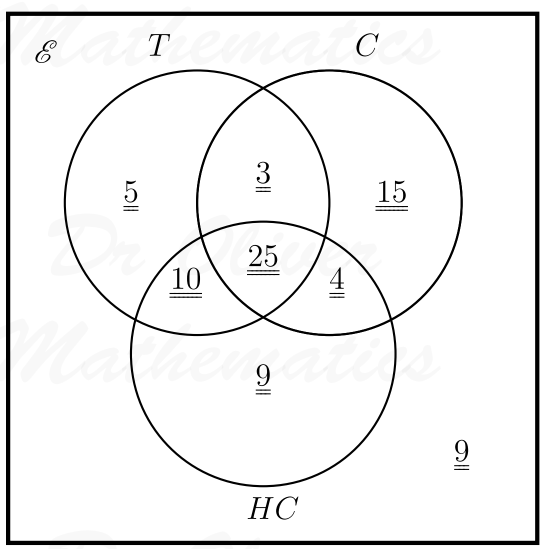
$$48 - (10 + 25 + 4) = 9.$$

47 liked coffee:

$$47 - (3 + 25 + 4) = 15.$$

Liked none:

$$80 - (5 + 3 + 15 + 10 + 25 + 4 + 9) = 9.$$



(b) Hence determine how many people said that they did **not** like any of the drinks. (2)

Solution

$\underline{\underline{9}}$.

12. The point $A(1, 3)$ lies on a circle with centre $(4, 5)$.

(a) Determine the equation of the circle. (2)

Solution

Let $E(4, 5)$ be the centre of the circle. Well,

$$\begin{aligned}(4-1)^2 + (5-3)^2 &= 3^2 + 2^2 \\ &= 13\end{aligned}$$

and, hence, the equation of the circle is

$$\underline{\underline{(x-4)^2 + (y-5)^2 = 13}}.$$

B is a point on the circle such that AB is a diameter of the circle.

(b) Find the coordinates of B. (2)

Solution

$$\begin{aligned}\overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB} \\ &= \overrightarrow{OA} + 2\overrightarrow{AE} \\ &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 4-1 \\ 5-3 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 6 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 7 \\ 7 \end{pmatrix};\end{aligned}$$

hence,

$$\underline{\underline{B(7, 7)}}.$$

D is the point $(2, 8)$.

(c) Show that AD and DB are perpendicular. (3)

Solution
Now,

$$m_{AD} = \frac{8-3}{2-1} = 5$$

and

$$m_{DB} = \frac{7-8}{7-2} = -\tfrac{1}{5};$$

so

$$m_{AD} \times m_{DB} = -1;$$

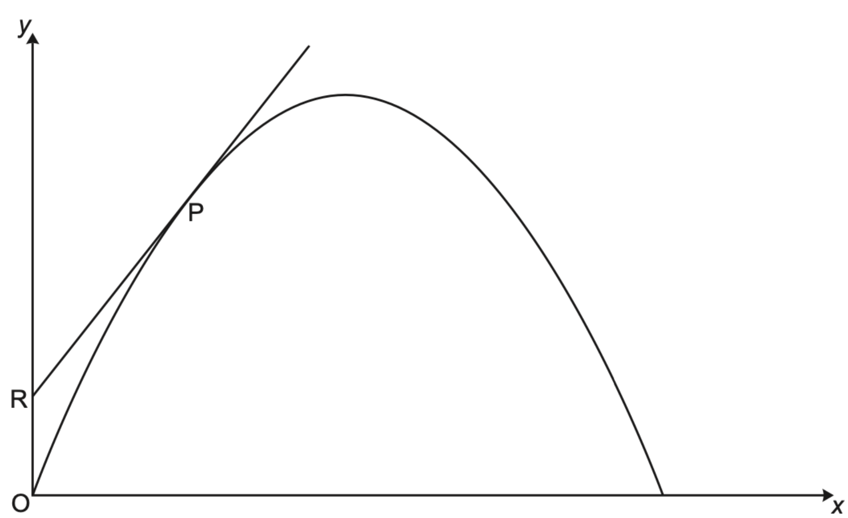
and this tells you that AD and DB are perpendicular.

(d) Explain what this tells you about the point D. (1)

Solution
E.g., $\triangle ABD$ is a right-angled triangle with $\angle ADB = 90°$.

13. The point P with coordinates $(2, 12)$ lies on the curve

$$y = 8x - x^2.$$



- The tangent to this curve at P meets the y-axis at the point R, as shown in the diagram.
- The origin is O.

(a) Determine the coordinates of the point R. (5)

Solution

Well,

$$y = 8x - x^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 8 - 2x$$

and

$$\begin{aligned} x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 8 - 2(2) \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 4. \end{aligned}$$

Now, the tangent line is

$$\begin{aligned} y - 12 = 4(x - 2) &\Rightarrow y - 12 = 4x - 8 \\ &\Rightarrow y = 4x + 4; \end{aligned}$$

hence, $\underline{\underline{R(0,4)}}$.

(b) Determine the exact area of the region OPR that is bounded by the curve from O to P, the tangent PR, and the y-axis. (6)

Solution

Let $S(2,0)$. Now,

$$\begin{aligned} \text{Area} &= \text{area of } ORPS - \text{area under the curve} \\ &= \left[\tfrac{1}{2} \times (4 + 12) \times 2\right] - \int_0^2 (8x - x^2)\,\mathrm{d}x \\ &= 16 - \left[4x^2 - \tfrac{1}{3}x^3\right]_{x=0}^{2} \\ &= 16 - \left\{\left(16 - \tfrac{8}{3}\right) - (0 - 0)\right\} \\ &= \underline{\underline{2\tfrac{2}{3}}}. \end{aligned}$$

14. Sarah brings a saucepan of water to the boil.
She leaves the water to cool, measuring its temperature every 10 minutes for 30 minutes.

The results are shown in the table below.

Time (t minutes)	0	10	20	30
Temperature (T°C)	100	60	40	30

Sarah believes that the temperature of the water as it cools can be modelled by the equation

$$T - 20 = A \times 2^{-\frac{t}{b}},$$

where A and b are constants.

(a) (i) Explain the significance of the number 20 in this equation. (1)

Solution
E.g., it was room temperature.

(ii) Use the fact that the initial temperature of the water is 100°C to determine the value of A. (2)

Solution

$$\begin{aligned} t = 0, T = 100 &\Rightarrow 100 - 20 = A \times 1 \\ &\Rightarrow \underline{\underline{A = 80}}. \end{aligned}$$

(b) By taking logs of both sides of Sarah's equation, show that plotting (3)

$$\log_{10}(T - 20)$$

against t will give a straight line.

Solution
Well,

$$\begin{aligned} T - 20 = A \times 2^{-\frac{t}{b}} &\Rightarrow \log_{10}(T-20) = \log_{10}[80 \times 2^{-\frac{t}{b}}] \\ &\Rightarrow \log_{10}(T-20) = \log_{10} 80 + \log_{10} 2^{-\frac{t}{b}} \\ &\Rightarrow \log_{10}(T-20) = \log_{10} 80 - \tfrac{t}{b} \log_{10} 2; \end{aligned}$$

comparing it with $y = mx + c$, $\log_{10}(T-20)$ against t will give a straight line.

(c) Complete the table below. (2)

Time (t minutes)	0	10	20	30
Temperature (T°C)	100	60	40	30
$T - 20$				
$\log_{10}(T-20)$				

Solution

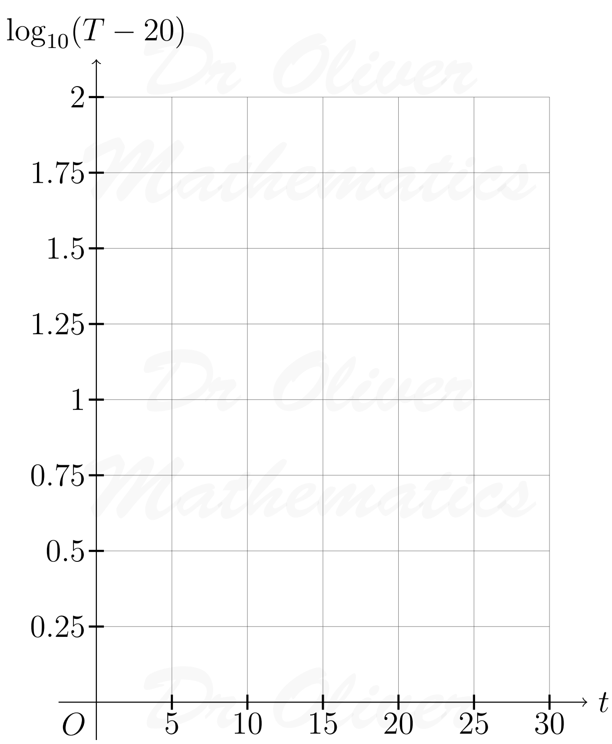
We will take 3 significant figure:

Time (t minutes)	0	10	20	30
Temperature (T°C)	100	60	40	30
$T - 20$	$\underline{\underline{80}}$	$\underline{\underline{40}}$	$\underline{\underline{20}}$	$\underline{\underline{10}}$
$\log_{10}(T-20)$	$\underline{\underline{1.90}}$	$\underline{\underline{1.60}}$	$\underline{\underline{1.30}}$	$\underline{\underline{1}}$

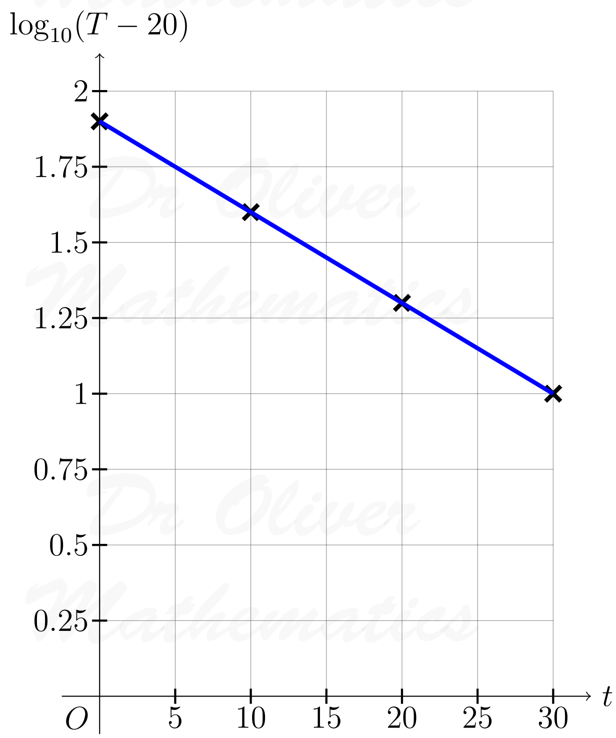
(d) Plot the values of (1)

$$\log_{10}(T-20)$$

against t on the grid below.



Solution



(e) Hence estimate the value of b. (2)

Solution

Well, the straight line goes through $(0, 1.9)$ and $(30, 1)$. Now,

$$\begin{aligned} m &= \frac{1.9 - 1}{0 - 30} \\ &= -0.03. \end{aligned}$$

Next, the equation of the best fit line is

$$\begin{aligned} \log_{10}(T - 20) - 1.90 = -0.03(t - 0) &\Rightarrow \log_{10}(T - 20) - 1.90 = -0.03t \\ &\Rightarrow \log_{10}(T - 20) = -0.03t + 1.90. \end{aligned}$$

Well, the coefficients of t are the same:

$$\begin{aligned}
-\tfrac{1}{b}\log_{10} 2 = -0.03 &\Rightarrow \tfrac{1}{b}\log_{10} 2 = 0.03\\
&\Rightarrow \tfrac{1}{b} = \frac{0.03}{\log_{10} 2}\\
&\Rightarrow b = \frac{\log_{10} 2}{0.03}\\
&\Rightarrow b = 10.034\,333\,19 \text{ (FCD)}\\
&\Rightarrow \underline{\underline{b = 10.0 \quad \text{(3 sf)}}}.
\end{aligned}$$

15. **In this question you must show detailed reasoning.**
You are given that the curve

$$y = 2x^3 + 3x^2 - 12x + 8$$

has two stationary points.

(a) (i) Show that one of the stationary points has coordinates $(1, 1)$. (4)

Solution

$$y = 2x^3 + 3x^2 - 12x + 8 \Rightarrow \frac{dy}{dx} = 6x^2 + 6x - 12$$

and

$$\begin{aligned}
\frac{dy}{dx} = 0 &\Rightarrow 6x^2 + 6x - 12 = 0\\
&\Rightarrow 6(x^2 + x - 2) = 0
\end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & +1 \\ \text{multiply to:} & -2 \end{array}\right\} +2,\ -1$$

$$\begin{aligned}
&\Rightarrow 6(x + 2)(x - 1) = 0\\
&\Rightarrow x = -2 \text{ or } x = 1\\
&\Rightarrow y = 28 \text{ or } y = 1;
\end{aligned}$$

hence, one of the stationary points has coordinates $\underline{\underline{(1, 1)}}$.

(ii) Determine the nature of this stationary point. (2)

Solution
Well,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + 6x - 12 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x + 6$$

and

$$x = 1 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 18 > 0$$

so the stationary point is a minimum.

(b) Find the coordinates of the other stationary point. (2)

Solution
$(-2, 28)$.

16. I can drive my motor boat at a maximum speed of 4 kilometres per hour in still water.

One day, I drive at maximum speed up a river from a point A to a point B, a distance of 9 km.

The constant speed of the current down the river is r kilometres per hour.

(a) Show that the time it takes me to drive up the river from A to B is (2)

$$\left(\frac{9}{4-r}\right) \text{ hours.}$$

Solution
Well,

$$\begin{aligned}\text{boat speed} &= \text{my current speed} - \text{current speed}\\ &= 4 - r\end{aligned}$$

and

$$\begin{aligned}\text{speed} = \frac{\text{distance}}{\text{time}} &\Rightarrow 4 - r = \frac{9}{t}\\ &\Rightarrow t = \frac{9}{4-r},\end{aligned}$$

as required.

(b) Write down, in terms of r, the time it takes me to drive down the river from B to A. (1)

Solution

$t = \frac{9}{4+r}$.

(c) Given that the difference between the time to drive up the river (a) and the time to drive down the river (b) is 1.2 hours, form an equation in r and show that it simplifies to (4)

$$r^2 + 15r - 16 = 0.$$

Solution

We want to solve

$$\frac{9}{4-r} - \frac{9}{4+r} = 1.2.$$

Well,

$\times$	4	$-r$
4	16	$-4r$
$+r$	$+4r$	$-r^2$

and

$$\begin{aligned}\frac{9}{4-r} - \frac{9}{4+r} = 1.2 &\Rightarrow 9(4+r) - 9(4-r) = 1.2(4-r)(4+r) \\ &\Rightarrow 36 + 9r - 36 + 9r = 1.2(16 - r^2) \\ &\Rightarrow 18r = 1.2(16 - r^2)\end{aligned}$$

multiply by $\frac{5}{6}$:

$$\begin{aligned}&\Rightarrow 15r = 16 - r^2 \\ &\Rightarrow \underline{\underline{r^2 + 15r - 16 = 0}},\end{aligned}$$

as required.

(d) Hence find the speed of the current down the river. (3)

Solution

$$\left.\begin{array}{ll}\text{add to:} & +15 \\ \text{multiply to:} & -16\end{array}\right\} +16, -1$$

$$\begin{aligned} r^2 + 15r - 16 = 0 &\Rightarrow (r+16)(r-1) = 0 \\ &\Rightarrow r = -16 \text{ or } r = 1; \end{aligned}$$

but $r > 0$!

Hence,

$$\text{speed of the current} = \underline{\underline{1 \text{ kmh}^{-1}}}.$$