Dr Oliver Mathematics
Further Mathematics
Conic Sections: Parabolas
Past Examination Questions

This booklet consists of 26 questions across a variety of examination topics.
The total number of marks available is 284.

1. The curve C has equation $y^2 = 4ax$, where a is a positive constant.

   (a) Show that an equation of the tangent to C the point $P(ap^2, 2ap)$, $p \neq 0$, is (4)

   $$yp = x + ap^2.$$

   The point $Q(aq^2, 2aq)$ is on C where $p \neq q$ and $q \neq 0$. The chord PQ passes through the focus of C. Show that

   (b) $pq = -1$, (5)

   (c) the tangent to C at P and the tangent to C at Q meet on the directrix of C. (4)

2. The line with equation $y = mx + c$ is a tangent to the parabola with equation $y^2 = 8x$.

   (a) Show that $mc = 2$. (5)

   The lines l_1 and l_2 are tangents to both the parabola with equation $y^2 = 4ax$ and the circle with equation $x^2 + y^2 = 2$.

   (b) Find the equations of l_1 and l_2. (9)

3. The point P lies on the parabola with equation $y^2 = 4ax$, where a is a positive constant.

   (a) Show that an equation of the tangent to the parabola at $P(ap^2, 2ap)$ is (5)

   $$py = x + ap^2.$$

   The tangents at the points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, where $p \neq 0$, $q \neq 0$, and $p \neq q$, meet at the point N.

   (b) Find the coordinates of N. (4)

   Given further that N lies on the directrix of the parabola,

   (c) write down a relationship between p and q. (2)

4. A line joins the point $A(-4a, 0)$ to the point $P(at^2, 2at)$, where a is a positive constant. As t varies the locus of the midpoint of the line AP is a parabola, C.

   (a) Find an equation of C in cartesian form. (5)

(b) Sketch C. (2)

(c) Write down the equation of the directrix of C. (1)

(d) Write down the coordinates of the focus of C. (1)

5. The curve C has equation $y^2 = 4ax$ where a is a positive constant.

(a) Show that an equation of the normal to C at the point $P(ap^2, 2ap)$, $p \neq 0$, is (6)

$$y + px = 2ap + ap^3.$$

The normal at P meets C again the point $Q(aq^2, 2aq)$.

(b) Find q in terms of p. (6)

Given that the midpoint of PQ has coordinates $\left(\frac{125}{28}a, -3a\right)$,

(c) use your answer to part (b), or otherwise, to find the value of p. (5)

6. The point $P(ap^2, 2ap)$ lies on the parabola M with equation $y^2 = 4ax$, where a is a positive constant.

(a) Show that an equation of the tangent to the parabola at M at P is (3)

$$py = x + ap^2.$$

The point $Q(16ap^2, 8ap)$ also lies on M.

(b) Write down an equation of the tangent to M at Q. (2)

The tangent at P and the tangent at Q intersect at the point V.

(c) Show that, as p varies, the locus of V is a parabola N with equation (4)

$$4y^2 = 25ax.$$

(d) Find the coordinates of the focus of N, and find an equation of the directrix of N. (2)

(e) Sketch M and N on the same diagram, labelling each of them. (2)

7. A parabola C has equation $y^2 = 4ax$, where a is a constant.

(a) Show that an equation of the normal to C at the point $P(ap^2, 2ap)$ is (4)

$$y + px = 2ap + ap^3.$$

The normals to C at the points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, $p \neq q$, meet at the point R.

(b) Find, in terms of a, p, and q, the coordinates of R. (5)

The points P and Q vary such that $pq = 3$.

(c) Find, in the form $y^2 = \mathrm{f}(x)$, an equation of the locus of R. (4)

8. The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, $p \neq q$, lies on the parabola C with equation $y^2 = 4ax$, where a is a constant.

(a) Show that an equation for the chord PQ is (3)

$$(p+q)y = 2(x + apq).$$

The normals to C at P and Q meet at the point R.

(b) Show that the coordinates of R are (7)

$$(a(p^2 + q^2 + pq + 2), -apq(p+q)).$$

Given that the points P and Q vary such that PQ always passes through the point $(5a, 0)$,

(c) find, in the form $y^2 = \mathrm{f}(x)$, an equation for the locus of R. (5)

9. The parabola C has equation $y^2 = 4ax$, where a is a positive constant. The point P has coordinates $(ap^2, 2ap)$.

(a) Show that an equation of the normal to C at the point $P(ap^2, 2ap)$ is (4)

$$y + px = 2ap + ap^3.$$

The normal to C at P meets the curve again at Q.

(b) Show that the y-coordinate of Q is $-2a\left(\dfrac{2+p^2}{p}\right)$. (5)

(c) Show that, as p varies, the least distance from P to Q is $6\sqrt{3}a$. (7)

10. A parabola C has equation $y^2 = 4ax$, where $a > 0$, and the line l has equation $y = mx + c$. Given that l is a tangent to C,

(a) show that $c = \dfrac{a}{m}$. (4)

The point P has coordinates $(4a, 5a)$.

(b) Find equations of the two tangents from P to C. (5)

The tangents from P to C meet at the point R and Q.

(c) Find the distance RQ. (5)

11. A parabola has equation $y^2 = 4ax$, $a > 0$. The point $Q(aq^2, 2aq)$ lies on the parabola.

(a) Show that an equation of the tangent to the parabola at Q is (4)

$$yq = x + aq^2.$$

This tangent meets the y-axis at the point R.

(b) Find an equation of the line l which passes through R and is perpendicular to the tangent at Q. (3)

(c) Show that l passes through the focus of the parabola. (1)

(d) Find the coordinates of the point where l meets the directrix of the parabola. (2)

12. The parabola C has equation $y^2 = 16x$.

(a) Verify that the point $P(4t^2, 8t)$ is a general point on C. (1)

(b) Write down the coordinates of the focus S of C. (1)

(c) Show that the normal to C at P has equation (5)

$$y + tx = 8t + 4t^3.$$

The normal to C at P meets the x-axis at the point N.

(d) Find the area of triangle PSN in terms of t, giving your answer in its simplest form. (4)

13. Figure 1 shows a sketch of part of the parabola with equation $y^2 = 12x$.

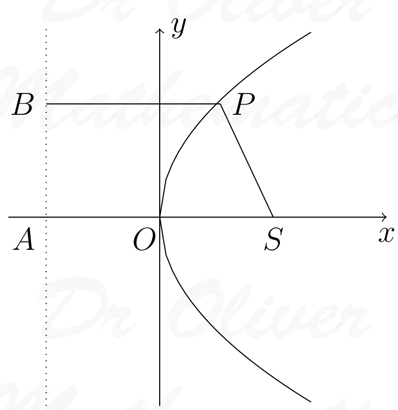

Figure 1: $y^2 = 12x$

The point P on the parabola has x-coordinate $\frac{1}{3}$. The point S is the focus of the parabola.

(a) Write down the coordinates of S. (1)

The points A and B lie on the directrix of the parabola. The point A is on the x-axis and the y-coordinate of B is positive. Given that $ABPS$ is a trapezium,

(b) calculate the perimeter of $ABPS$. (5)

14. The parabola C has equation $y^2 = 20x$.

(a) Verify that the point $P(5t^2, 10t)$ is a general point on C. (1)

The point A on C has parameter $t = 4$. The line l passes through A and also passes through the focus of C.

(b) Find the gradient of l. (4)

15. Figure 2 shows a sketch of the parabola C with equation $y^2 = 36x$.

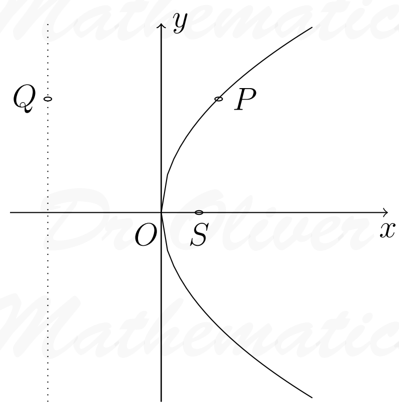

Figure 2: $y^2 = 36x$

The point S is the focus of C.

(a) Find the coordinates of S. (1)

(b) Write down the equation of the directrix of C. (1)

Figure 2 shows the point P which lies on C, where $y > 0$, and the point Q which lies on the the directrix of C. The line segment QP is parallel to the x-axis. Given that the distance QP is 25,

(c) write down the distance QP, (1)

(d) find the coordinates of P, (3)

(e) find the area of the trapezium $OSPQ$. (2)

16. The parabola C has equation $y^2 = 48x$. The point $P(12t^2, 24t)$ is a general point on C.

(a) Find an equation of the directrix of C. (2)

(b) Show that the equation of the tangent to C at $P(12t^2, 24t)$ is (4)

$$x - ty + 12t^2 = 0.$$

The tangent to C at the point $(3, 12)$ meets the directrix of C at the point X.

(c) Find the coordinates of X. (4)

17. The parabola C has equation $y^2 = 16x$. The point $P(4t^2, 8t)$ is a general point on C.

(a) Write down the coordinates of the focus F and the equation of the directrix of C. (3)

(b) Show that the equation of the normal to C is $y + tx = 8t + 4t^3$. (5)

18. Figure 3 shows a sketch of the parabola C with equation $y^2 = 8x$.

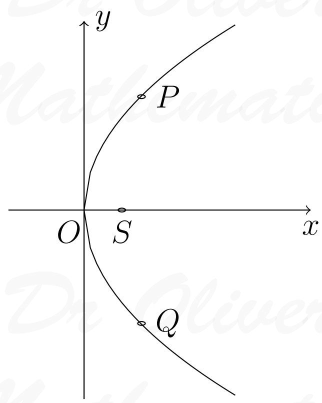

Figure 3: $y^2 = 8x$

The point P lies on C, where $y > 0$, and the point Q lies on C, where $y < 0$. The line segment PQ is parallel to the y-axis. Given that the distance PQ is 12,

(a) write down the y-coordinate of P, (1)

(b) find the x-coordinate of P. (2)

Figure 3 shows the point S which is the focus of C. The line l passes through the point P and the point S.

(c) Find an equation for l in the form $ax + by + c = 0$, where a, b, and c are integers. (4)

19. Figure 4 shows a sketch of the of the parabola with equation $y^2 = 36x$.

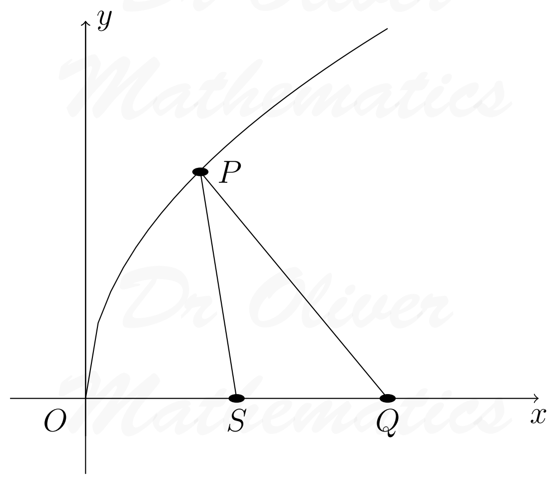

Figure 4: $y^2 = 36x$

The point $P(4, 12)$ lies on the parabola.

(a) Find an equation for the normal to the parabola at P. (5)

This normal meets the x-axis at the point N and S is the focus of the parabola, as shown in Figure 4.

(b) Find the area of triangle PST. (4)

20. A parabola has equation $y^2 = 4ax$, $a > 0$. The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ lie on C, where $p \neq 0$, $q \neq 0$, and $p \neq q$.

(a) Show that the equation of the tangent to parabola at P is (4)

$$py - x = ap^2.$$

(b) Write down the equation of the tangent at Q. (1)

The tangent at P meets the tangent at Q at the point R.

(c) Find, in terms of p and q, the coordinates of R, giving your answer in their simplest form. (4)

Given that R lies on the directrix of C,

(d) find the value of pq. (2)

21. A parabola C has equation $y^2 = 4ax$, where a is a positive constant. The point $P(at^2, 2at)$ is a general point on C.

(a) Show that the equation of the tangent to C to parabola at $P(at^2, 2at)$ is (4)

$$ty = x + at^2.$$

The tangent to C at P meets the y-axis at a point Q.

(b) Find the coordinates of Q. (1)

Given that the point S is the focus of C,

(c) show that PQ is perpendicular to SQ. (3)

22. The points $P(4k^2, 8k)$ and $Q(k^2, 4k)$, where k is a constant, lie on the parabola C with equation $y^2 = 16x$. The straight line l_1 passes through the points P and Q.

(a) Show that an equation of the line l_1 is given by (4)

$$3ky - 4x = 8k^2.$$

The line l_2 is perpendicular to the line l_1 and passes through the focus of the parabola C. The line l_2 meets the directrix of C at the point R.

(b) Find, in terms of k, the y-coordinate of the point R. (7)

23. A parabola C has cartesian equation $y^2 = 4ax$, $a > 0$. The points $P(ap^2, 2ap)$ and $P'(ap^2, -2ap)$ lie on C.

(a) Show that an equation of the normal to C at the point $P(ap^2, 2ap)$ is (5)

$$y + px = 2ap + ap^3.$$

(b) Write down an equation of the normal to C at the point P'. (1)

The normal to C at P meets the normal to P' at point Q.

(c) Find, in terms of a and p, the coordinates of Q. (2)

Given that S is the focus of the parabola,

(d) find the area of the quadrilateral $SPQP'$. (3)

24. The point $P(3p^2, 6p)$ lies on the parabola with equation $y^2 = 12x$ and the point S is the focus of this parabola.

(a) Prove that $SP = 3(1 + p^2)$. (3)

The point $Q(3q^2, 6q)$, $p \neq q$, also lies on this parabola. The tangent to the parabola at the point P and the tangent to the parabola at the point Q meet at the point R.

(b) Find the equations of these two tangents and hence find the coordinates of the point R, giving the coordinates in their simplest form. (8)

(c) Prove that $SR^2 = SP \times SQ$. (3)

25. Points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, where $p^2 \neq q^2$, lie on the parabola $y^2 = 4ax$.

(a) Show that an equation for the chord PQ is (5)

$$(p + q)y = 2(x + apq).$$

Given that this chord passes through the focus of the parabola,

(b) show that $pq = -1$. (1)

(c) Using calculus, find the gradient of the tangent to the parabola at P. (2)

(d) Show that the tangent to the parabola at P and the tangent to the parabola at Q are perpendicular. (2)

26. The parabola C has equation $y^2 = 4ax$, where a is a constant and $a > 0$. The point $Q(aq^2, 2aq)$, $q > 0$, lies on the parabola C.

(a) Show that an equation of the tangent to C at Q is (4)

$$qy = x + aq^2.$$

The tangent to C at the point Q meets the x-axis at the point $X(-\frac{1}{4}, 0)$ and meets the directrix of C at the point D.

(b) Find, in terms of a, the coordinates of D. (4)

Given that the point F is the focus of the parabola C,

(c) find the area, in terms of a, of the triangle FXD, giving your answer in its simplest form. (2)