

**Dr Oliver Mathematics**  
**Mathematics: Advanced Higher**  
**2022 Paper 2: Calculator**  
**2 hours**

The total number of marks available is 65.

You must write down all the stages in your working.

1. Express (3)

$$\frac{3x^2 - 3x + 5}{x(x^2 + 5)}$$

in partial fractions.

2. Find the exact value of (2)

$$\int_0^3 \frac{4}{2x + 1} dx.$$

3. Use the Euclidean algorithm to find integers  $a$  and  $b$  such that (3)

$$634a + 87b = 1.$$

4. Use integration by parts to find (3)

$$\int (x + 2)(2x + 7)^{\frac{1}{2}} dx.$$

5. Matrix  $\mathbf{A}$  is given by (3)

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & k & 3 \\ k & 18 & -7 \end{pmatrix}.$$

Find the values of  $k$  so that the matrix  $\mathbf{A}$  is singular.

6. The first three terms of a sequence are defined algebraically by

$$x + 5, 3x + 2, 5x - 1,$$

where  $x \in \mathbb{N}$ .

- (a) Show that these three terms form the start of an arithmetic sequence. (2)

- (b) Find a simplified expression for the 15th term of this sequence. (2)

- (c) Given that the sum of the first 20 terms of this sequence is 1 130, find the value of  $x$ . (2)

7. The complex number

$$z = 3 + i$$

is a root of

$$z^2 - 6z + a = 0,$$

where  $a$  is a real number.

(a) State the second root of

$$z^2 - 6z + a = 0. \tag{1}$$

(b) Hence, or otherwise, find the value of  $a$ . (2)

The expression

$$z^2 - 6z + a$$

is a factor of

$$z^3 - z^2 - 20z + b,$$

where  $b$  is a real number.

(c) Find the value of  $b$ . (1)

8. (a) Differentiate

$$x \ln x - x \tag{2}$$

with respect to  $x$ .

(b) Hence find the general solution of the differential equation (4)

$$\frac{dy}{dx} + y \ln x = x^{-x}.$$

9. The matrix  $\mathbf{A}$  is given by (5)

$$\begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}.$$

Prove by induction that

$$\mathbf{A}^n = \begin{pmatrix} 3^n & 1 - 3^n \\ 0 & 1 \end{pmatrix}.$$

10. Solve the differential equation (9)

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 9\sin x + 13\cos x,$$

given that  $y = 5$  and  $\frac{dy}{dx} = 0$  when  $x = 0$ .

11. A curve defined parametrically has the following properties: (4)

- $x = \tan^{-1} 2t$ ,
- $\frac{dy}{dx} = 6t(1 + 4t^2)$ , and
- $y = 5$  when  $t = 1$ .

Find  $y$  in terms of  $t$ .

12. Let

$$z = \cos \theta + i \sin \theta.$$

(a) Use de Moivre's theorem to state an expression for  $z^4$ . (1)

(b) State and simplify the binomial expansion of  $(\cos \theta + i \sin \theta)^4$ . (3)

$$(\cos \theta + i \sin \theta)^4.$$

(c) Hence show that:

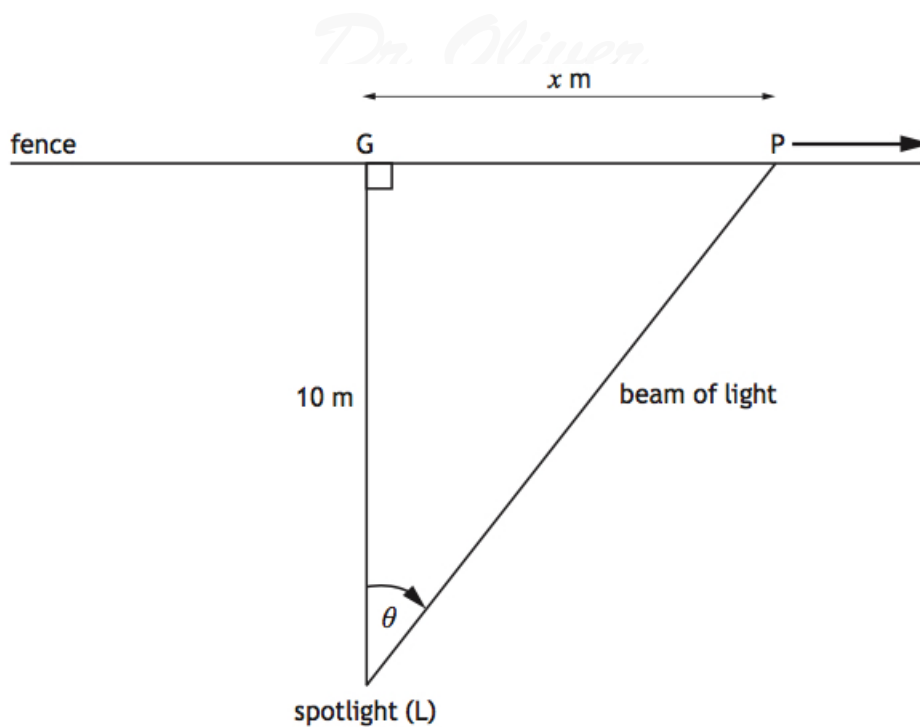
(i)  $\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1$ . (2)

(ii)  $\sin \theta \cot 4\theta$  can be written in terms of  $\cos \theta$  only. (2)

13. A security spotlight is situated 10 metres from a straight fence. The spotlight rotates at a constant speed and makes one full revolution every 12 seconds.

The situation at time  $t$  seconds is modelled in the diagram below, where:

- $L$  is the position of the spotlight,
- $G$  is the point on the fence nearest to the spotlight,
- $P$  is the position where the light hits the fence,
- $\theta$  is the angle between  $LG$  and  $LP$ , and
- $x$  is the distance in metres from  $G$  to  $P$ .



(a) Show that:

(i)  $\frac{d\theta}{dt} = \frac{1}{6}\pi$  radians per second, (1)

(ii)  $\frac{dx}{dt} = \frac{5}{3}\pi \sec^2 \theta$  metres per second. (4)

(b) Prove that (1)

$$1 + \tan^2 \theta = \sec^2 \theta.$$

(c) Hence, or otherwise, find the exact value of  $\frac{dx}{dt}$  when  $P$  is 5 metres from  $G$ . (3)

*Dr. Oliver*  
*Mathematics*

*Dr. Oliver*  
*Mathematics*