Core Mathematics 3

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Mathematics

Algebraic Fractions

You should be able to add, subtract, multiply, and divide algebraic fractions, giving the answer in its simplest form. If you cannot, then since this is GCSE work, you should consider finding another A2 course.

Mappings

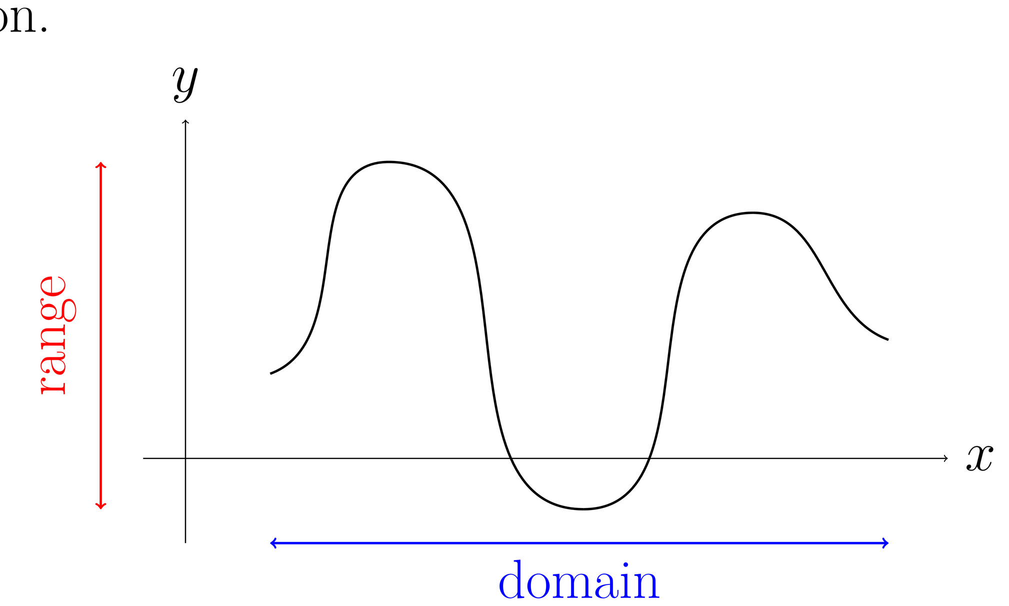
Let A and B be two sets. A *mapping* $\mathrm{f} : A \to B$ is a way of relating an element of the set A to one or more elements in the set B.

Domain and Range

Let $\mathrm{f} : A \to B$ be a mapping.

- The set A is the *domain* of the mapping.
- The set B is the *codomain* of the mapping.
- The set $\{\mathrm{f}(a) : a \in A\}$ is the *range* of the mapping.

The domain and range of a mapping are easily determined if you have the Cartesian graph of the function.



Functions

Let $\mathrm{f} : A \to B$ be a mapping. f is a *function* if every element of the domain is mapped to exactly one element of the range. The important thing is that a one-to-one function f has an inverse function, denoted by f^{-1}.

The domain of the function f is precisely the same as the range of the inverse function f^{-1} and range of the function f is precisely the same as the domain of the inverse function f^{-1}.

To get the graph of $y = \mathrm{f}^{-1}(x)$, simply reflect the graph of $y = \mathrm{f}(x)$ in the line $y = x$.

Natural logarithm

We define the natural logarithm, denoted by $\ln x$, by

$$\ln x = \int_0^x \frac{1}{t}\,\mathrm{d}t,\ x > 0.$$

This is a logarithm and therefore satisfies all of the logarithmic properties that you learned in Core Mathematics 2.

Exponential function

The number $e = 2.718\,281\,828\ldots$ is the base of the natural logarithm and the function $y = e^x$ is called the *exponential function*. The natural logarithm and exponential functions are the inverse of each other.

Iteration

A scheme of the form $x_{n+1} = \mathrm{f}(x_n)$ is called an *iterative scheme* and is used to generate a sequence of numbers x_1, x_2, x_3, … that (hopefully) gives increasingly accurate approximations to the solution to an equation.

If $\mathrm{f}(x)$ is a continuous function (this is important: you need to make the point in an examination answer that the function is continuous, at least in the region under consideration) such that $\mathrm{f}(a) < 0$ and $\mathrm{f}(b) > 0$, then there is a value $a < x < b$ such that $\mathrm{f}(x) = 0$.

In particular, to show that $x = 2.87$ (2 dp) is a solution to the equation $\mathrm{f}(x) = 0$, evaluate $\mathrm{f}(2.865)$ and $\mathrm{f}(2.875)$ and show that there has been a change in the sign of a continuous function; hence there is a solution x such that $2.865 < x < 2.875$ and so $x = 2.87$ (2 dp), as required.

Modulus function

Given a function $y = \mathrm{f}(x)$ you need to be able to sketch the graphs of both $y = |\,\mathrm{f}(x)|$ (simply reflect anything below the x-axis so that it is now above) and $y = \mathrm{f}(|x|)$ (simply throw away anything to the left of the y-axis and reflect what is in on the right in the y-axis).

Addition Formulae

$$\begin{aligned}
\sin(A \pm B) &\equiv \sin A \cos B \pm \sin B \cos A \\
\cos(A \pm B) &\equiv \cos A \cos B \mp \sin A \sin B \\
\tan(A \pm B) &\equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{aligned}$$

Double Angle Formulae

$$\begin{aligned}
\sin 2A &\equiv 2 \sin A \cos B \\
\cos 2A &\equiv \cos^2 A - \sin^2 A \\
&\equiv 2\cos^2 A - 1 \\
&\equiv 1 - 2\sin^2 A \\
\tan 2A &\equiv \frac{2\tan A}{1 - \tan^2 A}
\end{aligned}$$

Products, Sums, and Differences

$$\begin{aligned}
2\sin A \cos B &\equiv \sin(A+B) + \sin(A-B) \\
2\cos A \sin B &\equiv \sin(A+B) - \sin(A-B) \\
2\cos A \cos B &\equiv \cos(A+B) + \cos(A-B) \\
2\sin A \sin B &\equiv -\left[\cos(A+B) - \cos(A-B)\right]
\end{aligned}$$

Factor Formulae

$$\begin{aligned}
\sin P + \sin Q &\equiv 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right) \\
\sin P - \sin Q &\equiv 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right) \\
\cos P + \cos Q &\equiv 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right) \\
\cos P - \cos Q &\equiv -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)
\end{aligned}$$

$a\sin\theta + b\cos\theta$

An expression of the form $a\sin\theta + b\cos\theta$ can be written in the form $R\sin(\theta \pm \alpha)$ or $R\cos(\theta \pm \alpha)$. R is easy to find as $R = \sqrt{a^2 + b^2}$. You are advised to write out the appropriate addition formula to ensure that you correctly pair up like terms. For example,

$$\begin{aligned}
3\sin\theta + 4\cos\theta &\equiv R\sin(\theta + \alpha) \\
&\equiv R\sin\theta\cos\alpha + R\sin\alpha\cos\theta
\end{aligned}$$

gives the equations $R\cos\alpha = 3$ and $R\sin\alpha = 4$. We can then solve for α since $\tan\alpha = \frac{4}{3}$.

Product Rule

If u and v are functions of x then

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Quotient Rule

If u and v are functions of x then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}.$$

Generally speaking, it is easier to differentiate a genuine quotient, i.e., where one function of x is divided by another using the quotient rule rather than rewriting the expression as a product (although this will always work).

Differentiation: the exponential function and the natural logarithm

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(e^x) &= e^x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) &= \frac{1}{x}
\end{aligned}$$

Differentiation: trigonometry

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) &= \cos x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) &= -\cos x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) &= \sec^2 x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} x) &= -\operatorname{cosec} x \cot x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) &= \sec x \tan x \\
\frac{\mathrm{d}}{\mathrm{d}x}(\cot x) &= -\operatorname{cosec}^2 x
\end{aligned}$$

Given the derivatives of $\sin x$ and $\cos x$ you must be able to derive the other four derivatives.

Chain Rule

There are two main forms for the chain rule:

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}.$$