Dr Oliver Mathematics
Further Mathematics
Conic Sections: Parabolas
Past Examination Questions

This booklet consists of 26 questions across a variety of examination topics.
The total number of marks available is 284.

1. The curve C has equation $y^2 = 4ax$, where a is a positive constant.

   (a) Show that an equation of the tangent to C the point $P(ap^2, 2ap)$, $p \neq 0$, is (4)

$$yp = x + ap^2.$$

Solution

$$y^2 = 4ax \Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y}$$

and, at the point $P(ap^2, 2ap)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{2ap} = \frac{1}{p}.$$

Now,

$$y - 2ap = \frac{1}{p}(x - ap^2) \Rightarrow yp - 2ap^2 = x - ap^2$$
$$\Rightarrow \underline{\underline{yp = x + ap^2}}.$$

The point $Q(aq^2, 2aq)$ is on C where $p \neq q$ and $q \neq 0$. The chord PQ passes through the focus of C. Show that

(b) $pq = -1$, (5)

Solution

The tangent at Q is $yq = x + aq^2$ and $S(a, 0)$ (why?). Now,

$$\begin{aligned} m_{PQ} &= \frac{2aq - 2ap}{aq^2 - ap^2} \\ &= \frac{2a(q-p)}{a(q+p)(q-p)} \\ &= \frac{2}{q+p} \end{aligned}$$

and

$$\begin{aligned} m_{PS} &= \frac{2ap - 0}{ap^2 - a} \\ &= \frac{2p}{p^2 - 1}. \end{aligned}$$

Now, the two expressions are equal:

$$\begin{aligned} \frac{2}{q+p} = \frac{2p}{p^2-1} &\Rightarrow p^2 - 1 = p(q+p) \\ &\Rightarrow p^2 - 1 = pq + p^2 \\ &\Rightarrow \underline{\underline{pq = -1}}, \end{aligned}$$

as required.

(c) the tangent to C at P and the tangent to C at Q meet on the directrix of C. (4)

Solution
The tangent at Q is $yq = x + aq^2$. Subtract:

$$\begin{aligned} yp - yq = ap^2 - aq^2 &\Rightarrow y(p-q) = a(p+q)(p-q) \\ &\Rightarrow y = a(p+q) \\ &\Rightarrow x = ap(p+q) + ap^2 \\ &\Rightarrow x = apq \\ &\Rightarrow x = -a; \end{aligned}$$

they $\underline{\underline{\text{meet on the directrix of } C}}$.

2. The line with equation $y = mx + c$ is a tangent to the parabola with equation $y^2 = 8x$.

(a) Show that $mc = 2$. (5)

Solution

$$\begin{aligned}(mx+c)^2 = 8x &\Rightarrow m^2x^2 + 2mcx + c^2 = 8x\\ &\Rightarrow m^2x^2 + (2mc-8)x + c^2 = 0.\end{aligned}$$

The line with equation $y = mx + c$ is a tangent which means '$b^2 - 4ac = 0$':

$$\begin{aligned}(2mc-8)^2 - 4\times m^2 \times c^2 = 0 &\Rightarrow (4m^2c^2 - 32mc + 64) - 4m^2c^2 = 0\\ &\Rightarrow 32mc = 64\\ &\Rightarrow \underline{\underline{mc = 2}}.\end{aligned}$$

The lines l_1 and l_2 are tangents to both the parabola with equation $y^2 = 4ax$ and the circle with equation $x^2 + y^2 = 2$.

(b) Find the equations of l_1 and l_2. (9)

Solution

$$\begin{aligned}c = \frac{2}{m} &\Rightarrow x^2 + \left(mx + \frac{2}{m}\right)^2 = 2\\ &\Rightarrow x^2 + \left(m^2x^2 + 4x + \frac{4}{m^2}\right) = 2\\ &\Rightarrow (1+m^2)x^2 + 4x + \frac{4}{m^2} - 2 = 0.\end{aligned}$$

The line with equation $y = mx + c$ is a tangent which means '$b^2 - 4ac = 0$':

$$\begin{aligned}& 16 - 4\times(1+m^2)\times\left(\frac{4}{m^2} - 2\right) = 0\\ \Rightarrow\quad & 4 = (1+m^2)\left(\frac{4}{m^2} - 2\right)\\ \Rightarrow\quad & 4 = \frac{4}{m^2} + 4 - 2 - 2m^2\\ \Rightarrow\quad & 4m^2 = 4 + 2m^2 - 2m^4\\ \Rightarrow\quad & 2m^4 + 2m^2 - 4 = 0\\ \Rightarrow\quad & m^4 + m^2 - 2 = 0\\ \Rightarrow\quad & (m^2+2)(m^2-1) = 0\\ \Rightarrow\quad & m = \pm 1.\end{aligned}$$

$\underline{m = 1}$:

$$m = 1 \Rightarrow c = 2 \Rightarrow \underline{\underline{y = x + 2}}.$$

$\underline{m = -1}$:

$$m = -1 \Rightarrow c = -2 \Rightarrow \underline{\underline{y = -x - 2}}.$$

3. The point P lies on the parabola with equation $y^2 = 4ax$, where a is a positive constant.

(a) Show that an equation of the tangent to the parabola at $P(ap^2, 2ap)$ is (5)

$$py = x + ap^2.$$

Solution

$$y^2 = 4ax \Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y}$$

and, at the point $P(ap^2, 2ap)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{2ap} = \frac{1}{p}.$$

Now,

$$y - 2ap = \frac{1}{p}(x - ap^2) \Rightarrow py - 2ap^2 = x - ap^2$$
$$\Rightarrow \underline{\underline{py = x + ap^2}}.$$

The tangents at the points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, where $p \neq 0$, $q \neq 0$, and $p \neq q$, meet at the point N.

(b) Find the coordinates of N. (4)

Solution

So the tangents at P and Q are given by

$$py = x + ap^2 \text{ and } qy = x + aq^2$$

respectively. If we now subtract the second from the first:

$$\begin{aligned} py - qy = (x + ap^2) - (x + aq^2) &\Rightarrow y(p - q) = ap^2 - aq^2 \\ &\Rightarrow y(p - q) = a(p + q)(p - q) \\ &\Rightarrow y = a(p + q); \end{aligned}$$

But now $x = py - ap^2$:

$$\begin{aligned} x = py - ap^2 &\Rightarrow x = ap(p + q) - ap^2 \\ &\Rightarrow x = ap^2 + apq - ap^2 \\ &\Rightarrow x = apq; \end{aligned}$$

it is $\underline{\underline{N(apq, a[p + q])}}$.

Given further that N lies on the directrix of the parabola,

(c) write down a relationship between p and q. (2)

Solution

The directrix is $x = -a$ which gives

$$apq = -a \Rightarrow \underline{\underline{pq = -1}}.$$

4. A line joins the point $A(-4a, 0)$ to the point $P(at^2, 2at)$, where a is a positive constant. As t varies the locus of the midpoint of the line AP is a parabola, C.

(a) Find an equation of C in cartesian form. (5)

Solution

$x = \frac{1}{2}(-4a + at^2)$ and $y = \frac{1}{2}(0 + 2at) = at$. Now,

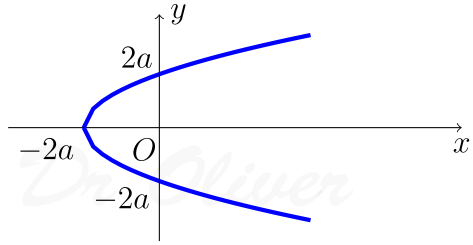
$$\begin{aligned} y^2 &= (at)^2 \\ &= a^2 \times \frac{2x + 4a}{a} \\ &= a(2x + 4a); \end{aligned}$$

hence,

$$\underline{\underline{y^2 = 2ax + 4a^2}}.$$

(b) Sketch C. (2)

Solution



(c) Write down the equation of the directrix of C. (1)

Solution

$x = -\frac{5}{2}a$.

(d) Write down the coordinates of the focus of C. (1)

Solution

$(-\frac{3}{2}a, 0)$.

5. The curve C has equation $y^2 = 4ax$ where a is a positive constant.

(a) Show that an equation of the normal to C at the point $P(ap^2, 2ap)$, $p \neq 0$, is (6)

$$y + px = 2ap + ap^3.$$

Solution

$$y^2 = 4ax \Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y}$$

and, at the point $P(ap^2, 2ap)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{2ap} = \frac{1}{p}.$$

This is when the gradient of the normal is $-p$ and

$$y - 2ap = -p(x - ap^2) \Rightarrow y - 2ap = -px + ap^3$$
$$\Rightarrow y + px = 2ap + ap^3.$$

The normal at P meets C again the point $Q(aq^2, 2aq)$.

(b) Find q in terms of p. (6)

Solution

Substitute Q into the normal:

$$\begin{aligned} 2aq + paq^2 = 2ap + ap^3 &\Rightarrow pap^2 - ap^3 = 2ap - 2aq \\ &\Rightarrow ap(q^2 - p^2) = 2a(p - q) \\ &\Rightarrow ap(q - p)(q + p) = 2a(p - q) \\ &\Rightarrow p(q + p) = -2 \\ &\Rightarrow q + p = -\tfrac{2}{p} \\ &\Rightarrow \underline{\underline{q = -\tfrac{2}{p} - p}}. \end{aligned}$$

Given that the midpoint of PQ has coordinates $\left(\frac{125}{28}a, -3a\right)$,

(c) use your answer to part (b), or otherwise, to find the value of p. (5)

Solution

The midpoint of PQ is

$$\left(\tfrac{a}{2}(p^2 + q^2), a(p + q)\right).$$

So,

$$\begin{aligned} a(p + q) = -3a &\Rightarrow p - \tfrac{2}{p} - p = -3 \\ &\Rightarrow -\tfrac{2}{p} = -3 \\ &\Rightarrow \underline{\underline{p = \tfrac{2}{3}}}. \end{aligned}$$

6. The point $P(ap^2, 2ap)$ lies on the parabola M with equation $y^2 = 4ax$, where a is a positive constant.

(a) Show that an equation of the tangent to the parabola at M at P is (3)

$$py = x + ap^2.$$

Solution

$$
\begin{aligned}
y^2 = 4ax \Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 4a \\
\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2a}{y}
\end{aligned}
$$

and, at the point $P(ap^2, 2ap)$,

$$
\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{2ap} = \frac{1}{p}.
$$

Now,

$$
\begin{aligned}
y - 2ap = \frac{1}{p}(x - ap^2) &\Rightarrow py - 2ap^2 = x - ap^2 \\
&\Rightarrow \underline{\underline{py = x + ap^2}}.
\end{aligned}
$$

The point $Q(16ap^2, 8ap)$ also lies on M.

(b) Write down an equation of the tangent to M at Q. (2)

Solution
At Q, the parameter equal $4p$. Hence

$$
(4p)y = x + a(4p)^2 \Rightarrow \underline{\underline{4py = x + 16ap^2}}.
$$

The tangent at P and the tangent at Q intersect at the point V.

(c) Show that, as p varies, the locus of V is a parabola N with equation (4)

$$
4y^2 = 25ax.
$$

Solution

$$
\begin{aligned}
4py &= x + 16ap^2 \\
py &= x + ap^2
\end{aligned}
$$

Subtract:

$$
3py = 15ap^2 \Rightarrow y = 5ap \Rightarrow x = 4ap^2.
$$

Now,

$$\begin{aligned} 4y^2 &= 4(5ap)^2 \\ &= 100a^2p^2 \\ &= 25a(4ap^2) \\ &= \underline{\underline{25ax}}. \end{aligned}$$

(d) Find the coordinates of the focus of N, and find an equation of the directrix of N. (2)

Solution

$$4y^2 = 25ax \Rightarrow y^2 = \frac{25}{4}ax = 4\left(\frac{25a}{16}\right)x.$$

The focus is

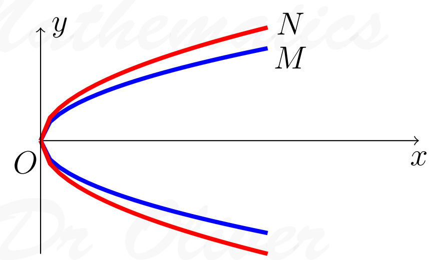
$$\underline{\underline{\left(\frac{25a}{16}, 0\right)}}$$

and the directrix is

$$\underline{\underline{x = -\frac{25a}{16}}}.$$

(e) Sketch M and N on the same diagram, labelling each of them. (2)

Solution



7. A parabola C has equation $y^2 = 4ax$, where a is a constant.

(a) Show that an equation of the normal to C at the point $P(ap^2, 2ap)$ is (4)

$$y + px = 2ap + ap^3.$$

Solution

$$y^2 = 4ax \Rightarrow 2y\,\frac{\mathrm{d}y}{\mathrm{d}x} = 4a$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y}$$

and, at the point $P(ap^2, 2ap)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{2ap} = \frac{1}{p}.$$

This is when the gradient of the normal is $-p$ and

$$\begin{aligned} y - 2ap = -p(x - ap^2) &\Rightarrow y - 2ap = -px + ap^3 \\ &\Rightarrow \underline{\underline{y + px = 2ap + ap^3}}. \end{aligned}$$

The normals to C at the points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, $p \neq q$, meet at the point R.

(b) Find, in terms of a, p, and q, the coordinates of R. (5)

Solution

So the normals at P and Q are given by

$$y + px = 2ap + ap^3 \text{ and } y + qx = 2aq + aq^3$$

respectively. If we now subtract the second from the first:

$$\begin{aligned} & (p - q)x = 2a(p - q) + a(p^3 - q^3) \\ \Rightarrow\ & (p - q)x = 2a(p - q) + a(p - q)(p^2 + pq + q^2) \\ \Rightarrow\ & x = 2a + a(p^2 + pq + q^2) \\ \Rightarrow\ & y = -p[2a + a(p^2 + pq + q^2)] + 2ap + ap^3 \\ \Rightarrow\ & y = -2ap - ap(p^2 + pq + q^2) + 2ap + ap^3 \\ \Rightarrow\ & y = -ap(p^2 + pq + q^2) + ap^3 \\ \Rightarrow\ & y = -ap^2q - apq^2 \\ \Rightarrow\ & y = -apq(p + q); \end{aligned}$$

hence $\underline{\underline{R(2a + a(p^2 + pq + q^2), -apq(p + q))}}$.

The points P and Q vary such that $pq = 3$.

(c) Find, in the form $y^2 = \mathrm{f}(x)$, an equation of the locus of R. (4)

Solution

$$pq = 3 \Rightarrow x = 2a + a(p^2 + 3 + q^2), y = -3a(p+q)$$

and

$$\begin{aligned} y^2 &= 9a^2(p+q)^2 \\ &= 9a^2(p^2 + 2pq + q^2) \\ &= 9a^2(p^2 + 6 + q^2) \\ &= 9a^2\left(\frac{x-5a}{a} + 6\right) \\ &= 9a^2\left(\frac{x+a}{a}\right) \\ &= \underline{\underline{9a(x+a)}} \end{aligned}$$

and

$$\begin{aligned} x = 2a + a(p^2 + 3 + q^2) &\Rightarrow x - 2a = a(p^2 + 3 + q^2) \\ &\Rightarrow \frac{x-2a}{a} = p^2 + 3 + q^2 \\ &\Rightarrow \frac{x-2a}{a} - 3 = p^2 + q^2 \\ &\Rightarrow \frac{x-2a}{a} - \frac{3a}{a} = p^2 + q^2 \\ &\Rightarrow \frac{x-5a}{a} = p^2 + q^2. \end{aligned}$$

8. The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, $p \neq q$, lies on the parabola C with equation $y^2 = 4ax$, where a is a constant.

(a) Show that an equation for the chord PQ is (3)

$$(p+q)y = 2(x + apq).$$

Solution

The gradient of the chord is

$$\frac{2ap - 2aq}{ap^2 - aq^2} = \frac{2a(p-q)}{a(p-q)(p+q)} = \frac{2}{p+q}$$

as $p \neq q$. Now,

$$\begin{aligned} y - 2ap = \frac{2}{p+q}(x - ap^2) &\Rightarrow (p+q)[y - 2ap] = 2(x - ap^2) \\ &\Rightarrow (p+q)y - 2ap(p+q) = 2x - 2ap^2 \\ &\Rightarrow (p+q)y = 2x - 2ap^2 + 2ap(p+q) \\ &\Rightarrow (p+q)y = 2x + 2apq \\ &\Rightarrow \underline{\underline{(p+q)y = 2(x + apq)}}, \end{aligned}$$

as required.

The normals to C at P and Q meet at the point R.

(b) Show that the coordinates of R are (7)

$$(a(p^2 + q^2 + pq + 2), -apq(p+q)).$$

Solution

$$\begin{aligned} y^2 = 4ax &\Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y} \end{aligned}$$

and, at the point $P(ap^2, 2ap)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{2ap} = \frac{1}{p}.$$

This is when the gradient of the normal is $-p$ and

$$\begin{aligned} y - 2ap = -p(x - ap^2) &\Rightarrow y - 2ap = -px + ap^3 \\ &\Rightarrow y + px = 2ap + ap^3 \end{aligned}$$

and

$$y + qx = 2aq + aq^3.$$

If we now subtract the second from the first:

$$\begin{aligned}
& (p-q)x = 2a(p-q) + a(p^3 - q^3) \\
\Rightarrow\quad & (p-q)x = 2a(p-q) + a(p-q)(p^2 + pq + q^2) \\
\Rightarrow\quad & x = 2a + a(p^2 + pq + q^2) \\
\Rightarrow\quad & x = a(p^2 + q^2 + pq + 2) \\
\Rightarrow\quad & y = -p[a(p^2 + q^2 + pq + 2)] + 2ap + ap^3 \\
\Rightarrow\quad & y = -2ap - ap(p^2 + pq + q^2) + 2ap + ap^3 \\
\Rightarrow\quad & y = -ap(p^2 + pq + q^2) + ap^3 \\
\Rightarrow\quad & y = -ap^2q - apq^2 \\
\Rightarrow\quad & y = -apq(p+q);
\end{aligned}$$

hence $\underline{\underline{R(a(p^2 + q^2 + pq + 2), -apq(p+q))}}$.

Given that the points P and Q vary such that PQ always passes through the point $(5a, 0)$,

(c) find, in the form $y^2 = \mathrm{f}(x)$, an equation for the locus of R. (5)

Solution

Use $(p+q)y = 2(x + apq)$:

$$y = 0 \Rightarrow 0 = 2(5a + apq) \Rightarrow pq = -5$$

and then

$$x = a(p^2 + q^2 - 3) \text{ and } y = -5a(p+q).$$

Now,

$$\begin{aligned}
y^2 &= [-5a(p+q)]^2 \\
&= 25a^2(p+q)^2 \\
&= 25a^2(p^2 + 2pq + q^2) \\
&= 25a^2(p^2 + q^2 - 10) \\
&= 25a^2\left[\left(\frac{x}{a} + 3\right) - 10\right] \\
&= 25a^2\left[\left(\frac{x+3a}{a}\right) - 10\right] \\
&= 25a^2\left(\frac{x-7a}{a}\right) \\
&= \underline{\underline{25a(x-7a)}}.
\end{aligned}$$

9. The parabola C has equation $y^2 = 4ax$, where a is a positive constant. The point P has coordinates $(ap^2, 2ap)$.

(a) Show that an equation of the normal to C at the point $P(ap^2, 2ap)$ is (4)

$$y + px = 2ap + ap^3.$$

Solution

$$\begin{aligned} y^2 = 4ax &\Rightarrow 2y\,\frac{\mathrm{d}y}{\mathrm{d}x} = 4a \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y} \end{aligned}$$

and, at the point $P(ap^2, 2ap)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{2ap} = \frac{1}{p}.$$

This is when the gradient of the normal is $-p$ and

$$\begin{aligned} y - 2ap = -p(x - ap^2) &\Rightarrow y - 2ap = -px + ap^3 \\ &\Rightarrow \underline{\underline{y + px = 2ap + ap^3}}. \end{aligned}$$

The normal to C at P meets the curve again at Q.

(b) Show that the y-coordinate of Q is $-2a\left(\frac{2+p^2}{p}\right)$. (5)

Solution

$$\begin{aligned} y + px = 2ap + ap^3 &\Rightarrow y + \frac{py^2}{4a} = 2ap + ap^3 \\ &\Rightarrow 4ay + py^2 = 8a^2p + 4a^2p^3 \\ &\Rightarrow py^2 + 4ay - 8a^2p - 4a^2p^3 = 0 \\ &\Rightarrow (y - 2ap)(py + 4a + 2p^2) = 0 \\ &\Rightarrow y = 2ap \text{ or } \underline{\underline{y = -2a\left(\frac{2+p^2}{p}\right)}}. \end{aligned}$$

(c) Show that, as p varies, the least distance from P to Q is $6\sqrt{3}a$. (7)

Solution

$$y = -2a\left(\frac{2+p^2}{p}\right) \Rightarrow x = \frac{1}{4a}\left[-2a\left(\frac{2+p^2}{p}\right)\right]^2$$
$$\Rightarrow x = a\left(\frac{2+p^2}{p}\right)^2.$$

Now,

$$\begin{aligned}
PQ^2 &= \left[ap^2 - a\left(\frac{2+p^2}{p}\right)^2\right]^2 + \left[2ap + 2a\left(\frac{2+p^2}{p}\right)\right]^2 \\
&= \left[ap^2 - \frac{a}{p^2}(4+4p^2+p^4)\right]^2 + \left[2ap + \frac{2a}{p}(2+p^2)\right]^2 \\
&= \left[ap^2 - \frac{4a}{p^2} - 4a - ap^2\right]^2 + \left[2ap + \frac{4a}{p} + 2ap\right]^2 \\
&= \left[-\frac{4a}{p^2} - 4a\right]^2 + \left[4ap + \frac{4a}{p}\right]^2 \\
&= \left(\frac{16a^2}{p^4} + \frac{32a^2}{p^2} + 16a^2\right) + \left(16a^2p^2 + 32a^2 + \frac{16a^2}{p^2}\right) \\
&= \frac{a^2}{p^4}\left\{\left(16+32p^2+16p^4\right) + \left(16p^6+32p^4+16p^2\right)\right\} \\
&= \frac{a^2}{p^4}\left(16p^6+48p^4+48p^2+16\right) \\
&= \frac{16a^2}{p^4}\left(p^6+3p^4+3p^2+1\right) \\
&= \frac{16a^2(p^2+1)^3}{p^4}.
\end{aligned}$$

Now,

$$\frac{\mathrm{d}}{\mathrm{d}x}(PQ^2) = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{16a^2(p^2+1)^3}{p^4}\right) = 0$$
$$\Rightarrow \frac{16a^2\left[p^4 \times 6p(p^2+1)^2 - (p^2+1)^3 \times 4p^3\right]}{p^8} = 0$$
$$\Rightarrow \frac{16a^2(p^2+1)^2\left[6p^2 - 4(p^2+1)\right]}{p^5} = 0$$
$$\Rightarrow \frac{16a^2(p^2+1)^2(2p^2-4)}{p^5} = 0$$
$$\Rightarrow \frac{32a^2(p^2+1)^2(p+\sqrt{2})(p-\sqrt{2})}{p^5} = 0$$
$$\Rightarrow p = \pm\sqrt{2}.$$

Substituting $p = \pm\sqrt{2}$ (either one will work), we have

$$PQ^2_{\min} = \frac{16a^2(2+1)^3}{4} \Rightarrow PQ^2_{\min} = 108a^2$$
$$\Rightarrow \underline{\underline{PQ_{\min} = 6\sqrt{3}a}}.$$

10. A parabola C has equation $y^2 = 4ax$, where $a > 0$, and the line l has equation $y = mx+c$. Given that l is a tangent to C,

(a) show that $c = \dfrac{a}{m}$. (4)

Solution

$$y^2 = 4ax \Rightarrow (mx+c)^2 = 4ax$$
$$\Rightarrow m^2x^2 + 2cmx + c^2 = 4ax$$
$$\Rightarrow m^2x^2 + (2cm - 4a)x + c^2 = 0.$$

Now, '$b^2 - 4ac = 0$':

$$(2cm-4a)^2 - 4c^2m^2 = 0 \Rightarrow (4c^2m^2 - 16acm + 16a^2) - 4c^2m^2 = 0$$
$$\Rightarrow -16acm + 16a^2 = 0$$
$$\Rightarrow 16acm = 16a^2$$
$$\Rightarrow \underline{\underline{c = \frac{a}{m}}}.$$

The point P has coordinates $(4a, 5a)$.

(b) Find equations of the two tangents from P to C. (5)

Solution
The line
$$y = mx + \frac{a}{m}$$
goes though $(4a, 5a)$:
$$\begin{aligned} 5a = 4am + \frac{a}{m} &\Rightarrow 5 = 4m + \frac{1}{m} \\ &\Rightarrow 5m = 4m^2 + 1 \\ &\Rightarrow 4m^2 - 5m + 1 = 0 \\ &\Rightarrow (4m-1)(m-1) = 0 \\ &\Rightarrow m = \tfrac{1}{4} \text{ or } m = 1. \end{aligned}$$
$\underline{m = \frac{1}{4}}$:
$$c = \frac{a}{\frac{1}{4}} = 4a \Rightarrow \underline{\underline{y = \tfrac{1}{4}x + 4a}}.$$
$\underline{m = 1}$:
$$c = \frac{a}{1} = a \Rightarrow \underline{\underline{y = x + a}}.$$

The tangents from P to C meet at the point R and Q.

(c) Find the distance RQ. (5)

Solution
$\underline{y = \frac{1}{4}x + 4a}$:
$$\begin{aligned} y^2 = 4ax &\Rightarrow (\tfrac{1}{4}x + 4a)^2 = 4ax \\ &\Rightarrow \tfrac{1}{16}x^2 + 2ax + 16a^2 = 4ax \\ &\Rightarrow \tfrac{1}{16}x^2 - 2ax + 16a^2 = 0 \\ &\Rightarrow (\tfrac{1}{4}x - 4a)^2 = 0 \\ &\Rightarrow x = 16a \\ &\Rightarrow y = \tfrac{1}{4}(16a) + 4a = 8a; \end{aligned}$$
the point $R(16a, 8a)$.

$\underline{y = x + a}$:

$$\begin{aligned} y^2 = 4ax \Rightarrow (x+a)^2 &= 4ax \\ \Rightarrow x^2 + 2ax + a^2 &= 4ax \\ \Rightarrow x^2 - 2ax + a^2 &= 0 \\ \Rightarrow (x-a)^2 &= 0 \\ \Rightarrow x &= a \\ \Rightarrow y &= a + a = 2a; \end{aligned}$$

the point $Q(a, 2a)$. Finally,

$$RQ = \sqrt{(16a - a)^2 + (8a - 2a)^2} = \underline{\underline{3\sqrt{29}a}}.$$

11. A parabola has equation $y^2 = 4ax$, $a > 0$. The point $Q(aq^2, 2aq)$ lies on the parabola.

(a) Show that an equation of the tangent to the parabola at Q is (4)

$$yq = x + aq^2.$$

Solution

$$\begin{aligned} y^2 = 4ax \Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 4a \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2a}{y} \end{aligned}$$

and, at the point $Q(aq^2, 2aq)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{2aq} = \frac{1}{q}.$$

Now,

$$\begin{aligned} y - 2aq = \frac{1}{q}(x - aq^2) \Rightarrow yq - 2aq^2 &= x - aq^2 \\ \Rightarrow \underline{\underline{yq = x + aq^2}}. \end{aligned}$$

This tangent meets the y-axis at the point R.

(b) Find an equation of the line l which passes through R and is perpendicular to the tangent at Q. (3)

Solution

$$x = 0 \Rightarrow yq = aq^2 \Rightarrow y = aq$$

and so $R(0, aq)$. Now, $m_T = -q$ and so

$$y - aq = -q(x - 0) \Rightarrow \underline{\underline{y = -qx + aq}}.$$

(c) Show that l passes through the focus of the parabola. (1)

Solution

$$y = 0 \Rightarrow 0 = -qx + aq \Rightarrow x = a;$$

the line l passes through the $\underline{\underline{\text{focus of the parabola}}}$.

(d) Find the coordinates of the point where l meets the directrix of the parabola. (2)

Solution

$$x = -a \Rightarrow y = -q(-a) + aq = 2aq$$

so the coordinates are $\underline{\underline{(-a, 2aq)}}$.

12. The parabola C has equation $y^2 = 16x$.

(a) Verify that the point $P(4t^2, 8t)$ is a general point on C. (1)

Solution

$$y^2 = (8t)^2 = 64t^2 = 16(4t^2);$$

$\underline{\underline{\text{the point does lie on } C}}$.

(b) Write down the coordinates of the focus S of C. (1)

Solution

$y^2 = 4 \times 4 \times x$: $\underline{\underline{(4, 0)}}$.

(c) Show that the normal to C at P has equation (5)

$$y + tx = 8t + 4t^3.$$

Solution

$$y^2 = 16x \Rightarrow 2y\,\frac{\mathrm{d}y}{\mathrm{d}x} = 16$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{16}{y}$$

and, at the point $P(4t^2, 8t)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{8t}{8t^2} = \frac{1}{t}.$$

This is when the gradient of the normal is $-t$ and

$$y - 8t = -t(x - 4t^2) \Rightarrow y - 8t = -tx + 4t^3$$
$$\Rightarrow \underline{\underline{y + tx = 8t + 4t^3}}.$$

The normal to C at P meets the x-axis at the point N.

(d) Find the area of triangle PSN in terms of t, giving your answer in its simplest form. (4)

Solution

$$y = 0 \Rightarrow x = 8 + 4t^2.$$

Now,

$$\text{base of the triangle} = (8 + 4t^2) - 4 = 4 + 4t^2$$

and

$$\text{area of the triangle } PSN = \tfrac{1}{2} \times (4 + 4t^2) \times |8t|$$
$$= \underline{\underline{16|t|(1 + t^2)}}.$$

13. Figure 1 shows a sketch of part of the parabola with equation $y^2 = 12x$.

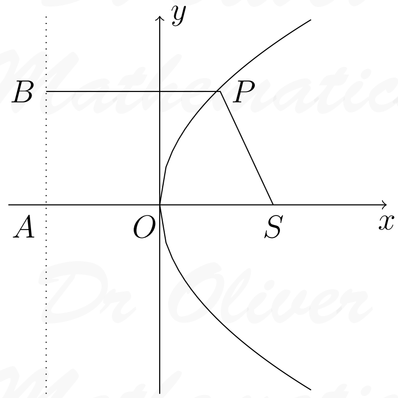


Figure 1: $y^2 = 12x$

The point P on the parabola has x-coordinate $\frac{1}{3}$. The point S is the focus of the parabola.

(a) Write down the coordinates of S. (1)

Solution
$y^2 = 4 \times 3 \times x$ and $\underline{\underline{S(3, 0)}}$.

The points A and B lie on the directrix of the parabola. The point A is on the x-axis and the y-coordinate of B is positive. Given that $ABPS$ is a trapezium,

(b) calculate the perimeter of $ABPS$. (5)

Solution
$A(-3, 0)$ (why?) and

$$x = \tfrac{1}{3} \Rightarrow y^2 = 4 \Rightarrow y = \pm 2;$$

hence $B(-3, 2)$ and

$$BP = SP = 3\tfrac{1}{3}.$$

So

$$\begin{aligned} \text{perimeter} &= 6 + 3\tfrac{1}{3} + 3\tfrac{1}{3} + 2 \\ &= \underline{\underline{14\tfrac{2}{3}}}. \end{aligned}$$

14. The parabola C has equation $y^2 = 20x$.

(a) Verify that the point $P(5t^2, 10t)$ is a general point on C. (1)

Solution

$$y^2 = (10t)^2 = 100t^2 = 20(5t^2);$$

the point does lie on C.

The point A on C has parameter $t = 4$. The line l passes through A and also passes through the focus of C.

(b) Find the gradient of l. (4)

Solution

$A(80, 40)$ and, for $y^2 = 4 \times 5 \times x$, $S(5, 0)$. Now,

$$\text{gradient} = \frac{40 - 0}{80 - 5} = \frac{8}{15}.$$

15. Figure 2 shows a sketch of the parabola C with equation $y^2 = 36x$.

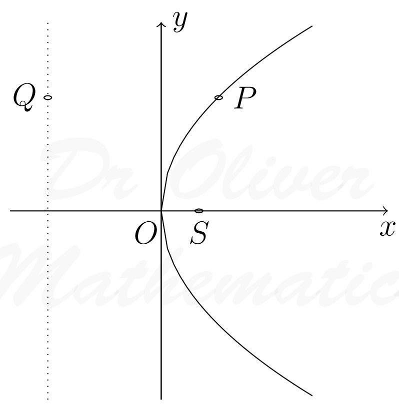


Figure 2: $y^2 = 36x$

The point S is the focus of C.

(a) Find the coordinates of S. (1)

Solution

$y^2 = 4 \times 9 \times x$ and hence $S(9, 0)$.

(b) Write down the equation of the directrix of C. (1)

Solution

$\underline{\underline{x = -9}}$.

Figure 2 shows the point P which lies on C, where $y > 0$, and the point Q which lies on the the directrix of C. The line segment QP is parallel to the x-axis. Given that the distance QP is 25,

(c) write down the distance QP, (1)

Solution

$\underline{\underline{QP = 25}}$.

(d) find the coordinates of P, (3)

Solution

$x = -9 + 25 = 16$ and

$$y^2 = 36 \times 16 = 576 \Rightarrow y = \pm 24;$$

hence $\underline{\underline{(16, 24)}}$.

(e) find the area of the trapezium $OSPQ$. (2)

Solution

$$\begin{aligned}\text{Area of the trapezium} &= \tfrac{1}{2} \times (9 + 25) \times 24 \\ &= \underline{\underline{408}}.\end{aligned}$$

16. The parabola C has equation $y^2 = 48x$. The point $P(12t^2, 24t)$ is a general point on C.

(a) Find an equation of the directrix of C. (2)

Solution

$y^2 = 4 \times 12 \times x$ and hence $\underline{\underline{x = -12}}$.

(b) Show that the equation of the tangent to C at $P(12t^2, 24t)$ is (4)

$$x - ty + 12t^2 = 0.$$

Solution

$$y^2 = 48x \Rightarrow 2y\,\frac{\mathrm{d}y}{\mathrm{d}x} = 48$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{24}{y}$$

and, at the point $P(12t^2, 24t)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{24}{24t} = \frac{1}{t}.$$

Now,

$$y - 24t = \frac{1}{t}(x - 12t^2) \Rightarrow ty - 24t^2 = x - 12t^2$$
$$\Rightarrow \underline{\underline{x - ty + 12t^2 = 0}}.$$

The tangent to C at the point $(3, 12)$ meets the directrix of C at the point X.

(c) Find the coordinates of X. (4)

Solution

$$x = 3 \Rightarrow 12t^2 = 3 \Rightarrow t = \pm\tfrac{1}{2};$$

now, $t = \frac{1}{2}$ (why?) which mean

$$x - \tfrac{1}{2}y + 3 = 0.$$

Now,

$$x = -12 \Rightarrow -12 - \tfrac{1}{2}y + 3 = 0 \Rightarrow y = -18;$$

so $\underline{\underline{X(-12, -18)}}$.

17. The parabola C has equation $y^2 = 16x$. The point $P(4t^2, 8t)$ is a general point on C.

(a) Write down the coordinates of the focus F and the equation of the directrix of C. (3)

Solution

$$y^2 = 4 \times 4 \times x;$$

hence $\underline{\underline{F(4, 0)}}$ and $\underline{\underline{x = -4}}$.

(b) Show that the equation of the normal to C is $y + tx = 8t + 4t^3$. (5)

Solution

$$y^2 = 16x \Rightarrow 2y\,\frac{\mathrm{d}y}{\mathrm{d}x} = 16$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{16}{y}$$

and, at the point $P(4t^2, 8t)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{8t}{8t^2} = \frac{1}{t}.$$

This is when the gradient of the normal is $-t$ and

$$y - 8t = -t(x - 4t^2) \Rightarrow y - 8t = -tx + 4t^3$$
$$\Rightarrow \underline{\underline{y + tx = 8t + 4t^3}}.$$

18. Figure 3 shows a sketch of the parabola C with equation $y^2 = 8x$.

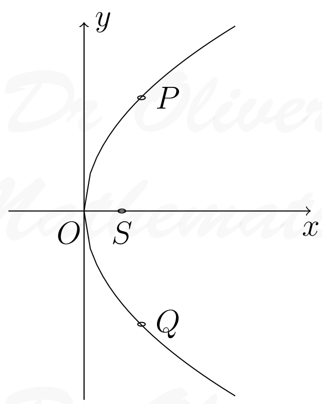


Figure 3: $y^2 = 8x$

The point P lies on C, where $y > 0$, and the point Q lies on C, where $y < 0$. The line segment PQ is parallel to the y-axis. Given that the distance PQ is 12,

(a) write down the y-coordinate of P, (1)

Solution

$PQ = 12 \Rightarrow \underline{\underline{y = 6}}$.

(b) find the x-coordinate of P. (2)

Solution

$$8x = 36 \Rightarrow \underline{\underline{x = 4.5}}.$$

Figure 3 shows the point S which is the focus of C. The line l passes through the point P and the point S.

(c) Find an equation for l in the form $ax + by + c = 0$, where a, b, and c are integers. (4)

Solution

$y^2 = 4 \times 2 \times x$ and $S(2, 0)$. Now,

$$\text{gradient} = \frac{6 - 0}{4.5 - 2} = \frac{12}{5}$$

and

$$\begin{aligned} y - 0 = \tfrac{12}{5}(x - 2) &\Rightarrow 5y = 12x - 24 \\ &\Rightarrow \underline{\underline{12x - 5y - 24 = 0}}. \end{aligned}$$

19. Figure 4 shows a sketch of the of the parabola with equation $y^2 = 36x$.

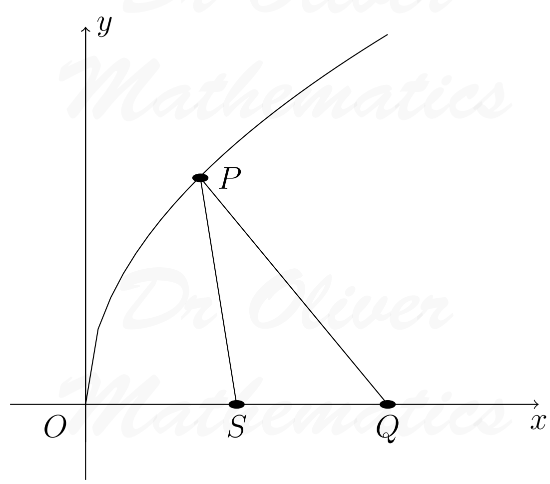


Figure 4: $y^2 = 36x$

The point $P(4, 12)$ lies on the parabola.

(a) Find an equation for the normal to the parabola at P. (5)

Solution

$$\begin{aligned} y^2 = 36x &\Rightarrow 2y\,\frac{\mathrm{d}y}{\mathrm{d}x} = 36 \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{18}{y} \end{aligned}$$

and, at the point $P(4, 12)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{18}{12} = \frac{3}{2}.$$

This is when the gradient of the normal is $-\frac{2}{3}$ and

$$\begin{aligned} y - 12 = -\tfrac{2}{3}(x - 4) &\Rightarrow 3y - 36 = -2x + 8 \\ &\Rightarrow \underline{\underline{2x + 3y - 44 = 0}}. \end{aligned}$$

This normal meets the x-axis at the point N and S is the focus of the parabola, as shown in Figure 4.

(b) Find the area of triangle PST. (4)

Solution

$$y = 0 \Rightarrow 2x - 44 = 0$$

and hence $N(22, 0)$. Now, $S(9, 0)$ and

$$\begin{aligned} \text{area of the triangle} &= \tfrac{1}{2} \times (22 - 9) \times 12 \\ &= \underline{\underline{78}}. \end{aligned}$$

20. A parabola has equation $y^2 = 4ax$, $a > 0$. The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ lie on C, where $p \neq 0$, $q \neq 0$, and $p \neq q$.

(a) Show that the equation of the tangent to parabola at P is (4)

$$py - x = ap^2.$$

Solution

$$y^2 = 4ax \Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y}$$

and, at the point P,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{2ap} = \frac{1}{p}.$$

Now,

$$y - 2ap = \frac{1}{p}(x - ap^2) \Rightarrow py - 2ap^2 = x - ap^2$$
$$\Rightarrow \underline{\underline{py - x = ap^2}}.$$

(b) Write down the equation of the tangent at Q. (1)

Solution
$\underline{\underline{qy - x = aq^2}}$.

The tangent at P meets the tangent at Q at the point R.

(c) Find, in terms of p and q, the coordinates of R, giving your answer in their simplest form. (4)

Solution
Subtract:

$$(p - q)y = a(p^2 - q^2) \Rightarrow (p - q)y = a(p + q)(p - q)$$
$$\Rightarrow y = a(p + q)$$

and

$$x = py - ap^2 = ap(p + q) - ap^2 = apq;$$

hence $\underline{\underline{R(apq, a(p + q))}}$.

Given that R lies on the directrix of C,

(d) find the value of pq. (2)

Solution

$$-a = apq \Rightarrow \underline{\underline{pq = -1}}.$$

21. A parabola C has equation $y^2 = 4ax$, where a is a positive constant. The point $P(at^2, 2at)$ is a general point on C.

(a) Show that the equation of the tangent to C to parabola at $P(at^2, 2at)$ is (4)

$$ty = x + at^2.$$

Solution

$$\begin{aligned} y^2 = 4ax &\Rightarrow 2y\,\frac{\mathrm{d}y}{\mathrm{d}x} = 4a \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y} \end{aligned}$$

and, at the point P,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{2at} = \frac{1}{t}.$$

Now,

$$\begin{aligned} y - 2at = \frac{1}{p}(x - at^2) &\Rightarrow ty - 2at^2 = x - at^2 \\ &\Rightarrow \underline{\underline{ty = x + at^2}}. \end{aligned}$$

The tangent to C at P meets the y-axis at a point Q.

(b) Find the coordinates of Q. (1)

Solution

$\underline{\underline{Q(0, at)}}$.

Given that the point S is the focus of C,

(c) show that PQ is perpendicular to SQ. (3)

Solution

$S(a, 0)$,

$$\frac{2at - at}{at^2 - 0} = \frac{1}{t},$$

and

$$\frac{0 - at}{a - 0} = -t;$$

clearly, $\underline{\underline{PQ \text{ is perpendicular to } SQ}}$.

22. The points $P(4k^2, 8k)$ and $Q(k^2, 4k)$, where k is a constant, lie on the parabola C with equation $y^2 = 16x$. The straight line l_1 passes through the points P and Q.

(a) Show that an equation of the line l_1 is given by (4)

$$3ky - 4x = 8k^2.$$

Solution

Well,

$$\text{gradient} = \frac{8k - 4k}{4k^2 - k^2} = \frac{4k}{3k^2} = \frac{4}{3k}$$

and

$$\begin{aligned} y - 8k = \frac{4}{3k}(x - 4k^2) &\Rightarrow 3k(y - 8k) = 4(x - 4k^2) \\ &\Rightarrow 3ky - 24k^2 = 4x - 16k^2 \\ &\Rightarrow \underline{\underline{3ky - 4x = 8k^2}}. \end{aligned}$$

The line l_2 is perpendicular to the line l_1 and passes through the focus of the parabola C. The line l_2 meets the directrix of C at the point R.

(b) Find, in terms of k, the y-coordinate of the point R. (7)

Solution

The line l_2 has gradient $-\frac{3k}{4}$ and it goes through $S(4, 0)$ (why?). Now,

$$y - 0 = -\frac{3k}{4}(x - 4) \Rightarrow 4y = -3k(x - 4) \Rightarrow 4y = -3kx + 12k.$$

The x-coordinate of R is -4 (why?) and

$$y = -\frac{3k}{4}(-4 - 4) = \underline{\underline{6k}}.$$

23. A parabola C has cartesian equation $y^2 = 4ax$, $a > 0$. The points $P(ap^2, 2ap)$ and $P'(ap^2, -2ap)$ lie on C.

(a) Show that an equation of the normal to C at the point $P(ap^2, 2ap)$ is (5)

$$y + px = 2ap + ap^3.$$

Solution

$$\begin{aligned} y^2 = 4ax &\Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y} \end{aligned}$$

and, at the point $P(ap^2, 2ap)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{2ap} = \frac{1}{p}.$$

This is when the gradient of the normal is $-p$ and

$$\begin{aligned} y - 2ap = -p(x - ap^2) &\Rightarrow y - 2ap = -px + ap^3 \\ &\Rightarrow \underline{\underline{y + px = 2ap + ap^3}}. \end{aligned}$$

(b) Write down an equation of the normal to C at the point P'. (1)

Solution

$\underline{\underline{y - px = -2ap - ap^3}}$.

The normal to C at P meets the normal to P' at point Q.

(c) Find, in terms of a and p, the coordinates of Q. (2)

Solution

Add:

$$2y = 0 \Rightarrow y = 0$$

and

$$-px = -2ap - ap^3 \Rightarrow x = 2a + ap^2;$$

$\underline{\underline{R(2a + ap^2, 0)}}$.

Given that S is the focus of the parabola,

(d) find the area of the quadrilateral $SPQP'$. (3)

Solution

$S(a, 0)$ (why?) and

$$\begin{aligned}\text{area of the quadrilateral } SPQP' &= \text{area of the triangle } SPQ \\ &= 2 \times \tfrac{1}{2} \times (a + ap^2) \times 2ap \\ &= \underline{\underline{2a^2p(1 + p^2)}}.\end{aligned}$$

24. The point $P(3p^2, 6p)$ lies on the parabola with equation $y^2 = 12x$ and the point S is the focus of this parabola.

(a) Prove that $SP = 3(1 + p^2)$. (3)

Solution

EITHER The focus is $S(3, 0)$ and hence

$$\begin{aligned}SP &= \sqrt{(3p^2 - 3)^2 + (6p - 0)^2} \\ &= \sqrt{9p^4 - 18p^2 + 9 + 36p^2} \\ &= \sqrt{9p^4 + 18p^2 + 9} \\ &= \sqrt{9(p^2 + 1)^2} \\ &= \underline{\underline{3(p^2 + 1)}},\end{aligned}$$

as required.

OR The directrix has equation $x = -3$ and, if N is the foot of the perpendicular from P to the directrix then $N(-3, 6p)$. So

$$SP = PN = 3p^2 - (-3) = 3p^2 + 3 = \underline{\underline{3(1 + p^2)}}.$$

The point $Q(3q^2, 6q)$, $p \neq q$, also lies on this parabola. The tangent to the parabola at the point P and the tangent to the parabola at the point Q meet at the point R.

(b) Find the equations of these two tangents and hence find the coordinates of the point R, giving the coordinates in their simplest form. (8)

Solution

$$y^2 = 12x \Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 12 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6}{y}$$

and so the gradient of the tangent at P is $\frac{1}{p}$. So the tangents at P and Q are given by

$$y - 6p = \tfrac{1}{p}(x - 3p^2) \text{ and } y - 6q = \tfrac{1}{q}(x - 3q^2)$$

respectively. If we now subtract the second from the first:

$$\begin{aligned}-6p + 6q = \tfrac{1}{p}(x - 3p^2) - \tfrac{1}{q}(x - 3q^2) &\Rightarrow -6p + 6q = (\tfrac{1}{p} - \tfrac{1}{q})x - 3p + 3q\\ &\Rightarrow 6(q - p) = \tfrac{q-p}{pq}x + 3(q - p)\\ &\Rightarrow 6 = \tfrac{1}{pq}x + 3 \text{ (since } p \neq q)\\ &\Rightarrow x = 3pq.\end{aligned}$$

Using the equation of the tangent at P,

$$y - 6p = \tfrac{1}{p}(3pq - 3p^2) \Rightarrow y - 6p = 3q - 3p \Rightarrow y = 3p + 3q.$$

Hence $\underline{\underline{R(3pq, 3(p + q))}}$.

(c) Prove that $SR^2 = SP \times SQ$. (3)

Solution

Using part (a), $SQ = 3(1 + q^2)$. Now

$$\begin{aligned}SR^2 &= (3pq - 3)^2 + (3p + 3q)^2\\ &= (9p^2q^2 - 12pq + 9) + (9p^2 + 12pq + 9q^2)\\ &= 9 + 9p^2 + 9q^2 + 9p^2q^2\\ &= 3(1 + p^2) \times 3(1 + q^2)\\ &= \underline{\underline{SP \times SQ}},\end{aligned}$$

as required.

25. Points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, where $p^2 \neq q^2$, lie on the parabola $y^2 = 4ax$.

(a) Show that an equation for the chord PQ is (5)

$$(p + q)y = 2(x + apq).$$

Solution

The gradient of the chord is

$$\frac{2ap - 2aq}{ap^2 - aq^2} = \frac{2a(p-q)}{a(p-q)(p+q)} = \frac{2}{p+q}$$

as $p \neq q$. Now,

$$\begin{aligned} y - 2ap = \frac{2}{p+q}(x - ap^2) &\Rightarrow (p+q)[y - 2ap] = 2(x - ap^2) \\ &\Rightarrow y(p+q) - 2ap(p+q) = 2x - 2ap^2 \\ &\Rightarrow y(p+q) = 2x - 2ap^2 + 2ap(p+q) \\ &\Rightarrow \underline{\underline{y(p+q) = 2x + 2apq}}, \end{aligned}$$

as required.

Given that this chord passes through the focus of the parabola,

(b) show that $pq = -1$. (1)

Solution

$S(a, 0)$ and

$$0 = 2a + 2apq \Rightarrow \underline{\underline{pq = -1}}.$$

(c) Using calculus, find the gradient of the tangent to the parabola at P. (2)

Solution

$$\begin{aligned} y^2 = 4ax &\Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y} \end{aligned}$$

and, at the point $P(ap^2, 2ap)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{2ap} = \underline{\underline{\frac{1}{p}}}.$$

(d) Show that the tangent to the parabola at P and the tangent to the parabola at Q are perpendicular. (2)

Solution

At $Q(aq^2, 2aq)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{q}.$$

The products of the two tangents are

$$\frac{1}{p} \times \frac{1}{q} = -1;$$

hence, the two tangents are perpendicular.

26. The parabola C has equation $y^2 = 4ax$, where a is a constant and $a > 0$. The point $Q(aq^2, 2aq)$, $q > 0$, lies on the parabola C.

(a) Show that an equation of the tangent to C at Q is (4)

$$qy = x + aq^2.$$

Solution

$$\begin{aligned} y^2 = 4ax \Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 4a \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2a}{y} \end{aligned}$$

and, at the point $Q(aq^2, 2aq)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{2aq} = \frac{1}{q}.$$

Now,

$$\begin{aligned} y - 2aq = \frac{1}{q}(x - aq^2) &\Rightarrow qy - 2aq^2 = x - aq^2 \\ &\Rightarrow \underline{\underline{qy = x + aq^2}}. \end{aligned}$$

The tangent to C at the point Q meets the x-axis at the point $X(-\frac{1}{4}, 0)$ and meets the directrix of C at the point D.

(b) Find, in terms of a, the coordinates of D. (4)

Solution

$X(-\frac{1}{4}, 0)$:

$$\begin{aligned} 0 = -\tfrac{1}{4}a + aq^2 &\Rightarrow q^2 = \tfrac{1}{4} \\ &\Rightarrow q = \tfrac{1}{2} \text{ (as } q > 0). \end{aligned}$$

Now, $x = -a$:

$$\begin{aligned} &\Rightarrow \tfrac{1}{2}y = -a + \tfrac{1}{4}a \\ &\Rightarrow \tfrac{1}{2}y = -\tfrac{3}{4}a \\ &\Rightarrow y = -\tfrac{3}{2}a. \end{aligned}$$

So, the point is $\underline{\underline{D(-a, -\frac{3}{2}a)}}$.

Given that the point F is the focus of the parabola C,

(c) find the area, in terms of a, of the triangle FXD, giving your answer in its simplest form. (2)

Solution

Well, $F(a, 0)$ and

$$\begin{aligned} \text{area of the } \triangle FXD &= \tfrac{1}{2} \times \tfrac{3}{2}a \times \tfrac{5}{4}a \\ &= \underline{\underline{\tfrac{15}{16}a^2}}. \end{aligned}$$