Dr Oliver Mathematics
Further Mathematics
Eigenvalues, Eigenvectors,
and 3×3 Determinants
Past Examination Questions

This booklet consists of 24 questions across a variety of examination topics.
The total number of marks available is 236.

1. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 4 & -1 \\ 3 & 0 & p \\ a & b & c \end{pmatrix},$$

where p, a, b, and c are constants and $a > 0$. Given that $\mathbf{M}\mathbf{M}^{\mathrm{T}} = k\mathbf{I}$ for some constant k, find

(a) the value of p, (2)

Solution

$$\begin{aligned} & \mathbf{M}\mathbf{M}^{\mathrm{T}} = k\mathbf{I} \\ \Rightarrow \quad & \begin{pmatrix} 1 & 4 & -1 \\ 3 & 0 & p \\ a & b & c \end{pmatrix} \begin{pmatrix} 1 & 3 & a \\ 4 & 0 & b \\ -1 & p & c \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} \\ \Rightarrow \quad & \begin{pmatrix} 18 & 3-p & a+4b-c \\ 3-p & 9+p^2 & 3a+cp \\ a+4b-c & 3a+cp & a^2+b^2+c^2 \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}; \end{aligned}$$

hence,

$$3 - p = 0 \Rightarrow \underline{\underline{p = 3}}.$$

(b) the value of k, (2)

Solution

$\underline{\underline{k = 18}}$.

(c) the values of a, b, and c, (6)

Solution

From $(3,2)$th entry,

$$3a + 3c = 0 \Rightarrow a = -c.$$

Now,

$$a + 4b - c = 0 \Rightarrow 2a + 4b = 0 \Rightarrow a = -2b$$

and

$$(-2b)^2 + b^2 + (2b)^2 = 18 \Rightarrow 9b^2 = 18 \Rightarrow \underline{\underline{b = -\sqrt{2}}}$$

and

$$\underline{\underline{a = 2\sqrt{2}}} \text{ and } \underline{\underline{c = -2\sqrt{2}}}.$$

(It is a that determines whether b is positive or not.)

(d) $|\det \mathbf{M}|$. (2)

Solution

As we are dealing with 3×3 matrices,

$$\begin{aligned} |\det \mathbf{M}\mathbf{M}^{\mathrm{T}}| = 18^3 &\Rightarrow |\det \mathbf{M}|^2 = 5\,832 \\ &\Rightarrow |\det \mathbf{M}| = \sqrt{5\,832} \\ &\Rightarrow |\det \mathbf{M}| = \underline{\underline{54\sqrt{2}}}. \end{aligned}$$

2. The transformation R is represented by the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

(a) Find the eigenvectors of $\mathbf{A}$. (5)

Solution

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) = 0 &\Rightarrow \det \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = 0 \\ &\Rightarrow (3-\lambda)^2 - 1 = 0 \\ &\Rightarrow (3-\lambda)^2 = 1 \\ &\Rightarrow 3-\lambda = 1 \text{ or } 3-\lambda = -1 \\ &\Rightarrow \lambda = 2 \text{ or } \lambda = 4. \end{aligned}$$

(Check: is it the case that $3 \times 3 - 1 \times 1 = 2 \times 4$? Yes.) $\underline{\lambda = 2}$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and we, for example, have

$$\underline{\underline{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}}.$$

$\underline{\lambda = 4}$:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and we, for example, have

$$\underline{\underline{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}}.$$

(b) Find an orthogonal matrix **P** and a diagonal matrix **D** such that (5)

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}.$$

Solution

The orthogonal matrix is

$$\underline{\underline{\mathbf{P} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}}$$

and the diagonal matrix is

$$\underline{\underline{\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}}}.$$

(c) Hence describe the transformation R as a combination of geometrical transformations, stating clearly their order. (4)

Solution

(i) Rotation of 45°, clockwise, about $(0, 0)$,
(ii) Stretch: by 2 in the x-direction and by 4 in the y-direction, and
(iii) Rotation of 45°, anticlockwise, about $(0, 0)$.

3.

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & k \end{pmatrix}.$$

(a) Show that $\det \mathbf{A} = 20 - 4k$. (4)

Solution

$$\begin{aligned} \det \mathbf{A} &= 3(0-4) - 2(2k-8) + 4(4-0) \\ &= -12 - 4k + 16 + 16 \\ &= \underline{\underline{20 - 4k}}, \end{aligned}$$

as required.

(b) Find $\mathbf{A}^{-1}$. (6)

Solution

Determinant: We have got $\det \mathbf{A} = 20 - 4k$.

Matrix of minors:

$$\begin{pmatrix} -4 & 2k-8 & 4 \\ 2k-8 & 3k-16 & -2 \\ 4 & -2 & -4 \end{pmatrix}$$

Matrix of cofactors:

$$\begin{pmatrix} -4 & -2k+8 & 4 \\ -2k+8 & 3k-16 & 2 \\ 4 & 2 & -4 \end{pmatrix}$$

Transpose:

$$\begin{pmatrix} -4 & -2k+8 & 4 \\ -2k+8 & 3k-16 & 2 \\ 4 & 2 & -4 \end{pmatrix}$$

Inverse:

$$\underline{\underline{\mathbf{A}^{-1} = \frac{1}{20-4k} \begin{pmatrix} -4 & -2k+8 & 4 \\ -2k+8 & 3k-16 & 2 \\ 4 & 2 & -4 \end{pmatrix}}}.$$

Given that $k = 3$ and that

$$\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

is an eigenvector of $\mathbf{A}$,

(c) find the corresponding eigenvalue. (2)

Solution

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

and $\underline{\underline{\lambda = -1}}$.

Given that the only other distinct eigenvalue of $\mathbf{A}$ is 8,

(d) find a corresponding eigenvector. (4)

Solution

$$\begin{aligned} \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= 8 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \Rightarrow \begin{pmatrix} 3x + 2y + 4y \\ 2x + 2z \\ 4x + 2y + 3z \end{pmatrix} &= \begin{pmatrix} 8x \\ 8y \\ 8z \end{pmatrix} \\ \Rightarrow \begin{pmatrix} -5x + 2y + 4y \\ 2x - 8y + 2z \\ 4x + 2y - 5z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Now, for example,

$$\underline{\underline{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}}},$$

or any non-zero scalar multiple.

4. A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} k & 2 \\ 2 & -1 \end{pmatrix},$$

where k is a constant. For the case $k = -4$,

(a) find the image under T of the line with equation $y = 2x + 1$. (2)

Solution

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ 2x+1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix};$$

the image is the point $\underline{\underline{(2, -1)}}$.

For the case $k = 2$, find

(b) the two eigenvalues of $\mathbf{A}$, (4)

Solution

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) = 0 &\Rightarrow \det \begin{pmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix} = 0 \\ &\Rightarrow (-1-\lambda)(2-\lambda) - 4 = 0 \\ &\Rightarrow \lambda^2 - \lambda - 6 = 0 \\ &\Rightarrow (\lambda - 3)(\lambda + 2) \\ &\Rightarrow \underline{\underline{\lambda = -2}} \text{ or } \underline{\underline{\lambda = 3}}. \end{aligned}$$

(c) a cartesian equation for each of the two lines passing through the origin which are invariant under T. (3)

Solution

$\underline{\lambda = -2}$:

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and we have

$$\underline{\underline{y = -2x}}.$$

$\underline{\lambda = 3}$:

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and we have

$$\underline{\underline{y = \tfrac{1}{2}x}}.$$

5.

$$\mathbf{A} = \begin{pmatrix} k & 1 & -2 \\ 0 & -1 & k \\ 9 & 1 & 0 \end{pmatrix},$$

where k is a real constant.

(a) Find values of k for which $\mathbf{A}$ is singular. (4)

Solution

$$\begin{aligned} \det(\mathbf{A}) = 0 &\Rightarrow k(0-k) - (0-9k) - 2(0-(-9)) = 0 \\ &\Rightarrow -k^2 + 9k - 18 = 0 \\ &\Rightarrow k^2 - 9k + 18 = 0 \\ &\Rightarrow (k-3)(k-6) = 0 \\ &\Rightarrow \underline{\underline{k=3}} \text{ or } \underline{\underline{k=6}}. \end{aligned}$$

Given that $\mathbf{A}$ is non-singular,

(b) find, in terms of k, $\mathbf{A}^{-1}$. (5)

Solution

Determinant: We have got $\det \mathbf{A} = -k^2 + 9k - 18$.

Matrix of minors:

$$\begin{pmatrix} -k & -9k & 9 \\ 2 & 18 & k-9 \\ k-2 & k^2 & -k \end{pmatrix}$$

Matrix of cofactors:

$$\begin{pmatrix} -k & 9k & 9 \\ -2 & 18 & -k+9 \\ k-2 & -k^2 & -k \end{pmatrix}$$

Transpose:

$$\begin{pmatrix} -k & -2 & k-2 \\ 9k & 18 & -k^2 \\ 9 & -k+9 & -k \end{pmatrix}$$

Inverse:

$$\underline{\underline{\mathbf{A}^{-1} = \frac{1}{-k^2+9k-18} \begin{pmatrix} -k & -2 & k-2 \\ 9k & 18 & -k^2 \\ 9 & -k+9 & -k \end{pmatrix}}}.$$

6. (5)

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove by induction, that for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & n & \frac{1}{2}(n^2+3n) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution

$\underline{n=1}$:

$$\mathbf{A}^1 = \begin{pmatrix} 1 & 1 & \frac{1}{2}(1^2+3\times 1) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\mathbf{A}^k = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{A}^{k+1} &= \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & k & \frac{1}{2}(k^2+3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & k+1 & \frac{1}{2}(k^2+3k)+k+2 \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & k+1 & \frac{1}{2}(k^2+5k+4) \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & k+1 & \frac{1}{2}\left[(k+1)^2+3(k+1)\right] \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

7. The eigenvalues of the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix},$$

and λ_1 and λ_2, where $\lambda_1 < \lambda_2$.

(a) Find the value of λ_1 and the value of λ_2. (3)

Solution

$$\begin{aligned} \det(\mathbf{M} - \lambda \mathbf{I}) = 0 &\Rightarrow (4-\lambda)(1-\lambda) + 2 = 0 \\ &\Rightarrow \lambda^2 - 5\lambda + 6 = 0 \\ &\Rightarrow (\lambda - 2)(\lambda - 3) = 0 \\ &\Rightarrow \underline{\underline{\lambda_1 = 2}} \text{ or } \underline{\underline{\lambda_2 = 3}}. \end{aligned}$$

(b) Find $\mathbf{M}^{-1}$. (2)

Solution

$$\det \mathbf{M} = 4 \times 1 - 1 \times (-2) = 6$$

and

$$\mathbf{M}^{-1} = \underline{\underline{\tfrac{1}{6}\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}}}.$$

(c) Verify that the eigenvalues of $\mathbf{M}^{-1}$ and λ_1^{-1} and λ_2^{-1}. (3)

Solution

$$\begin{aligned} \det(\mathbf{M}^{-1} - \lambda \mathbf{I}) = 0 &\Rightarrow (\tfrac{1}{6} - \lambda)(\tfrac{2}{3} - \lambda) + \tfrac{1}{18} = 0 \\ &\Rightarrow \lambda^2 - \tfrac{5}{6}\lambda + \tfrac{1}{6} = 0 \\ &\Rightarrow 6\lambda^2 - 5\lambda + 1 = 0 \\ &\Rightarrow (3\lambda - 1)(2\lambda - 1) = 0 \\ &\Rightarrow \underline{\underline{\lambda = \tfrac{1}{3}}} \text{ or } \underline{\underline{\lambda = \tfrac{1}{2}}}. \end{aligned}$$

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix $\mathbf{M}$. There are two lines, passing through the origin, each of which is mapped onto itself under the transformation T.

(d) Find cartesian equations for each of these lines. (4)

Solution
$\underline{\lambda = 2}$:

$$\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and we have

$$\underline{\underline{y = x}}.$$

$\underline{\lambda = 3}$:

$$\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and we have

$$\underline{\underline{y = \tfrac{1}{2}x}}.$$

8. Given that $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 3 & 4 & p \\ -1 & q & -4 \\ 1 & 1 & 3 \end{pmatrix},$$

(a) find the eigenvalue of $\mathbf{A}$ corresponding to $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, (2)

Solution

$$\begin{pmatrix} 3 & 4 & p \\ -1 & q & -4 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4-p \\ q+4 \\ -2 \end{pmatrix}$$

and it is $\underline{\underline{\lambda = 2}}$.

(b) find the value of p and the value of q. (4)

Solution

$$4 - p = 0 \Rightarrow \underline{\underline{p = 4}}$$

and

$$q + 4 = 2 \Rightarrow \underline{\underline{q = -2}}.$$

The image of the vector $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ when transformed by $\mathbf{A}$ is $\begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$.

(c) Using the values of p and q from part (b), find the values of the constants l, m, and n. (4)

Solution

$$\begin{pmatrix} 3 & 4 & 4 \\ -1 & -2 & -4 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}$$
$$\Rightarrow \quad \begin{pmatrix} 3l + 4m + 4n \\ -l - 2m - 4n \\ l + m + 3n \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ 3 \end{pmatrix}.$$

Adding the first and third rows:

$$2l + 2m = 6.$$

Adding the three times the second and four times the third rows:

$$l - 4m = 0.$$

Hence,

$$\begin{aligned} l = 2m &\Rightarrow 2(2m) + 2m = 6 \\ &\Rightarrow 6m = 6 \\ &\Rightarrow \underline{\underline{m = 1}} \\ &\Rightarrow \underline{\underline{l = 2}} \\ &\Rightarrow 3 \times 2 + 4 \times 1 + 4n = 10 \\ &\Rightarrow 4n = 0 \\ &\Rightarrow \underline{\underline{n = 0}}. \end{aligned}$$

9.

$$\mathbf{M} = \begin{pmatrix} 1 & p & 2 \\ 0 & 3 & q \\ 2 & p & 1 \end{pmatrix},$$

where p and q are constants. Given that $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{M}$,

(a) show that $q = 4p$. (3)

Solution

$$\begin{pmatrix} 1 & p & 2 \\ 0 & 3 & q \\ 2 & p & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2p+3 \\ q+6 \\ 2p+3 \end{pmatrix},$$

so

$$q + 6 = 2(2p+3) \Rightarrow q + 6 = 4p + 6 \Rightarrow \underline{\underline{q = 4p}}.$$

Given also that $\lambda = 5$ is an eigenvalue of $\mathbf{M}$, and $p < 0$ and $q < 0$, find

(b) the values of p and q, (4)

Solution

$$\begin{pmatrix} 1 & p & 2 \\ 0 & 3 & 4p \\ 2 & p & 1 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -4 & p & 2 \\ 0 & -2 & 4p \\ 2 & p & -4 \end{pmatrix}$$

and

$$\begin{vmatrix} -4 & p & 2 \\ 0 & -2 & 4p \\ 2 & p & -4 \end{vmatrix} = 0 \Rightarrow (-4)(8 - 4p^2) - p(0 - 8p) + 2(0 + 4) = 0$$

$$\Rightarrow -32 + 16p^2 + 8p^2 + 8 = 0$$
$$\Rightarrow 24p^2 = 24$$
$$\Rightarrow p^2 = 1$$
$$\Rightarrow \underline{\underline{p = -1}}$$
$$\Rightarrow \underline{\underline{q = -4}}.$$

(c) an eigenvector corresponding to the eigenvalue $\lambda = 5$. (3)

Solution

$$\begin{pmatrix} -4 & -1 & 2 \\ 0 & -2 & -4 \\ 2 & -1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\underline{\underline{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}}$$

is an eigenvector (or any non-zero multiple).

10.

$$\mathbf{A} = \begin{pmatrix} k & -2 \\ 1-k & k \end{pmatrix},$$

where k is constant. A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix $\mathbf{A}$.

(a) Find the value of k for which the line $y = 2x$ is mapped onto itself under T. (3)

Solution

$$\begin{pmatrix} k & -2 \\ 1-k & k \end{pmatrix}\begin{pmatrix} x \\ 2x \end{pmatrix} = \begin{pmatrix} (k-4)x \\ (\frac{1}{2}+\frac{1}{2}k)2x \end{pmatrix};$$

hence,

$$k - 4 = \tfrac{1}{2} + \tfrac{1}{2}k \Rightarrow \tfrac{1}{2}k = \tfrac{9}{2} \Rightarrow \underline{\underline{k = 9}}.$$

(b) Show that $\mathbf{A}$ is non-singular for all values of k. (3)

Solution

$$\begin{aligned} \det \mathbf{A} &= k \times k - (1-k) \times (-2) \\ &= k^2 - 2k + 2 \\ &= (k^2 - 2k + 1) + 1 \\ &= (k-1)^2 + 1 \\ &\geqslant 1; \end{aligned}$$

hence, $\underline{\underline{\mathbf{A} \text{ is non-singular for all values of } k}}$.

(c) Find $\mathbf{A}^{-1}$ in terms of k. (2)

Solution

$$\mathbf{A}^{-1} = \underline{\underline{\tfrac{1}{k^2-2k+2}\begin{pmatrix} k & 2 \\ k-1 & k \end{pmatrix}}}.$$

A point P is mapped onto a point Q under T. The point Q has position vector $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ relative to an origin O. Given that $k = 3$,

(d) find the position vector of P. (3)

Solution

$$\begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{5}\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ -3 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{5}\begin{pmatrix} 6 \\ -1 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1\frac{1}{5} \\ -\frac{1}{5} \end{pmatrix}}}.$$

11.

$$\mathbf{M} = \begin{pmatrix} 6 & 1 & -1 \\ 0 & 7 & 0 \\ 3 & -1 & 2 \end{pmatrix}.$$

(a) Show that 7 is an eigenvalue of the matrix **M** and find the other two eigenvalues of **M**. (5)

Solution

$$\begin{aligned}
& \det(\mathbf{M} - \lambda\mathbf{I}) = 0 \\
\Rightarrow\ & (6-\lambda)[(7-\lambda)(2-\lambda) - 0] - (0-0) + (-1)[0 - 3(7-\lambda)] = 0 \\
\Rightarrow\ & (6-\lambda)(7-\lambda)(2-\lambda) + 3(7-\lambda) = 0 \\
\Rightarrow\ & (7-\lambda)[(6-\lambda)(2-\lambda) + 3] = 0 \\
\Rightarrow\ & (7-\lambda)(\lambda^2 - 8\lambda + 15) = 0 \\
\Rightarrow\ & (7-\lambda)(\lambda-5)(\lambda-3) = 0 \\
\Rightarrow\ & \underline{\underline{\lambda = 3, 5, \text{ or } 7}}.
\end{aligned}$$

(b) Find an eigenvector corresponding to the eigenvalue 7. (4)

Solution

$$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 3 & -1 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\underline{\underline{\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}}}$$

is an eigenvector (or any non-zero multiple).

12. For $n \in \mathbb{Z}^+$, show, using mathematical induction, that (5)

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ n(n+2) & 2n & 1 \end{pmatrix}.$$

Solution

Let

$$\mathbf{A}^n = \begin{pmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ n(n+2) & 2n & 1 \end{pmatrix}.$$

$\underline{n = 1}$:

$$\mathbf{A}^1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1(1+2) & 2 \times 1 & 1 \end{pmatrix}$$

and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\mathbf{A}^k = \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ k(k+2) & 2k & 1 \end{pmatrix}.$$

Then

$$\begin{aligned}
\mathbf{A}^{k+1} &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ k(k+2) & 2k & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ k+1 & 1 & k+1 \\ 3+2k+k(k+2) & 2+2k & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ k+1 & 1 & 0 \\ k^2+4k+3 & 2(k+1) & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ k+1 & 1 & 0 \\ (k+1)(k+3) & 2(k+1) & 1 \end{pmatrix},
\end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

13.

$$\mathbf{M} = \begin{pmatrix} 11 & -5\sqrt{3} \\ -5\sqrt{3} & 1 \end{pmatrix}.$$

Given that λ_1 and λ_2 are the eigenvalues of $\mathbf{M}$ and $\lambda_1 > \lambda_2$,

(a) show that $\lambda_1 = 16$ and find the value of λ_2. (4)

Solution

$$\begin{aligned}
& \det(\mathbf{M} - \lambda\mathbf{I}) = 0 \\
\Rightarrow\ & (11 - \lambda)(1 - \lambda) - (-5\sqrt{3})(-5\sqrt{3}) = 0 \\
\Rightarrow\ & (\lambda^2 - 12\lambda + 11) - 75 = 0 \\
\Rightarrow\ & \lambda^2 - 12\lambda - 64 = 0 \\
\Rightarrow\ & (\lambda - 16)(\lambda + 4) = 0;
\end{aligned}$$

hence, $\underline{\underline{\lambda_1 = 16}}$ and $\underline{\underline{\lambda_2 = -4}}$.

(b) Find eigenvectors corresponding to the eigenvalues λ_1 and λ_2. (4)

Solution

$\underline{\lambda_1 = 16}$:

$$\begin{pmatrix} -5 & -5\sqrt{3} \\ -5\sqrt{3} & -15 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and we have

$$\underline{\underline{\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}}}.$$

$\underline{\lambda_2 = -4}$:

$$\begin{pmatrix} 15 & -5\sqrt{3} \\ -5\sqrt{3} & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and we have

$$\underline{\underline{\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}}}.$$

Given that there is an orthogonal matrix $\mathbf{M}$ such that $\mathbf{P}^{-1}\mathbf{MP}$ is the diagonal matrix $\mathbf{D}$, where

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

(c) find the matrix $\mathbf{P}$, (2)

Solution

$$\underline{\underline{\mathbf{P} = \tfrac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix}}}.$$

(d) verify that $\mathbf{P}^{-1}\mathbf{MP} = \mathbf{D}$. (4)

Solution

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{MP} &= \tfrac{1}{2}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix} \begin{pmatrix} 11 & -5\sqrt{3} \\ -5\sqrt{3} & 1 \end{pmatrix} \tfrac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix} \\ &= \tfrac{1}{4}\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix} \begin{pmatrix} 16\sqrt{3} & -4 \\ -16 & -4\sqrt{3} \end{pmatrix} \\ &= \tfrac{1}{4}\begin{pmatrix} 64 & 0 \\ 0 & -16 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 16 & 0 \\ 0 & -4 \end{pmatrix}}}, \end{aligned}$$

as required.

14.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ k & 0 & 1 \end{pmatrix},$$

where k is a constant. Given that $\begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix}$ is an eigenvector of $\mathbf{M}$,

(a) find the eigenvalue of $\mathbf{M}$ corresponding to $\begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix}$, (2)

Solution

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ k & 0 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix} = \lambda \begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 24 \\ 4 \\ 6k+6 \end{pmatrix} = \lambda \begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix};$$

hence,

$$24 = 6\lambda \Rightarrow \underline{\underline{\lambda = 4}}.$$

(b) show that $k = 3$, (2)

Solution

$$6k + 6 = 24 \Rightarrow 6k = 18 \Rightarrow \underline{\underline{k = 3}}.$$

(c) show that $\mathbf{M}$ has exactly two eigenvalues. (4)

Solution

$$\begin{aligned}\det(\mathbf{M}-\lambda\mathbf{I})=0 &\Rightarrow (1-\lambda)[(-2-\lambda)(1-\lambda)-0]-0+3[0-3(-2-\lambda)]=0\\ &\Rightarrow (-2-\lambda)(1-\lambda)^2-9(-2-\lambda)=0\\ &\Rightarrow (-2-\lambda)[(1-\lambda)^2-9]=0\\ &\Rightarrow (-2-\lambda)(\lambda^2-2\lambda^2-8)=0\\ &\Rightarrow (-2-\lambda)(\lambda-4)(\lambda+2)=0,\end{aligned}$$

and the eigenvalues are

$$\underline{\underline{-2 \text{ (twice) and } 4}}.$$

15. The matrix $\mathbf{M}$ is given by

$$\mathbf{M}=\begin{pmatrix} k & -1 & 1\\ 1 & 0 & -1\\ 3 & -2 & -1\end{pmatrix},$$

where $k \neq 1$.

(a) Show that $\det \mathbf{M} = 2-2k$. (2)

Solution

$$\begin{aligned}\det \mathbf{M} &= k(0-2)+(1+3)+(-2-0)\\ &= \underline{\underline{2-2k}}.\end{aligned}$$

(b) Find $\mathbf{M}^{-1}$, in terms of k. (5)

Solution

Matrix of minors:

$$\begin{pmatrix} -2 & 4 & -2\\ 1 & k-3 & -2k+3\\ 1 & -k-1 & 1\end{pmatrix}$$

Matrix of cofactors:

$$\begin{pmatrix} -2 & -4 & -2\\ -1 & k-3 & 2k-3\\ 1 & k+1 & 1\end{pmatrix}$$

Transpose:

$$\begin{pmatrix} -2 & -1 & 1\\ -4 & k-3 & k+1\\ -2 & 2k-3 & 1\end{pmatrix}$$

Inverse:

$$\mathbf{M}^{-1} = \underline{\underline{\tfrac{1}{2-2k}\begin{pmatrix} -2 & -1 & 1 \\ -4 & k-3 & k+1 \\ -2 & 2k-3 & 1 \end{pmatrix}}}.$$

16. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}.$$

(a) Show that 4 is an eigenvalue of $\mathbf{M}$, and find the other two eigenvalues. (5)

Solution

$$\begin{aligned}
& \det(\mathbf{M} - \lambda\mathbf{I}) = 0 \\
\Rightarrow\ & (2-\lambda)[(2-\lambda)(4-\lambda) - 0] - [(4-\lambda) - 0] + 0 = 0 \\
\Rightarrow\ & (2-\lambda)^2(4-\lambda) - (4-\lambda) = 0 \\
\Rightarrow\ & (4-\lambda)[(2-\lambda)^2 - 1] = 0 \\
\Rightarrow\ & (4-\lambda)(\lambda^2 - 4\lambda + 3) = 0 \\
\Rightarrow\ & (4-\lambda)(\lambda-1)(\lambda-3) = 0 \\
\Rightarrow\ & \underline{\underline{\lambda = 1, 3, \text{ or } 4}}.
\end{aligned}$$

(b) For the eigenvalue 4, find a corresponding eigenvector. (3)

Solution

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\underline{\underline{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}}$$

is an eigenvector (or any non-zero multiple).

17. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & a \\ 2 & b & c \\ -1 & 0 & 1 \end{pmatrix},$$

where a, b, and c are constants.

(a) Given that $\mathbf{j}+\mathbf{k}$ and $\mathbf{i}-\mathbf{k}$ are two of the eigenvectors of $\mathbf{M}$, find (8)

(i) the values of a, b, and c,

Solution

$$\begin{pmatrix} 1 & 1 & a \\ 2 & b & c \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = m\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1+a \\ b+c \\ 1 \end{pmatrix} = m\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Since $1 = m \times 1$,

$$1 + a = 0 \Rightarrow \underline{\underline{a = -1}}$$

and

$$b + c = 1.$$

$$\begin{pmatrix} 1 & 1 & a \\ 2 & b & c \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = n\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1-a \\ 2-c \\ -2 \end{pmatrix} = n\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Since $-2 = n \times (-1) \Rightarrow n = 2$,

$$2 - c = 0 \Rightarrow \underline{\underline{c = 2}}$$

and

$$b + 2 = 1 \Rightarrow \underline{\underline{b = -1}}.$$

(ii) the eigenvalues which correspond to the two given eigenvectors.

Solution

The eigenvector $\mathbf{j}+\mathbf{k}$ corresponds to $\underline{\underline{1}}$ and eigenvector $\mathbf{i}-\mathbf{k}$ corresponds to $\underline{\underline{2}}$.

The matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & d \\ -1 & 0 & 1 \end{pmatrix},$$

where d is constant and $d \neq 1$. Find

(b) (i) the determinant of $\mathbf{P}$ in terms of d, (5)

Solution

$$\det \mathbf{P} = 1(1-0) - 1(2+d) + 0 = \underline{\underline{-d-1}}.$$

(ii) the matrix $\mathbf{P}^{-1}$ in terms of d.

Solution

Matrix of minors:

$$\begin{pmatrix} 1 & d+2 & 1 \\ 1 & 1 & 1 \\ d & d & -1 \end{pmatrix}$$

Matrix of cofactors:

$$\begin{pmatrix} 1 & -d-2 & 1 \\ -1 & 1 & -1 \\ d & -d & -1 \end{pmatrix}$$

Transpose:

$$\begin{pmatrix} 1 & -1 & d \\ -d-2 & 1 & -d \\ 1 & -1 & -1 \end{pmatrix}$$

Inverse:

$$\mathbf{P}^{-1} = \underline{\underline{\frac{1}{-d-1}\begin{pmatrix} 1 & -1 & d \\ -d-2 & 1 & -d \\ 1 & -1 & -1 \end{pmatrix}}}.$$

18. It is given that $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ is an eigenvector of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 3 \\ 2 & b & 0 \\ a & 1 & 8 \end{pmatrix},$$

and a and b are constants.

(a) Find the eigenvalue of $\mathbf{A}$ corresponding to the eigenvector $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$. (3)

Solution

$$\begin{pmatrix} 4 & 2 & 3 \\ 2 & b & 0 \\ a & 1 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = m\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$
$$\Rightarrow \quad \begin{pmatrix} 8 \\ 2b+2 \\ a+2 \end{pmatrix} = \begin{pmatrix} m \\ 2m \\ 0 \end{pmatrix};$$

hence, $\underline{\underline{m = 8}}$.

(b) Find the values of a and b. (3)

Solution

$$a + 2 = 0 \Rightarrow \underline{\underline{a = -2}}$$

and

$$2b + 2 = 16 \Rightarrow 2b = 14 \Rightarrow \underline{\underline{b = 7}}.$$

(c) Find the other eigenvalues of $\mathbf{A}$. (5)

Solution

$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 3 \\ 2 & 7 & 0 \\ -2 & 1 & 8 \end{pmatrix}$$

and

$$\begin{aligned} & \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \\ \Rightarrow \quad & (4-\lambda)[(7-\lambda)(8-\lambda) - 0] - 2[2(8-\lambda) - 0] + 3[2 + 2(7-\lambda)] = 0 \\ \Rightarrow \quad & (4-\lambda)(\lambda^2 - 15\lambda + 56) - 4(8-\lambda) + 6 + 6(7-\lambda) = 0 \\ \Rightarrow \quad & -\lambda^3 + 19\lambda^2 - 116\lambda + 224 + 16 - 2\lambda = 0 \\ \Rightarrow \quad & \lambda^3 - 19\lambda^2 + 118\lambda - 240 = 0 \\ \Rightarrow \quad & (\lambda - 8)(\lambda^2 - 11\lambda + 30) = 0 \text{ (from part (a))} \\ \Rightarrow \quad & (\lambda - 8)(\lambda - 5)(\lambda - 6) = 0; \end{aligned}$$

hence, the other eigenvalues are $\underline{\underline{5}}$ and $\underline{\underline{6}}$.

19.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 5 & 1 \end{pmatrix}.$$

(a) Show that matrix **M** is not orthogonal. (2)

Solution

$$\mathbf{M}\mathbf{M}^{\mathrm{T}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 1 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 5 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 17 & 20 \\ 0 & 20 & 25 \end{pmatrix};$$

hence, **M is not orthogonal**.

(b) Using algebra, show that 1 is an eigenvalue of M and find the other two eigenvalues of **M**. (5)

Solution

$$\begin{aligned} & \det(\mathbf{M} - \lambda\mathbf{I}) = 0 \\ \Rightarrow\ & (1-\lambda)[(4-\lambda)(-\lambda) - 5] - 0 + 2[0 - 0] = 0 \\ \Rightarrow\ & (1-\lambda)(\lambda^2 - 4\lambda - 5) = 0 \\ \Rightarrow\ & (1-\lambda)(\lambda - 5)(\lambda + 1) = 0 \\ \Rightarrow\ & \underline{\underline{\lambda = -1, 1, \text{ or } 5}}. \end{aligned}$$

(c) Find an eigenvector of **M** which corresponds to the eigenvalue 1. (2)

Solution

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & 3 & 1 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\underline{\underline{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}}$$

is an eigenvector (or any non-zero multiple).

20. The symmetric matrix **M** has eigenvectors $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ with eigenvalues 5, 2, and -1 respectively.

(a) Find an orthogonal matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that (4)

$$\mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D}.$$

Solution

$$\sqrt{2^2 + 2^2 + 1^2} = 3.$$

Now,

$$\mathbf{P} = \frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Given that $\mathbf{P}^{-1} = \mathbf{P}^{\mathrm{T}}$,

(b) show that (2)

$$\mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}.$$

Solution

$$\begin{aligned} \mathbf{P}^{\mathrm{T}}\mathbf{M}\mathbf{P} = \mathbf{D} &\Rightarrow \mathbf{P}^{-1}\mathbf{M}\mathbf{P} = \mathbf{D} \\ &\Rightarrow \mathbf{P}\mathbf{P}^{-1}\mathbf{M}\mathbf{P}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \\ &\Rightarrow (\mathbf{P}\mathbf{P}^{-1})\mathbf{M}(\mathbf{P}\mathbf{P}^{-1}) = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \\ &\Rightarrow \mathbf{I}\mathbf{M}\mathbf{I} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \\ &\Rightarrow \mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}, \end{aligned}$$

as required.

(c) Hence find the matrix $\mathbf{M}$. (5)

Solution

$$\begin{aligned}
\mathbf{M} &= \mathbf{PDP}^{-1} \\
&= \tfrac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}\tfrac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix} \\
&= \tfrac{1}{9}\begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 10 & 10 & 5 \\ -4 & 2 & 4 \\ -1 & 2 & -2 \end{pmatrix} \\
&= \tfrac{1}{9}\begin{pmatrix} 27 & 18 & 0 \\ 18 & 18 & 18 \\ 0 & 18 & 9 \end{pmatrix} \\
&= \underline{\underline{\begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}}}.
\end{aligned}$$

21.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues of $\mathbf{A}$. (5)

Solution

$$\begin{aligned}
& \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \\
\Rightarrow\ & (2-\lambda)[(2-\lambda)^2 - 1] - [(2-\lambda) - 0] + 0 = 0 \\
\Rightarrow\ & (2-\lambda)[(2-\lambda)^2 - 2] = 0 \\
\Rightarrow\ & 2-\lambda = 0 \text{ or } (2-\lambda)^2 = 2 \\
\Rightarrow\ & \lambda = 2 \text{ or } 2-\lambda = \pm\sqrt{2} \\
\Rightarrow\ & \underline{\underline{\lambda = 2}} \text{ or } \underline{\underline{\lambda = 2 \pm \sqrt{2}}}.
\end{aligned}$$

(b) Find a normalised eigenvector for each of the eigenvalues of $\mathbf{A}$. (5)

Solution

$\underline{\lambda = 2}$:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\underline{\underline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}}$$

is an normalised eigenvector.

$\underline{\lambda = 2 + \sqrt{2}}$:

$$\begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\underline{\underline{\tfrac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}}}$$

is an normalised eigenvector.

$\underline{\lambda = 2 - \sqrt{2}}$:

$$\begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\underline{\underline{\tfrac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}}}$$

is an normalised eigenvector.

(c) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{D}$. (2)

Solution

$$\mathbf{P} = \underline{\underline{\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}}} \text{ and } \mathbf{D} = \underline{\underline{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2+\sqrt{2} & 0 \\ 0 & 0 & 2-\sqrt{2} \end{pmatrix}}}.$$

22. (4)

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & -3 \\ k & 1 & 3 \\ 2 & -1 & k \end{pmatrix},$$

where k is a constant. Given that the matrix $\mathbf{A}$ is singular, find the possible values of k.

Solution

$$\begin{aligned}\det \mathbf{A} = 0 &\Rightarrow -2(k+3) - (k^2-6) - 3(-k-2) = 0 \\ &\Rightarrow -2k - 6 - k^2 + 6 + 3k + 6 = 0 \\ &\Rightarrow k^2 - k - 6 = 0 \\ &\Rightarrow (k-3)(k+2) = 0 \\ &\Rightarrow \underline{\underline{k = -2}} \text{ or } \underline{\underline{k = 3}}.\end{aligned}$$

23.

$$\mathbf{M} = \begin{pmatrix} p & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & q \end{pmatrix},$$

where p and q are constants. Given that $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix $\mathbf{M}$,

(a) find the eigenvalue corresponding to this eigenvector, (3)

Solution

$$\begin{pmatrix} p & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & q \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2p+4 \\ -18 \\ 4+q \end{pmatrix}$$

and hence the eigenvalue is $\underline{\underline{9}}$.

(b) find the value of p and the value of q. (3)

Solution

$$2p + 4 = 18 \Rightarrow 2p = 14 \Rightarrow \underline{\underline{p = 7}}$$

and

$$4 + q = 9 \Rightarrow \underline{\underline{q = 5}}.$$

Given that 6 is another eigenvalue of $\mathbf{M}$,

(c) find a corresponding eigenvector. (2)

Solution

$$\begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\underline{\underline{\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}}}$$

is an eigenvector.

Given that $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ is a third eigenvector of $\mathbf{M}$ with eigenvalue 3,

(d) find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that (3)

$$\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{D}.$$

Solution

$$\mathbf{P} = \underline{\underline{\tfrac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}}} \text{ and } \mathbf{D} = \underline{\underline{\begin{pmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix}}}.$$

24. The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 1 & k & 0 \\ 2 & -2 & 1 \\ -4 & 1 & -1 \end{pmatrix}, \; k \in \mathbb{R}, \; k \neq \tfrac{1}{2}.$$

(a) Show that $\det \mathbf{M} = 1 - 2k$. (2)

Solution

$$\begin{aligned} \det \mathbf{M} &= 1(2-1) - k(-2+4) + 0 \\ &= \underline{\underline{1 - 2k}}, \end{aligned}$$

as required.

(b) Find $\mathbf{M}^{-1}$ in terms of k. (4)

Solution

Matrix of minors:

$$\begin{pmatrix} 1 & 2 & -6 \\ -k & -1 & 1+4k \\ k & 1 & -2k-2 \end{pmatrix}$$

Matrix of cofactors:

$$\begin{pmatrix} 1 & -2 & -6 \\ k & -1 & -4k-1 \\ k & -1 & -2k-2 \end{pmatrix}$$

Transpose:

$$\begin{pmatrix} 1 & k & k \\ -2 & -1 & -1 \\ -6 & -4k-1 & -2k-2 \end{pmatrix}$$

Inverse:

$$\mathbf{M}^{-1} = \underline{\underline{\frac{1}{1-2k}\begin{pmatrix} 1 & k & k \\ -2 & -1 & -1 \\ -6 & -4k-1 & -2k-2 \end{pmatrix}}}.$$