Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2004 June Paper 2: Calculator
2 hours

The total number of marks available is 80.
You must write down all the stages in your working.

1. The position vectors of the points A and B, relative to an origin O, are $\mathbf{i} - 7\mathbf{j}$ and $4\mathbf{i} + k\mathbf{j}$ respectively, where k is a scalar. (4)

   The unit vector in the direction of $\overrightarrow{AB}$ is $0.6\mathbf{i} + 0.8\mathbf{j}$.

   Find the value of k.

Solution

Well,

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ k+7 \end{pmatrix}$$

and, looking at the **i**-component, we can see there are

$$\frac{3}{0.6} = 5 \text{ steps.}$$

For the **j**-component,

$$-7 + (0.8 \times 5) = -7 + 4 = \underline{\underline{-3}}.$$

2. Given that x is measured in radians and $x > 10$, find the smallest value of x such that (4)

$$10 \cos\left(\frac{x+1}{2}\right) = 3.$$

Solution

Now,

$$\begin{aligned}10\cos\left(\frac{x+1}{2}\right) = 3 \Rightarrow \cos\left(\frac{x+1}{2}\right) &= \tfrac{3}{10}\\ \Rightarrow \frac{x+1}{2} &= \cos^{-1}(\tfrac{3}{10})\\ \Rightarrow \frac{x+1}{2} &= 1.266\,103\,673,\ 5.017\,081\,634,\ 7.549\,288\,98 \text{ (FCD)}\end{aligned}$$

the other two are not in range

$$\begin{aligned}&\Rightarrow x+1 = 15.098\,577\,96 \text{ (FCD)}\\ &\Rightarrow x = 14.098\,577\,96 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{x = 14.1 \text{ (3 sf)}}}.\end{aligned}$$

3. Given that

- $\mathscr{E}$ = {students in a college},
- A = {students who are over 180 cm tall},
- B = {students who are vegetarian}, and
- C = {students who are cyclists},

express in words each of the following

(a) $A \cap B = \varnothing$, (1)

Solution
$A \cap B = \varnothing$ means that there are $\underline{\underline{\text{no}}}$ students who are over 180 cm tall **and** vegetarian.

(b) $A \subset C'$. (1)

Solution
$A \subset C'$ means that $\underline{\underline{\text{no cyclists}}}$ who are over 180 cm tall.

Express in set notation the statement

(c) all students who are both vegetarians and cyclists are not over 180 cm tall. (2)

Solution

$\underline{\underline{B \cap C}}$.

4. Prove the identity (4)

$$(1 + \sec\theta)(\operatorname{cosec}\theta - \cot\theta) \equiv \tan\theta.$$

Solution

$$\begin{aligned}
& (1 + \sec\theta)(\operatorname{cosec}\theta - \cot\theta) \\
\equiv\ & \left(1 + \frac{1}{\cos\theta}\right)\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right) \\
\equiv\ & \left(\frac{\cos\theta + 1}{\cos\theta}\right)\left(\frac{1 - \cos\theta}{\sin\theta}\right)
\end{aligned}$$

$\times$	$\cos\theta$	$+1$
1	$\cos\theta$	$+1$
$-\cos\theta$	$-\cos^2\theta$	$-\cos\theta$

$$\begin{aligned}
\equiv\ & \frac{1 - \cos^2\theta}{\sin\theta\cos\theta} \\
\equiv\ & \frac{\sin^2\theta}{\sin\theta\cos\theta} \\
\equiv\ & \frac{\sin\theta}{\cos\theta} \\
\equiv\ & \underline{\underline{\tan\theta}},
\end{aligned}$$

as required.

5. The roots of the quadratic equation (5)

$$x^2 - \sqrt{20}\,x + 2 = 0$$

are c and d.

Without using a calculator, show that

$$\frac{1}{c} + \frac{1}{d} = \sqrt{5}.$$

Solution

Well,

$$c + d = \sqrt{20} \text{ and } cd = 2.$$

Now,

$$\begin{aligned}\frac{1}{c} + \frac{1}{d} &= \frac{d}{cd} + \frac{c}{cd} \\ &= \frac{c+d}{cd} \\ &= \frac{\sqrt{20}}{2} \\ &= \frac{\sqrt{4 \times 5}}{2} \\ &= \frac{\sqrt{4} \times \sqrt{5}}{2} \\ &= \frac{2\sqrt{5}}{2} \\ &= \underline{\underline{\sqrt{5}}},\end{aligned}$$

as required.

6. (a) Find the values of x for which (3)

$$2x^2 > 3x + 14.$$

Solution

$$2x^2 > 3x + 14 \Rightarrow 2x^2 - 3x - 14 > 0$$

$$\left.\begin{array}{ll}\text{add to:} & -3 \\ \text{multiply to:} & (+2) \times (-14) = -28\end{array}\right\} = 7,\ +4$$

e.g.,

$$\begin{aligned}&\Rightarrow 2x^2 - 7x + 4x - 14 > 0\\&\Rightarrow x(2x-7) + 2(2x-7) > 0\\&\Rightarrow (x+2)(2x-7) > 0.\end{aligned}$$

We need a 'table of signs':

	$x < -2$	$x = -2$	$-2 < x < \frac{7}{2}$	$x = \frac{7}{2}$	$x > \frac{7}{2}$
$x+2$	$-$	0	$+$	$+$	$+$
$2x-7$	$-$	$-$	$-$	0	$+$
$(x+2)(2x-7)$	$+$	0	$-$	0	$+$

Hence,

$$\underline{\underline{x < -2}} \text{ or } \underline{\underline{x > \tfrac{7}{2}}}.$$

(b) Find the values of k for which the line (3)

$$y + kx = 8$$

is a tangent to the curve

$$x^2 + 4y = 20.$$

Solution

Now,

$$y + kx = 8 \Rightarrow y = -kx + 8$$

and insert it into the $x^2 + 4y = 20$:

$$\begin{aligned}x^2 + 4y = 20 &\Rightarrow x^2 + 4(-kx + 8) = 20\\&\Rightarrow x^2 - 4kx + 32 = 20\\&\Rightarrow x^2 - 4kx + 12 = 0.\end{aligned}$$

Next, $b^2 - 4ac = 0$ for a tangent:

$$\begin{aligned}(-4k)^2 - 4 \times 1 \times 12 = 0 &\Rightarrow 16k^2 = 48\\&\Rightarrow k^2 = 3\\&\Rightarrow \underline{\underline{k = \pm\sqrt{3}}}.\end{aligned}$$

7. Functions f and g are defined for $x \in \mathbb{R}$ by

$$\mathrm{f}: x \mapsto \mathrm{e}^x,$$
$$\mathrm{g}: x \mapsto 2x - 3.$$

(a) Solve the equation (2)

$$\mathrm{f\,g}(x) = 7.$$

Solution

$$\begin{aligned}\mathrm{f\,g}(x) &= \mathrm{f}(\mathrm{g}(x)) \\ &= \mathrm{f}(2x-3) \\ &= \mathrm{e}^{2x-3}\end{aligned}$$

and

$$\begin{aligned}\mathrm{f\,g}(x) = 7 &\Rightarrow \mathrm{e}^{2x-3} = 7 \\ &\Rightarrow 2x - 3 = \ln 7 \\ &\Rightarrow 2x = 3 + \ln 7 \\ &\Rightarrow \underline{\underline{x = \tfrac{1}{2}(3 + \ln 7)}}.\end{aligned}$$

Function h is defined as g f.

(b) Express h in terms of x and state its range. (2)

Solution

$$\begin{aligned}\mathrm{h}(x) &= \mathrm{g\,f}(x) \\ &= \mathrm{g}(\mathrm{f}(x)) \\ &= \mathrm{g}(\mathrm{e}^x) \\ &= \underline{\underline{2\mathrm{e}^x - 3}}\end{aligned}$$

and

$$\underline{\underline{\mathrm{h}(x) > -3}}.$$

(c) Express h^{-1} in terms of x. (2)

Solution
Well,

$$\begin{aligned} y = 2\mathrm{e}^x - 3 &\Rightarrow y + 3 = 2\mathrm{e}^x \\ &\Rightarrow \tfrac{1}{2}(y+3) = \mathrm{e}^x \\ &\Rightarrow \ln(\tfrac{1}{2}(y+3)) = x \end{aligned}$$

and, hence,

$$\mathrm{h}^{-1}(x) = \underline{\underline{\ln(\tfrac{1}{2}(x+3))}}.$$

8. Solve

(a) $\log_3(2x+1) = 2 + \log_3(3x-11)$, (4)

Solution
Now,

$$\begin{aligned} \log_3(2x+1) = 2 + \log_3(3x-11) &\Rightarrow \log_3(2x+1) - \log_3(3x-11) = 2 \\ &\Rightarrow \log_3\left(\frac{2x+1}{3x-11}\right) = 2 \\ &\Rightarrow \frac{2x+1}{3x-11} = 3^2 \\ &\Rightarrow \frac{2x+1}{3x-11} = 9 \\ &\Rightarrow 2x+1 = 9(3x-11) \\ &\Rightarrow 2x+1 = 27x - 99 \\ &\Rightarrow 100 = 25x \\ &\Rightarrow \underline{\underline{x = 4}}. \end{aligned}$$

(b) $\log_4 y + \log_2 y = 9$. (4)

Solution

Now,

$$\begin{aligned}
\log_4 y + \log_2 y = 9 &\Rightarrow \frac{\log_2 y}{\log_2 4} + \log_2 y = 9 \\
&\Rightarrow \frac{\log_2 y}{\log_2 2^2} + \log_2 y = 9 \\
&\Rightarrow \tfrac{1}{2}\log_2 y + \log_2 y = 9 \\
&\Rightarrow \tfrac{3}{2}\log_2 y = 9 \\
&\Rightarrow \log_2 y = 6 \\
&\Rightarrow y = 2^6 \\
&\Rightarrow \underline{\underline{y = 64}}.
\end{aligned}$$

9. (a) Express (2)

$$6 + 4x - x^2$$

in the form

$$a - (x+b)^2,$$

where a and b are integers.

Solution

$$\begin{aligned}
6 + 4x - x^2 &= 6 - (x^2 - 4x) \\
&= 6 - [(x^2 - 4x + 4) - 4] \\
&= 6 - [(x-2)^2 - 4] \\
&= 6 - (x-2)^2 + 4 \\
&= \underline{\underline{10 - (x-2)^2}};
\end{aligned}$$

so, $\underline{\underline{a = 10}}$ and $\underline{\underline{b = -2}}$.

(b) Find the coordinates of the turning point of the curve (3)

$$y = 6 + 4x - x^2$$

and determine the nature of this turning point.

Solution

Well, the coordinates of the turning point of the curve are $\underline{\underline{(2, 10)}}$ and it is a $\underline{\underline{\text{maximum turning point}}}$ (upside-down parabola).

The function f is defined by

$$f : x \mapsto 6 + 4x - x^2,$$

for the domain $0 \leqslant x \leqslant 5$.

(c) Find the range of f. (2)

Solution
Well,

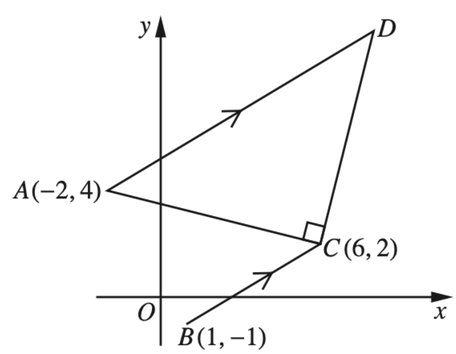
$$\begin{aligned} f(0) &= 6 \\ f(2) &= 10 \\ f(5) &= 1 \end{aligned}$$

and the range is $\underline{\underline{1 \leqslant f(x) \leqslant 10}}$.

(d) State, giving a reason, whether or not f has an inverse. (1)

Solution
$\underline{\underline{\text{No}}}$: e.g., $f(1) = 9 = f(3)$.

10. **Solutions to this question by accurate drawing will not be accepted.**



In the diagram, the points A, B, and C have coordinates $(-2, 4)$, $(1, -1)$, and $(6, 2)$ respectively.

The line AD is parallel to BC and angle $ACD = 90^\circ$.

(a) Find the equations of AD and CD. (6)

Solution

The equation of AD:

Well,

$$m_{BC} = \frac{2-(-1)}{6-1} = \tfrac{3}{5}$$

and the line AD has the same gradient. Now, the equation of AD is

$$y - 4 = \tfrac{3}{5}(x+2) \Rightarrow y - 4 = \tfrac{3}{5}x + \tfrac{6}{5} \Rightarrow \underline{\underline{y = \tfrac{3}{5}x + \tfrac{26}{5}}}.$$

The equation of CD:

Well,

$$m_{AC} = \frac{4-2}{-2-6} = \tfrac{2}{-8} = -\tfrac{1}{4}$$

and

$$m_{CD} = -\frac{1}{-\frac{1}{4}} = 4.$$

Now, the equation of CD is

$$y - 2 = 4(x-6) \Rightarrow y - 2 = 4x - 24 \Rightarrow \underline{\underline{y = 4x - 22}}.$$

(b) Find the coordinates of D. (2)

Solution

Solve:

$$4x - 22 = \tfrac{3}{5}x + \tfrac{26}{5} \Rightarrow \tfrac{17}{5}x = \tfrac{136}{5} \Rightarrow x = 8 \Rightarrow y = 10.$$

So, $\underline{\underline{D(8, 10)}}$.

(c) Show that triangle ACD is isosceles. (2)

Solution

Well,

$$\begin{aligned} AC &= \sqrt{[6-(-2)]^2 + (2-4)^2} \\ &= \sqrt{64+4} \\ &= \sqrt{68} \end{aligned}$$

and

$$\begin{aligned} CD &= \sqrt{(8-6)^2 + (10-2)^2} \\ &= \sqrt{4+64} \\ &= \sqrt{68}; \end{aligned}$$

so, $AC = CD$ and so the triangle ACD is $\underline{\underline{\text{isosceles}}}$.

11. It is given that

$$y = (x+1)(2x-3)^{\frac{3}{2}}.$$

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x}$ can be written in the form (4)

$$kx\sqrt{2x-3},$$

and state the value of k.

Solution

Product rule:

$$\begin{aligned} u = x+1 &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \\ v = (2x-3)^{\frac{3}{2}} &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 3(2x-3)^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (x+1)[3(2x-3)^{\frac{1}{2}}] + (1)[(2x-3)^{\frac{3}{2}}] \\ &= 3(x+1)(2x-3)^{\frac{1}{2}} + (2x-3)^{\frac{3}{2}} \\ &= (2x-3)^{\frac{1}{2}}[3(x+1) + (2x-3)] \\ &= (2x-3)^{\frac{1}{2}}(3x+3+2x-3) \\ &= \underline{\underline{5x\sqrt{2x-3}}};\end{aligned}$$

hence, $\underline{\underline{k=5}}$.

Hence

(b) find, in terms of p, an approximate value of y when $x = 6 + p$, where p is small, (3)

Solution

Well,

$$x = 6 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 5(6)\sqrt{9} = 90$$

and

$$\begin{aligned}\delta y &\approx \frac{\mathrm{d}y}{\mathrm{d}x} \times \delta x \\ &= 90 \times p \\ &= 90p.\end{aligned}$$

Finally,

$$\begin{aligned}x = 6 + p &\Rightarrow y = y(6) + \delta y \\ &\Rightarrow \underline{\underline{y = 189 + 90p}}\end{aligned}$$

(c) evaluate (3)

$$\int_2^6 x\sqrt{2x-3}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int_2^6 x\sqrt{2x-3}\,\mathrm{d}x &= \left[\tfrac{1}{5}(x+1)(2x-3)^{\frac{3}{2}}\right]_{x=2}^{6} \\ &= \tfrac{1}{5}(189-3) \\ &= \underline{\underline{37\tfrac{1}{5}}}.\end{aligned}$$

EITHER

12. A particle moves in a straight line so that, t s after leaving a fixed point O, its velocity, v ms^{-1}, is given by

$$v = 10(1 - \mathrm{e}^{-\frac{1}{2}t}).$$

(a) Find the acceleration of the particle when $v = 8$. (4)

Solution

$$\begin{aligned}v = 10(1 - \mathrm{e}^{-\frac{1}{2}t}) &\Rightarrow v = 10 - 10\mathrm{e}^{-\frac{1}{2}t} \\ &\Rightarrow a = 5\mathrm{e}^{-\frac{1}{2}t}\end{aligned}$$

and

$$\begin{aligned}v = 8 &\Rightarrow 8 = 10(1 - \mathrm{e}^{-\frac{1}{2}t}) \\ &\Rightarrow 8 = 10 - 10\mathrm{e}^{-\frac{1}{2}t} \\ &\Rightarrow 10\mathrm{e}^{-\frac{1}{2}t} = 2 \\ &\Rightarrow 5\mathrm{e}^{-\frac{1}{2}t} = 1 \\ &\Rightarrow \underline{\underline{a = 1 \text{ ms}^{-2}}}.\end{aligned}$$

(b) Calculate, to the nearest metre, the displacement of the particle from O when $t = 6$. (4)

Solution

Well,

$$v = 10 - 10\mathrm{e}^{-\frac{1}{2}t} \Rightarrow s = 10t + 20\mathrm{e}^{-\frac{1}{2}t} + c,$$

for some constant c. Now,

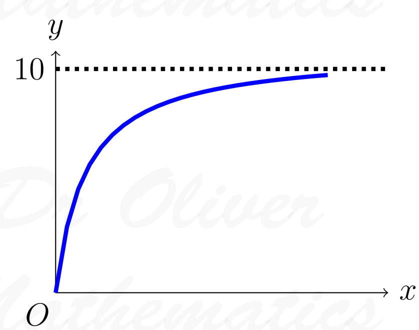
$$\begin{aligned}\text{displacement} &= s(6) - s(0) \\ &= (60 + 20\mathrm{e}^{-3} + c) - (20 + c) \\ &= 40.995\,741\,37 \text{ (FCD)} \\ &= \underline{\underline{41 \text{ m (nearest metre)}}}.\end{aligned}$$

(c) State the value which v approaches as t becomes very large. (1)

Solution
As $t \to \infty$, $\underline{\underline{v \to 10}}$.

(d) Sketch the velocity-time graph for the motion of the particle. (2)

Solution



OR

13. (a) By considering $\sec\theta$ as $(\cos\theta)^{-1}$, show that (2)

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\sec\theta) = \frac{\sin\theta}{\cos^2\theta}.$$

Solution
Well,

$$\begin{aligned}u = 1 &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 0 \\ v = \cos\theta &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = -\sin\theta\end{aligned}$$

and we apply the quotient rule:

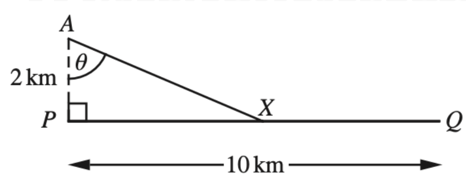
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}(\sec\theta) &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{\cos\theta}\right)\\ &= \frac{(\cos\theta)(0)-(1)(-\sin\theta)}{(\cos\theta)^2}\\ &= \underline{\underline{\frac{\sin\theta}{\cos^2\theta}}},\end{aligned}$$

as required.

The diagram shows a straight road joining two points, P and Q, 10 km apart.

A man is at point A, where AP is perpendicular to PQ and AP is 2 km.

The man wishes to reach Q as quickly as possible and travels across country in a straight line to meet the road at point X, where angle $PAX = \theta$ radians.



The man travels across country along AX at 3 ms^{-1} but on reaching the road he travels at 5 ms^{-1} along XQ.

Given that he takes T hours to travel from A to Q,

(b) show that (4)

$$T = \tfrac{2}{3}\sec\theta + 2 - \tfrac{2}{5}\tan\theta.$$

Solution

Now,

$$\begin{aligned}\cos = \frac{\text{adj}}{\text{hyp}} \Rightarrow \cos\theta &= \frac{2}{AX}\\ \Rightarrow AX &= \frac{2}{\cos\theta}\\ \Rightarrow AX &= 2\sec\theta.\end{aligned}$$

Next,

$$\tan = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan\theta = \frac{PX}{2}$$
$$\Rightarrow PX = 2\tan\theta$$

and

$$\begin{aligned} XQ &= PQ - PX \\ &= 10 - 2\tan\theta. \end{aligned}$$

Given that he takes T hours to travel from A to Q,

$$\begin{aligned} &\text{time taken from } AX + \text{time taken from } AQ = T \\ \Rightarrow\quad &\frac{\text{distance } AX}{\text{speed from } AX} + \frac{\text{distance } XQ}{\text{speed from } XQ} = T \\ \Rightarrow\quad &\frac{2\sec\theta}{3} + \frac{10 - 2\tan\theta}{5} = T \\ \Rightarrow\quad &\underline{\underline{\tfrac{2}{3}\sec\theta + 2 - \tfrac{2}{5}\tan\theta = T}}, \end{aligned}$$

as required.

(c) Given that θ can vary, show that T has a stationary value when $PX = 1.5$ km. (5)

Solution

Now,

$$\frac{\mathrm{d}T}{\mathrm{d}\theta} = \tfrac{2}{3}\sec\theta\tan\theta - \tfrac{2}{5}\sec^2\theta$$

and

$$
\begin{aligned}
\frac{\mathrm{d}T}{\mathrm{d}\theta} = 0 &\Rightarrow \tfrac{2}{3}\sec\theta\tan\theta - \tfrac{2}{5}\sec^2\theta = 0 \\
&\Rightarrow \tfrac{2}{15}\sec\theta(5\tan\theta - 3\sec\theta) = 0 \\
&\Rightarrow \sec\theta = 0 \text{ (impossible) or } 5\tan\theta - 3\sec\theta = 0 \\
&\Rightarrow \frac{5\sin\theta}{\cos\theta} - \frac{3}{\cos\theta} = 0 \\
&\Rightarrow \frac{1}{\cos\theta}(5\sin\theta - 3) = 0 \\
&\Rightarrow 5\sin\theta = 3 \\
&\Rightarrow \sin\theta = \tfrac{3}{5} \\
&\Rightarrow \tan\theta = \tfrac{3}{4} \\
&\Rightarrow PX = 2 \times \tfrac{3}{4} \\
&\Rightarrow \underline{\underline{PX = 1.5 \text{ m}}},
\end{aligned}
$$

as required.