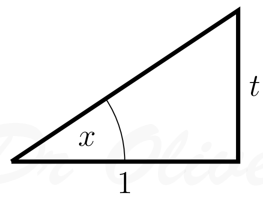
Dr Oliver Mathematics
The t-Substitutions

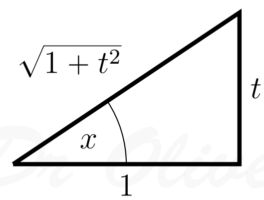
The t-substitutions are often referred to as "the world's sneakiest substitutions" and you need to recognise when they are needed.

1 $t = \tan x$

We start off with a right-angled triangle, with opposite side t, adjacent side 1, with the angle x:



We mark in the hypotenuse:



Now,

$$\begin{aligned} \sin x &= \frac{t}{\sqrt{1+t^2}} \\ \cos x &= \frac{1}{\sqrt{1+t^2}} \\ \tan x &= t; \end{aligned}$$

the expressions for $\operatorname{cosec} x$, $\sec x$, and $\cot x$ follow.

Finally, what about $\mathrm{d}x$? Now,

$$\begin{aligned} t = \tan x \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x} &= \sec^2 x \\ \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x} &= 1 + \tan^2 x \\ \Rightarrow \mathrm{d}x &= \frac{1}{1+t^2}\,\mathrm{d}t. \end{aligned}$$

We can summarise the formulas:

Expression	Replacement
$\sin x$	$\dfrac{t}{\sqrt{1+t^2}}$
$\cos x$	$\dfrac{1}{\sqrt{1+t^2}}$
$\tan x$	t
$\operatorname{cosec} x$	$\dfrac{\sqrt{1+t^2}}{t}$
$\sec x$	$\sqrt{1+t^2}$
$\cot x$	$\dfrac{1}{t}$
$\mathrm{d}x$	$\dfrac{1}{1+t^2}\,\mathrm{d}t$

Example 1
Find

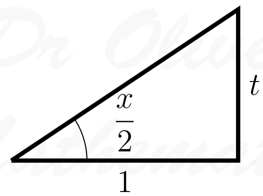
$$\int_0^{\frac{\pi}{4}} \frac{1}{3\cos^2 x + \sin^2 x}\,\mathrm{d}x.$$

Solution

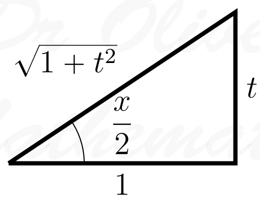
$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{1}{3\cos^2 x + \sin^2 x}\,\mathrm{d}x &= \int_0^1 \frac{1}{3\left(\frac{1}{1+t^2}\right) + \left(\frac{t^2}{1+t^2}\right)} \cdot \frac{1}{1+t^2}\,\mathrm{d}t \\ &= \int_0^1 \frac{1}{3+t^2}\,\mathrm{d}t \\ &= \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{t}{\sqrt{3}}\right)\right]_{t=0}^1 \\ &= \frac{1}{\sqrt{3}}\left(\frac{\pi}{6} - 0\right) \\ &= \underline{\underline{\frac{\pi\sqrt{3}}{18}}}.\end{aligned}$$

2 $t = \tan\left(\frac{x}{2}\right)$

This is also known as the *Weierstrass substitution*. We start off with a right-angled triangle, with opposite side t, adjacent side 1, with the angle $\frac{x}{2}$:



We mark in the hypotenuse:



Now,

$$\begin{aligned}
\sin\frac{x}{2} &= \frac{t}{\sqrt{1+t^2}} \\
\cos\frac{x}{2} &= \frac{1}{\sqrt{1+t^2}} \\
\tan\frac{x}{2} &= t.
\end{aligned}$$

Next, we use the double-angle formulas:

$$\begin{aligned}
\sin x &\equiv \sin 2\left(\frac{x}{2}\right) \\
&\equiv 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) \\
&\equiv 2 \times \frac{t}{\sqrt{1+t^2}} \times \frac{1}{\sqrt{1+t^2}} \\
&\equiv \frac{2t}{1+t^2},
\end{aligned}$$

$$
\begin{aligned}
\cos x &\equiv \cos 2\left(\frac{x}{2}\right) \\
&\equiv \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) \\
&\equiv \frac{1}{1+t^2} - \frac{t^2}{1+t^2} \\
&\equiv \frac{1-t^2}{1+t^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\tan x &\equiv \frac{\sin x}{\cos x} \\
&\equiv \frac{2t}{1+t^2} \times \frac{1+t^2}{1-t^2} \\
&\equiv \frac{2t}{1-t^2};
\end{aligned}
$$

the expressions for $\operatorname{cosec} x$, $\sec x$, and $\cot x$ follow.

Finally, what about $\mathrm{d}x$? Now,

$$
\begin{aligned}
t = \tan\left(\frac{x}{2}\right) &\Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x} = \tfrac{1}{2}\sec^2\left(\frac{x}{2}\right) \\
&\Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1+\tan^2\left(\frac{x}{2}\right)}{2} \\
&\Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1+t^2}{2} \\
&\Rightarrow \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t.
\end{aligned}
$$

We can summarise the formulas:

Expression	Replacement
$\sin x$	$\dfrac{2t}{1+t^2}$
$\cos x$	$\dfrac{1-t^2}{1+t^2}$
$\tan x$	$\dfrac{2t}{1-t^2}$
$\operatorname{cosec} x$	$\dfrac{1+t^2}{2t}$
$\sec x$	$\dfrac{1+t^2}{1-t^2}$
$\cot x$	$\dfrac{1-t^2}{2t}$
$\mathrm{d}x$	$\dfrac{2}{1+t^2}\,\mathrm{d}t$

Example 2
Find

$$\int \frac{1}{2\sin x - \cos x + 3}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}
\int \frac{1}{2\sin x - \cos x + 3}\,\mathrm{d}x &= \int \frac{1}{2\left(\frac{2t}{1+t^2}\right) - \left(\frac{1-t^2}{1+t^2}\right) + 3} \cdot \frac{2}{1+t^2}\,\mathrm{d}t \\
&= \int \frac{2}{2(2t) - (1-t^2) + 3(1+t^2)}\,\mathrm{d}t \\
&= \int \frac{2}{4t^2 + 4t + 2}\,\mathrm{d}t \\
&= \int \frac{2}{(2t+1)^2 + 1}\,\mathrm{d}t \\
&= \arctan(2t+1) + c \\
&= \underline{\underline{\arctan\left[1 + 2\tan\left(\frac{x}{2}\right)\right] + c.}}
\end{aligned}$$

3 Examples

Here are some examples for you to try.

1. (a) Using $t = \tan\left(\frac{x}{2}\right)$, find

$$\int \operatorname{cosec} x \, \mathrm{d}x.$$

Solution

$$\begin{aligned}
\int \operatorname{cosec} x \, \mathrm{d}x &= \int \frac{1+t^2}{2t} \cdot \frac{2}{1+t^2} \, \mathrm{d}t \\
&= \int \frac{1}{t} \, \mathrm{d}t \\
&= \ln|t| + c \\
&= \underline{\underline{\ln\left|\tan\left(\frac{x}{2}\right)\right| + c.}}
\end{aligned}$$

(b) Using $t = \tan x$, find

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \operatorname{cosec} 2x \, \mathrm{d}x.$$

Solution

$$\begin{aligned}
\int \operatorname{cosec} 2x \, \mathrm{d}x &= \int \frac{1}{\sin 2x} \, \mathrm{d}x \\
&= \int \frac{1}{2 \sin x \cos x} \, \mathrm{d}x \\
&= \int \frac{1}{2 \cdot \frac{t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}}} \cdot \frac{1}{1+t^2} \, \mathrm{d}t \\
&= \int \frac{1}{\frac{2t}{1+t^2}} \cdot \frac{1}{1+t^2} \, \mathrm{d}t \\
&= \int \frac{1}{2t} \, \mathrm{d}t \\
&= \tfrac{1}{2} \ln|t| + c \\
&= \tfrac{1}{2} \ln|\tan x| + c
\end{aligned}$$

and

$$\begin{aligned}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \operatorname{cosec} 2x \,\mathrm{d}x &= \tfrac{1}{2}\left[\ln|\tan x|\right]_{t=\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= \tfrac{1}{2}(\ln\sqrt{3} - \ln 1) \\ &= \tfrac{1}{2}\ln 3^{\frac{1}{2}} \\ &= \underline{\underline{\tfrac{1}{4}\ln 3}}.\end{aligned}$$

2. Using $t = \tan\left(\dfrac{x}{2}\right)$, find

$$\int \frac{1}{1+\sin x}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int \frac{1}{1+\sin x}\,\mathrm{d}x &= \int \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t \\ &= \int \frac{2}{(1+t^2)+2t}\,\mathrm{d}t \\ &= \int \frac{2}{(t+1)^2}\,\mathrm{d}t \\ &= -\frac{2}{t+1} + c \\ &= \underline{\underline{-\frac{2}{\tan\left(\frac{x}{2}\right)+1} + c}}.\end{aligned}$$

3. Using $t = \tan\left(\dfrac{x}{2}\right)$, find

$$\int \frac{1}{1+\sin x+\cos x}\,\mathrm{d}x.$$

Solution

$$
\begin{aligned}
\int \frac{1}{1+\sin x+\cos x}\,\mathrm{d}x &= \int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,\mathrm{d}t\\
&= \int \frac{2}{(1+t^2)+2t+(1-t^2)}\,\mathrm{d}t\\
&= \int \frac{2}{2+2t}\,\mathrm{d}t\\
&= \int \frac{1}{1+t}\,\mathrm{d}t\\
&= \ln|1+t|+c\\
&= \underline{\underline{\ln\left|1+\tan\left(\frac{x}{2}\right)\right|+c.}}
\end{aligned}
$$

4. Using $t=\tan\left(\frac{x}{2}\right)$, find

$$\int \frac{1}{\sin x+\tan x}\,\mathrm{d}x.$$

Solution

$$
\begin{aligned}
\int \frac{1}{\sin x+\tan x}\,\mathrm{d}x &= \int \frac{1}{\frac{2t}{1+t^2}+\frac{2t}{1-t^2}}\cdot\frac{2}{1+t^2}\,\mathrm{d}t\\
&= \int \frac{1}{\frac{2t(1-t^2)+2t(1+t^2)}{(1+t^2)(1-t^2)}}\cdot\frac{2}{1+t^2}\,\mathrm{d}t\\
&= \int \frac{(1+t^2)(1-t^2)}{2t(1-t^2)+2t(1+t^2)}\cdot\frac{2}{1+t^2}\,\mathrm{d}t\\
&= \int \frac{2(1-t^2)}{4t}\,\mathrm{d}t\\
&= \int \left(\frac{1}{2t}-\tfrac{1}{2}t\right)\mathrm{d}t\\
&= \tfrac{1}{2}\ln|t|-\tfrac{1}{2}t^2+c\\
&= \underline{\underline{\tfrac{1}{2}\ln\left|\tan\left(\frac{x}{2}\right)\right|-\tfrac{1}{2}\tan^2\left(\frac{x}{2}\right)+c.}}
\end{aligned}
$$

5. Using $t = \tan\left(\frac{x}{2}\right)$, find

$$\int \sec x \, \mathrm{d}x.$$

Solution

$$\begin{aligned}
\int \sec x \, \mathrm{d}x &= \int \frac{1+t^2}{1-t^2} \cdot \frac{2}{1+t^2} \, \mathrm{d}t \\
&= \int \frac{2}{1-t^2} \, \mathrm{d}t \\
&= \int \frac{2}{(1-t)(1+u)} \, \mathrm{d}t \\
&= \int \left(\frac{1}{1-t} + \frac{1}{1+u} \right) \, \mathrm{d}t \\
&= -\ln|1-t| + \ln|1+t| + c \\
&= \ln\left|\frac{1+t}{1-t}\right| + c \\
&= \underline{\underline{\ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right| + c}}.
\end{aligned}$$

Of course,

$$\begin{aligned}
\ln\left|\frac{1+t}{1-t}\right| + c &= \ln\left|\frac{1+t}{1-t} \cdot \frac{1+t}{1+t}\right| + c \\
&= \ln\left|\frac{(1+t)^2}{(1-t)(1+t)}\right| + c \\
&= \ln\left|\frac{1+2t+t^2}{1-t^2}\right| + c \\
&= \ln\left|\frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2}\right| + c \\
&= \underline{\underline{\ln|\sec x + \tan| + c}}.
\end{aligned}$$

6. Using $t = \tan\left(\frac{x}{2}\right)$, find

$$\int \frac{1}{1+\sec x} \, \mathrm{d}x.$$

Solution

$$\begin{aligned}
\int \frac{1}{1+\sec x}\,\mathrm{d}x &= \int \frac{1}{1+\frac{1+t^2}{1-t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t \\
&= \int \frac{1}{\frac{(1-t^2)+(1+t^2)}{1-t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t \\
&= \int \frac{1-t^2}{(1-t^2)+(1+t^2)} \cdot \frac{2}{1+t^2}\,\mathrm{d}t \\
&= \int \frac{1-t^2}{2} \cdot \frac{2}{1+t^2}\,\mathrm{d}t \\
&= \int \frac{1-t^2}{1+t^2}\,\mathrm{d}t \\
&= \int \frac{2-(1+t^2)}{1+t^2}\,\mathrm{d}t \\
&= \int \left(\frac{2}{1+t^2} - 1\right)\,\mathrm{d}t \\
&= 2\tan^{-1} t - t + c \\
&= 2\tan^{-1}\left(\tan\left(\frac{x}{2}\right)\right) - \tan\left(\frac{x}{2}\right) + c \\
&= 2 \cdot \frac{x}{2} - \tan\left(\frac{x}{2}\right) + c \\
&= \underline{\underline{x - \tan\left(\frac{x}{2}\right) + c}}.
\end{aligned}$$

7. (a) Given that

$$\frac{1}{(1+t)(1+t^2)} \equiv \frac{A}{1+t} + \frac{Bt+C}{1+t^2},$$

find the values of the constants A, B, and C.

Solution

$$\begin{aligned}
\frac{1}{(1+t)(1+t^2)} &\equiv \frac{A}{1+t} + \frac{Bt+C}{1+t^2} \\
&\equiv \frac{A(1+t^2)+(Bt+C)(1+t)}{(1+t)(1+t^2)}
\end{aligned}$$

and

$$1 \equiv A(1+t^2) + (Bt+C)(1+t).$$

$\underline{t=-1}$: $1 = 2A \Rightarrow A = \frac{1}{2}$.
$\underline{t=0}$: $1 = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$.
$\underline{t=1}$: $1 = 1 + 2B + 1 \Rightarrow B = -\frac{1}{2}$.
Hence,

$$\frac{1}{(1+t)(1+t^2)} \equiv \underline{\underline{\frac{\frac{1}{2}}{1+t} + \frac{-\frac{1}{2}t + \frac{1}{2}}{1+t^2}}}.$$

(b) Hence, using $t = \tan x$, find

$$\int \frac{1}{1+\tan x}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}
\int \frac{1}{1+\tan x}\,\mathrm{d}x &= \int \frac{1}{1+t} \cdot \frac{1}{1+t^2}\,\mathrm{d}t \\
&= \int \frac{1}{(1+t)(1+t^2)}\,\mathrm{d}t \\
&= \int \left(\frac{\frac{1}{2}}{1+t} + \frac{-\frac{1}{2}t + \frac{1}{2}}{1+t^2} \right)\,\mathrm{d}t \\
&= \tfrac{1}{2} \int \left(\frac{1}{1+t} \right)\,\mathrm{d}t + \tfrac{1}{2} \int \left(\frac{-t+1}{1+t^2} \right)\,\mathrm{d}t \\
&= \tfrac{1}{2} \int \left(\frac{1}{1+t} \right)\,\mathrm{d}t - \tfrac{1}{4} \int \left(\frac{2t}{1+t^2} \right)\,\mathrm{d}t + \tfrac{1}{2} \int \left(\frac{1}{1+t^2} \right)\,\mathrm{d}t \\
&= \tfrac{1}{2} \ln|1+t| - \tfrac{1}{4} \ln|1+t^2| + \tfrac{1}{2} \tan^{-1} t + c \\
&= \tfrac{1}{2} \ln|1+\tan x| - \tfrac{1}{4} \ln|1+\tan^2 x| + \tfrac{1}{2} \tan^{-1}(\tan x) + c \\
&= \underline{\underline{\tfrac{1}{2} \ln|1+\tan x| - \tfrac{1}{4} \ln|1+\tan^2 x| + \tfrac{1}{2} x + c}}.
\end{aligned}$$

8. (a) Given that

$$\frac{2}{3t^2 - 10t + 3} \equiv \frac{A}{3t-1} + \frac{B}{t-3},$$

find the values of the constants A and B.

Solution

$$\begin{aligned}\frac{2}{3t^2-10t+3} &\equiv \frac{A}{3t-1}+\frac{B}{t-3}\\ &\equiv \frac{A(t-3)+B(3t-1)}{(3t-1)(t-3)}\end{aligned}$$

and so

$$2 \equiv A(t-3)+B(3t-1).$$

$\underline{t=\frac{1}{3}}$: $2=-\frac{8}{3}A \Rightarrow A=-\frac{3}{4}$.
$\underline{t=3}$: $2=8B \Rightarrow B=\frac{1}{4}$.
Hence,

$$\frac{2}{3t^2-10t+3} \equiv \underline{\underline{\frac{-\frac{3}{4}}{3t-1}+\frac{\frac{1}{4}}{t-3}}}.$$

(b) Hence, using $t=\tan\left(\frac{x}{2}\right)$, find

$$\int \frac{1}{3-5\sin x}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int \frac{1}{3-5\sin x}\,\mathrm{d}x &= \int \frac{1}{3-5\left(\frac{2t}{1+t^2}\right)}\cdot\frac{2}{1+t^2}\,\mathrm{d}t\\ &= \int \frac{2}{3(1+t^2)-5(2t)}\,\mathrm{d}t\\ &= \int \frac{2}{3t^2-10t+3}\,\mathrm{d}t\\ &= \int \frac{2}{3t^2-10t+3}\,\mathrm{d}t\\ &= \int \frac{2}{(3t-1)(t-3)}\,\mathrm{d}t\\ &= \int \left(\frac{-\frac{3}{4}}{3t-1}+\frac{\frac{1}{4}}{t-3}\right)\mathrm{d}t\\ &= -\tfrac{1}{4}\ln|3t-1|+\tfrac{1}{4}\ln|t-3|+c\\ &= \underline{\underline{-\tfrac{1}{4}\ln\left|3\tan\left(\frac{x}{2}\right)-1\right|+\tfrac{1}{4}\ln\left|\tan\left(\frac{x}{2}\right)-3\right|+c}},\end{aligned}$$

or equivalent.

9. Find

$$\int \frac{1}{4 + 12\cos^2 x}\,\mathrm{d}x.$$

Solution

We will go for $t = \tan x$:

$$\begin{aligned}
\int \frac{1}{4 + 12\cos^2 x}\,\mathrm{d}x &= \int \frac{1}{4 + 12\left(\frac{1}{\sqrt{1+t^2}}\right)^2} \cdot \frac{1}{1+t^2}\,\mathrm{d}t \\
&= \int \frac{1}{4 + \frac{12}{1+t^2}} \cdot \frac{1}{1+t^2}\,\mathrm{d}t \\
&= \int \frac{1}{4(1+t^2) + 12}\,\mathrm{d}t \\
&= \int \frac{1}{16 + 4t^2}\,\mathrm{d}t \\
&= \tfrac{1}{4}\int \frac{1}{2^2 + t^2}\,\mathrm{d}t \\
&= \tfrac{1}{4}\tan^{-1}(\tfrac{1}{2}t) + c \\
&= \underline{\underline{\tfrac{1}{4}\tan^{-1}(\tfrac{1}{2}\tan x) + c}}.
\end{aligned}$$

10. Find

$$\int \frac{1}{5 + 3\cos x}\,\mathrm{d}x.$$

Solution

We will go for $t = \tan\left(\dfrac{x}{2}\right)$:

$$\begin{aligned}
\int \frac{1}{5+3\cos x}\,\mathrm{d}x &= \int \frac{1}{5+3\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\,\mathrm{d}t \\
&= \int \frac{2}{5(1+t^2)+3(1-t^2)}\,\mathrm{d}t \\
&= \int \frac{2}{8+2t^2}\,\mathrm{d}t \\
&= \int \frac{1}{2^2+t^2}\,\mathrm{d}t \\
&= \tfrac{1}{2}\tan^{-1}(\tfrac{1}{2}t) + c \\
&= \underline{\underline{\tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}\tan\left(\frac{x}{2}\right)\right) + c.}}
\end{aligned}$$

11. Find

$$\int \frac{1}{5-3\cos x}\,\mathrm{d}x.$$

Solution

We will go for $t = \tan\left(\dfrac{x}{2}\right)$:

$$\begin{aligned}
\int \frac{1}{5-3\cos x}\,\mathrm{d}x &= \int \frac{1}{5-3\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\,\mathrm{d}t \\
&= \int \frac{2}{5(1+t^2)-3(1-t^2)}\,\mathrm{d}t \\
&= \int \frac{2}{2+8t^2}\,\mathrm{d}t \\
&= \int \frac{1}{1+4t^2}\,\mathrm{d}t \\
&= \tfrac{1}{4}\int \frac{1}{(\frac{1}{2})^2+t^2}\,\mathrm{d}t \\
&= \tfrac{1}{4}\cdot 2\tan^{-1}(2t) + c \\
&= \underline{\underline{\tfrac{1}{2}\tan^{-1}\left(2\tan\left(\frac{x}{2}\right)\right) + c.}}
\end{aligned}$$

12. Find

$$\int \frac{1}{2+\cos x}\,\mathrm{d}x.$$

Solution

We will go for $t = \tan\left(\frac{x}{2}\right)$:

$$\begin{aligned}
\int \frac{1}{2+\cos x}\,\mathrm{d}x &= \int \frac{1}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t \\
&= \int \frac{2}{2(1+t^2)+(1-t^2)}\,\mathrm{d}t \\
&= \int \frac{2}{3+t^2}\,\mathrm{d}t \\
&= 2\int \frac{1}{(\sqrt{3})^2+t^2}\,\mathrm{d}t \\
&= \tfrac{2\sqrt{3}}{3}\tan^{-1}(\tfrac{\sqrt{3}}{3}t)+c \\
&= \underline{\underline{\tfrac{2\sqrt{3}}{3}\tan^{-1}\left(\tfrac{\sqrt{3}}{3}\tan\left(\frac{x}{2}\right)\right)+c.}}
\end{aligned}$$

13. Find, to 3 decimal places,

$$\int_0^{\frac{\pi}{4}} \frac{1}{5\cos^2 x+3\cos^2 x}\,\mathrm{d}x.$$

Solution

We will go for $t = \tan x$:

$$\begin{aligned}
\int \frac{1}{5\cos^2 x + 3\sin^2 x}\,\mathrm{d}x &= \int \frac{1}{5\left(\frac{1}{\sqrt{1+t^2}}\right)^2 + 3\left(\frac{t}{\sqrt{1+t^2}}\right)^2} \cdot \frac{1}{1+t^2}\,\mathrm{d}t \\
&= \int \frac{1}{5+3t^2}\mathrm{d}t \\
&= \tfrac{1}{3}\int \frac{1}{\left(\sqrt{\frac{5}{3}}\right)^2 + t^2}\mathrm{d}t \\
&= \tfrac{1}{3} \cdot \sqrt{\tfrac{3}{5}}\tan^{-1}\left(\sqrt{\tfrac{3}{5}}t\right) + c \\
&= \tfrac{\sqrt{15}}{15}\tan^{-1}\left(\sqrt{\tfrac{3}{5}}\tan x\right) + c
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{1}{5\cos^2 x + 3\cos^2 x}\,\mathrm{d}x &= \left[\tfrac{\sqrt{15}}{15}\tan^{-1}\left(\sqrt{\tfrac{3}{5}}\tan x\right)\right]_{x=0}^{\frac{\pi}{4}} \\
&= 0.170\,168\,053\,1 - 0 \text{ (FCD)} \\
&= \underline{\underline{0.170 \text{ (3 dp)}}}.
\end{aligned}$$

14. (a) Given that

$$\frac{2}{(5-t)(1+5t)} \equiv \frac{A}{5-t} + \frac{B}{1+5t},$$

find the values of the constants A and B.

Solution

$$\begin{aligned}
\frac{2}{(5-t)(1+5t)} &\equiv \frac{A}{5-t} + \frac{B}{1+5t} \\
&\equiv \frac{A(1+5t) + B(5-t)}{(5-t)(1+5t)}
\end{aligned}$$

and so

$$2 \equiv A(1+5t) + B(5-t).$$

$\underline{t=5}$: $2 = 26A \Rightarrow A = \frac{1}{13}$.
$\underline{t=-\frac{1}{5}}$: $2 = \frac{26}{5}B \Rightarrow B = \frac{5}{13}$.

Hence,

$$\frac{2}{(5-t)(5+t)} \equiv \underline{\underline{\frac{\frac{1}{13}}{5-t} + \frac{\frac{5}{13}}{1+5t}}}.$$

(b) Find

$$\int \frac{1}{12\sin x + 5\cos x}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}
\int \frac{1}{12\sin x + 5\cos x}\,\mathrm{d}x &= \int \frac{1}{12\left(\frac{2t}{1+t^2}\right) + 5\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\,\mathrm{d}t \\
&= \int \frac{2}{24t + 5(1-t^2)}\,\mathrm{d}t \\
&= \int \frac{2}{5 + 24t - 5t^2}\,\mathrm{d}t \\
&= \int \frac{2}{(5-t)(1+5t)}\,\mathrm{d}t \\
&= \int \left(\frac{\frac{1}{13}}{5-t} + \frac{\frac{5}{13}}{1+5t}\right)\,\mathrm{d}t \\
&= \tfrac{1}{13}\left[-\ln|5-t| + 5\ln|1+5t|\right] + c \\
&= \underline{\underline{\tfrac{1}{13}\left[-\ln\left|5 - \tan\left(\frac{x}{2}\right)\right| + 5\ln\left|1 + 5\tan\left(\frac{x}{2}\right)\right|\right] + c.}}
\end{aligned}$$

(c) Find, to 3 decimal places,

$$\int_3^{3.1} \frac{1}{12\sin x + 5\cos x}\,\mathrm{d}x.$$

Solution

$$
\begin{aligned}
& \int_3^{3.1} \frac{1}{12\sin x + 5\cos x}\,\mathrm{d}x \\
= \quad & \tfrac{1}{13}\left[-\ln\left|5 - \tan\left(\frac{x}{2}\right)\right| + 5\ln\left|1 + 5\tan\left(\frac{x}{2}\right)\right|\right]_{x=3}^{3.1} \\
= \quad & \tfrac{1}{13}\left[(-3.763\ldots + 27.432\ldots) - (-2.208\ldots + 21.348\ldots)\right] \\
& = 0.348\,349\,422\,1 \text{ (FCD)} \\
& = \underline{\underline{0.348 \text{ (3 dp)}}}.
\end{aligned}
$$