Dr Oliver Mathematics
Further Mathematics
Conic Sections: Rectangular Hyperbolas
Past Examination Questions

This booklet consists of 21 questions across a variety of examination topics.
The total number of marks available is 205.

1. The rectangular hyperbola C has equation $xy = c^2$ where c is a positive constant.

   (a) Show that an equation of the tangent to C at the point $P\left(cp, \dfrac{c}{p}\right)$ is (4)

   $$x + yp^2 = 2cp.$$

   The tangent to C at P meets the x-axis at the point X. The point Q on C has coordinates $\left(cq, \dfrac{c}{q}\right)$, $p \neq q$, such that QX is parallel to the y-axis.

   (b) Show that $q = 2p$. (3)

   M is the midpoint of PQ.

   (c) Find, in Cartesian form, an equation of the locus of M as p varies. (5)

2. The linr $y = mx + c$ is a tangent to the rectangular hyperbola with equation $xy = -9$.

   (a) Show that $c = \pm 6\sqrt{m}$. (4)

   (b) Hence, or otherwise, find the equations of the tangents from the point $(4, -2)$ to the rectangular hyperbola $xy = -9$. (5)

3. A hyperbola C has equations

   $$x = ct, y = \frac{c}{t}, t \neq 0,$$

   where c is a positive constant and t is a parameter.

   (a) Show that an equation of the normal to C at the point where $t = p$ is given by (6)

   $$py + cp^4 = p^3x + c.$$

   (b) Verify that this normal meets C again at the point at which $t = q$, where (3)

   $$qp^3 + 1 = 0.$$

4. The rectangular hyperbola C has equation $xy = c^2$ where c is a positive constant.

(a) Show that the tangent to C at the point $P(cp, \frac{c}{p})$, $p \neq 0$, has equation (3)

$$p^2 y = -x + 2cp.$$

The point $Q(cq, \frac{c}{q})$, $q \neq 0$, $q \neq p$, also lies on C. The tangents to C at P and Q meet at N. Given that $p + q \neq 0$,

(b) show that the y-coordinate of N is $\frac{2c}{p+q}$. (3)

The line joining N to the origin O is perpendicular to the chord PQ.

(c) Find the numerical value of p^2q^2. (6)

5. The parametric equations of a hyperbola are

$$x = \frac{3}{2}\left(t + \frac{1}{t}\right), y = \frac{5}{2}\left(t - \frac{1}{t}\right) .t \neq 0.$$

(a) Find a cartesian equation of the hyperbola. (5)

(b) Sketch the hyperbola, stating the coordinates of an points of intersection with the coordinate axes. (2)

6. The point $P\left(2p, \dfrac{2}{p}\right)$ and the point $Q\left(2q, \dfrac{2}{q}\right)$, where $p \neq q$, lie on the rectangular hyperbola with equation $xy = 4$. The tangents to the curve at the points P and Q meets at the point R.

(a) Show that at the point R, (8)

$$x = \frac{4pq}{p+q} \text{ and } y = \frac{4}{p+q}.$$

As p and q vary, the locus of R has equation $xy = 3$.

(b) Find the relationship between p and q in the form $q = \mathrm{f}(x)$. (5)

7. (a) Show that the normal to the rectangular hyperbola $xy = c^2$, at the point $P(ct, \frac{c}{t})$, $t \neq 0$, is (5)

$$y = t^2x + \frac{c}{t} - ct^3.$$

The normal to the hyperbola at P meets the hyperbola again at Q.

(b) Find, in terms of t, the coordinates of the point Q. (5)

Given that the mid-point of PQ is (X, Y) and that $t \neq \pm 1$,

(c) show that $\dfrac{X}{Y} = -\dfrac{1}{t^2}$, (2)

(d) show that, at t varies, the locus of the mid-point of PQ is given by the equation (2)

$$4xy + c^2\left(\frac{y}{x} - \frac{x}{y}\right)^2 = 0.$$

8. The rectangular hyperbola, H, has parametric equations $x = 5t, y = \frac{5}{t}, t \neq 0$.

   (a) Write the cartesian equation of H in the form $xy = c^2$. (1)

   Points A and B on the hyperbola have parameters $t = 1$ and $t = 5$ respectively.

   (b) Find the coordinates of the mid-point of AB. (3)

9. The rectangular hyperbola H has equation $xy = c^2$, where c is a constant. The point $P\left(ct, \frac{c}{t}\right)$ is a general point on H.

   (a) Show that the tangent to H at the point P has equation (4)

$$t^2y + x = 2ct.$$

   The tangents to H at the points A and B meets at the point $(15c, -c)$.

   (b) Find, in terms of c, the coordinates of A and B. (5)

10. The rectangular hyperbola H has equation $xy = c^2$, where c is a positive constant. The point A on H has x-coordinate $3c$.

   (a) Write down the y-coordinate of A. (1)

   (b) Show that an equation of the normal to H at A is (5)

$$3y = 27x - 80c.$$

   The normal to H at A meets H again at the point B.

   (c) Find, in terms of c, the coordinates of B. (5)

11. The point $P\left(6t, \frac{6}{t}\right)$, $t \neq 0$, lies on the rectangular hyperbola H has equation $xy = 36$.

   (a) Show that an equation of the tangent to H at P is (5)

$$y = -\frac{1}{t^2}x + \frac{12}{t}.$$

   The tangent to H at the point A and the tangent to H at the point B meet the point $(-9, 12)$.

(b) Find the coordinates of A and B. (7)

12. The rectangular hyperbola H has cartesian equation $xy = 9$. The points $P\left(3p, \frac{3}{p}\right)$ and $Q\left(3q, \frac{3}{q}\right)$ lie on H, where $p = \pm\, q$.

(a) Show that the equation of the tangent at P is (4)

$$x + p^2 y = 6p.$$

(b) Write down the equation of the tangent at Q. (1)

The tangent at the point P and the tangent at the point Q meet the point R.

(c) Find, as single fractions in their simplest form, the coordinates of R in terms of p and q. (4)

13. The rectangular hyperbola H has equation $xy = c^2$, where c is a positive constant. The point $P\left(ct, \frac{c}{t}\right)$, $t \neq 0$, is a general point on H.

(a) Show that an equation of the tangent to H at P is (4)

$$t^2 y + x = 2ct.$$

The tangent to H at the point P meets the x-axis at the point A and the y-axis at the point B. Given that the area of the triangle OAB, where O is the origin, is 36,

(b) find the exact value of c, expressing your answer in the form $k\sqrt{2}$, where k is an integer. (4)

14. The rectangular hyperbola H has cartesian equation $xy = 9$. The point $P\left(3p, \frac{3}{p}\right)$, and $Q\left(3q, \frac{3}{q}\right)$, where $p \neq 0$, $q \neq 0$, $p \neq q$, are points on the rectangular hyperbola H.

(a) Show that an equation of the tangent to H at the point P is (4)

$$p^2 y + x = 10p.$$

(b) Write down the equation of the tangent at Q. (1)

The tangents at P and Q meet at the point N. Given that $p + q \neq 0$,

(c) show that the point N has coordinates $\left(\frac{10pq}{p+q}, \frac{10}{p+q}\right)$. (4)

The line joining N to the origin is perpendicular to the line PQ.

(d) Find the value of p^2q^2. (5)

15. The rectangular hyperbola H has Cartesian equation $xy = 4$. The point $P\left(2t, \dfrac{2}{t}\right)$ lies on H, where $t \neq 0$.

(a) Show that an equation of the normal to H at the point P is (5)

$$ty - t^3x = 2 - 2t^4.$$

The normal to H at the point where $t = -\frac{1}{2}$ meets H again at the point Q.

(b) Find the coordinates of the point Q. (4)

16. Figure 1 shows a rectangular hyperbola H with parametric equations

$$x = 3t, y = \frac{3}{t}, t \neq 0.$$

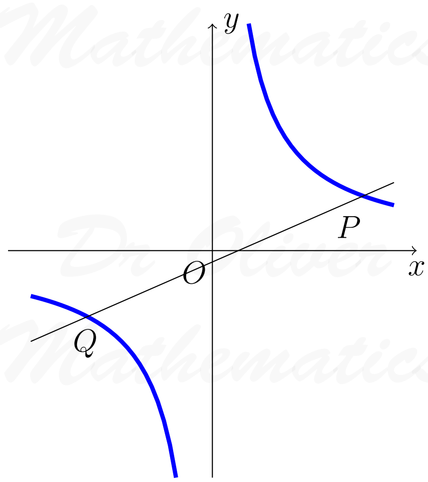

Figure 1: $x = 3t, y = \frac{3}{t}, t \neq 0$

The line L with equation $6y = 4x - 15$ intersects H at the point P and at the point Q, as shown in Figure 1.

(a) Show that L intersects H where $4t^2 - 5t - 6 = 0$. (3)

(b) Hence, or otherwise, find the coordinates of points P and Q. (5)

17. The rectangular hyperbola H has Cartesian equation $xy = c^2$. The point $P\left(2t, \dfrac{2}{t}\right)$, $t > 0$, is a general point on H.

(a) Show that an equation of the tangent to H at the point P is (4)

$$t^2y + x = 2ct.$$

An equation of the normal to H at the point P is

$$t^3x - ty = ct^4 - c.$$

Given that the normal to H at P meets the x-axis at the point A and the tangent to H at P meets the x-axis at the point B,

(b) find, in terms of c and t, the coordinates of A and the coordinates of B. (2)

Given that $c = 4$,

(c) find, in terms of t, the area of the triangle APB. Give your answer in its simplest form. (3)

18. The rectangular hyperbola H has Cartesian equation $xy = c^2$, where c is a positive constant. (7)
The point $P\left(ct, \dfrac{c}{t}\right)$, $t \neq 0$, is a general point on H.
An equation of the tangent to H at P is

$$y = -\frac{1}{t^2}x + \frac{2c}{t}.$$

The points A and B lie on H.
The tangent to H at A and the tangent to H at B meet at the point $\left(-\dfrac{6c}{7}, \dfrac{12c}{7}\right)$.
Find, in terms of c, the coordinates of A and the coordinates of B.

19. The rectangular hyperbola H has equation $xy = 9$. The point A on H has coordinates $(6, \frac{3}{2})$.

(a) Show that the normal to H at the point A has equation (5)

$$2y - 8x + 45 = 0.$$

The normal at A meets H again at the point B.

(b) Find the coordinates of B. (4)

20. The rectangular hyperbola, H, has cartesian equation $xy = 25$.

(a) Show that an equation of the normal to H at the point $P(5p, \frac{5}{p})$, $p \neq 0$, is (5)

$$y - p^2x = \frac{5}{p} - 5p^3.$$

This normal meets the line with equation $y = -x$ at the A.

(b) Show that the coordinates of A are (5)

$$\left(-\frac{5}{p} + 5p, \frac{5}{p} - 5p\right).$$

The point M is the midpoint of the line segment AP. Given that M lies on the positive x-axis,

(c) find exact value of the x-coordinate of point M. (3)

21. The rectangular hyperbola H has parametric equations

$$x = 4t,\ y = \frac{4}{t},\ t \neq 0.$$

The points P and Q on this hyperbola have parameters $t = \frac{1}{4}$ and $t = 2$ respectively. The line l passes through the origin O and is perpendicular to the line PQ.

(a) Find an equation for l. (3)

(b) Find a cartesian equation for H. (1)

(c) Find the exact coordinates of the two points where l intersects H. Give your answers in their simplest form. (3)