Dr Oliver Mathematics
Further Mathematics
Conic Sections: Rectangular Hyperbolas
Past Examination Questions

This booklet consists of 21 questions across a variety of examination topics.
The total number of marks available is 205.

1. The rectangular hyperbola C has equation $xy = c^2$ where c is a positive constant.

   (a) Show that an equation of the tangent to C at the point $P\left(cp, \dfrac{c}{p}\right)$ is (4)

$$x + yp^2 = 2cp.$$

Solution

$$\begin{aligned} xy = c^2 \Rightarrow y &= \frac{c^2}{x} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{c^2}{x^2}, \end{aligned}$$

and, at the $P\left(cp, \dfrac{c}{p}\right)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{c^2p^2} = -\frac{1}{p^2}.$$

Now,

$$\begin{aligned} y - \frac{c}{p} = -\frac{1}{p^2}(x - cp) &\Rightarrow yp^2 - cp = -x + cp \\ &\Rightarrow \underline{\underline{x + yp^2 = 2cp}}. \end{aligned}$$

The tangent to C at P meets the x-axis at the point X. The point Q on C has coordinates $\left(cq, \dfrac{c}{q}\right)$, $p \neq q$, such that QX is parallel to the y-axis.

   (b) Show that $q = 2p$. (3)

Solution

$X(cq, 0)$ and

$$cq + 0 = 2cp \Rightarrow \underline{\underline{q = 2p}}.$$

M is the midpoint of PQ.

(c) Find, in Cartesian form, an equation of the locus of M as p varies. (5)

Solution

Now, $q = 2p$:

$$x = \frac{cp + 2cp}{2} = \frac{3cp}{2}$$

and

$$y = \frac{\frac{c}{p} + \frac{c}{2p}}{2} = \frac{3c}{4p}.$$

Finally,

$$xy = \frac{3cp}{2} \times \frac{3c}{4p} \Rightarrow \underline{\underline{xy = \frac{9c^2}{8}}}.$$

2. The linr $y = mx + c$ is a tangent to the rectangular hyperbola with equation $xy = -9$.

(a) Show that $c = \pm 6\sqrt{m}$. (4)

Solution

$$\begin{aligned} xy = -9 &\Rightarrow x(mx + c) = -9 \\ &\Rightarrow mx^2 + cx + 9 = 0. \end{aligned}$$

Now, it is a tangent and so '$b^2 - 4ac = 0$':

$$c^2 - 4 \times m \times 9 = 0 \Rightarrow c^2 = 36m \Rightarrow \underline{\underline{c = \pm 6\sqrt{m}}}.$$

(b) Hence, or otherwise, find the equations of the tangents from the point $(4, -2)$ to the rectangular hyperbola $xy = -9$. (5)

Solution

$$
\begin{aligned}
-2 = 4m + c \Rightarrow c &= -4m - 2 \\
&\Rightarrow (-4m-2)^2 = 36m \\
&\Rightarrow 16m^2 + 16m + 4 = 36m \\
&\Rightarrow 16m^2 - 20m + 4 = 0 \\
&\Rightarrow 4m^2 - 5m + 1 = 0 \\
&\Rightarrow (4m-1)(m-1) = 0 \\
&\Rightarrow m = \tfrac{1}{4} \text{ or } m = 1.
\end{aligned}
$$

$\underline{m = \frac{1}{4}}$:

$$m = \tfrac{1}{4} \Rightarrow c = -3 \Rightarrow \underline{\underline{y = \tfrac{1}{4}x - 3}}.$$

$\underline{m = 1}$:

$$m = 1 \Rightarrow c = -6 \Rightarrow \underline{\underline{y = x - 6}}.$$

3. A hyperbola C has equations

$$x = ct, y = \frac{c}{t}, t \neq 0,$$

where c is a positive constant and t is a parameter.

(a) Show that an equation of the normal to C at the point where $t = p$ is given by (6)

$$py + cp^4 = p^3x + c.$$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{c}{t^2}, \frac{\mathrm{d}x}{\mathrm{d}t} = c, \text{ and } \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{t^2}.$$

Now, at $y = \dfrac{c}{t}$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{p^2} \Rightarrow m_T = p^2$$

and

$$
\begin{aligned}
y - \frac{c}{p} = p^2(x - cp) &\Rightarrow py - c = p^3(x - cp) \\
&\Rightarrow py - c = p^3x - cp^4 \\
&\Rightarrow \underline{\underline{py + cp^4 = p^3x + c}}.
\end{aligned}
$$

(b) Verify that this normal meets C again at the point at which $t = q$, where (3)

$$qp^3 + 1 = 0.$$

Solution

We find

$$x = cq \text{ and } y = \frac{c}{q}:$$

$$\begin{aligned} py + cp^4 = p^3x + c &\Rightarrow p\left(\frac{c}{q}\right) + cp^4 = p^3(cq) + c \\ &\Rightarrow \frac{p}{q} + p^4 = p^3q + 1 \\ &\Rightarrow p + p^4q = p^3q^2 + q \\ &\Rightarrow p - q + p^4q - p^3q^2 = 0 \\ &\Rightarrow (p - q) + p^3q(p - q) = 0 \\ &\Rightarrow (p - q)(1 + p^3q) = 0; \end{aligned}$$

if $p \neq q$, we have

$$\underline{\underline{qp^3 + 1 = 0}},$$

as required.

4. The rectangular hyperbola C has equation $xy = c^2$ where c is a positive constant.

(a) Show that the tangent to C at the point $P(cp, \frac{c}{p})$, $p \neq 0$, has equation (3)

$$p^2y = -x + 2cp.$$

Solution

The point $Q(cq, \frac{c}{q})$, $q \neq 0$, $q \neq p$, also lies on C. The tangents to C at P and Q meet at N. Given that $p + q \neq 0$,

(b) show that the y-coordinate of N is $\frac{2c}{p+q}$. (3)

Solution

The line joining N to the origin O is perpendicular to the chord PQ.

(c) Find the numerical value of p^2q^2. (6)

Solution

5. The parametric equations of a hyperbola are

$$x = \frac{3}{2}\left(t + \frac{1}{t}\right), y = \frac{5}{2}\left(t - \frac{1}{t}\right).t \neq 0.$$

(a) Find a cartesian equation of the hyperbola. (5)

Solution

(b) Sketch the hyperbola, stating the coordinates of an points of intersection with the coordinate axes. (2)

Solution

6. The point $P\left(2p, \frac{2}{p}\right)$ and the point $Q\left(2q, \frac{2}{q}\right)$, where $p \neq q$, lie on the rectangular hyperbola with equation $xy = 4$. The tangents to the curve at the points P and Q meets at the point R.

(a) Show that at the point R, (8)

$$x = \frac{4pq}{p+q} \text{ and } y = \frac{4}{p+q}.$$

Solution

$$\begin{aligned} xy = 4 \Rightarrow y &= \frac{4}{x} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{4}{x^2}, \end{aligned}$$

and, at the $P\left(2p, \frac{2}{p}\right)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{4p^2} = -\frac{1}{p^2}.$$

Now,

$$\begin{aligned} y - \frac{2}{p} = -\frac{1}{p^2}(x - 2p) \Rightarrow y - \frac{2}{p} &= -\frac{1}{p^2}x + \frac{2}{p} \\ \Rightarrow y &= -\frac{1}{p^2}x + \frac{4}{p} \end{aligned}$$

and

$$y = -\frac{1}{q^2}x + \frac{4}{q}.$$

Eliminate y:

$$\begin{aligned}
-\frac{1}{p^2}x + \frac{4}{p} = -\frac{1}{q^2}x + \frac{4}{q} &\Rightarrow \frac{1}{p^2}x - \frac{1}{q^2}x = \frac{4}{p} - \frac{4}{q} \\
&\Rightarrow \frac{q^2 - p^2}{p^2q^2}x = \frac{4(q-p)}{pq} \\
&\Rightarrow \frac{(p+q)(p-q)}{p^2q^2}x = \frac{4(q-p)}{pq} \\
&\Rightarrow \underline{\underline{x = \frac{4pq}{p+q}}} \\
&\Rightarrow y = -\frac{4pq}{p^2(p+q)} + \frac{4}{p} \\
&\Rightarrow y = \frac{4p(p+q) - 4pq}{p^2(p+q)} \\
&\Rightarrow y = \frac{4p^2}{p^2(p+q)} \\
&\Rightarrow \underline{\underline{y = \frac{4}{p+q}}}.
\end{aligned}$$

As p and q vary, the locus of R has equation $xy = 3$.

(b) Find the relationship between p and q in the form $q = \mathrm{f}(x)$. (5)

Solution

$$\begin{aligned}
xy = 3 &\Rightarrow \frac{4pq}{p+q} \times \frac{4}{p+q} = 3 \\
&\Rightarrow 16pq = 3(p+q)^2 \\
&\Rightarrow 16pq = 3p^2 + 6pq + 3q^2 \\
&\Rightarrow 3p^2 - 10pq + 3q^2 = 0 \\
&\Rightarrow (3p - q)(p - 3q) = 0 \\
&\Rightarrow \underline{\underline{q = 3p}} \text{ or } \underline{\underline{q = \frac{p}{3}}}.
\end{aligned}$$

7. (a) Show that the normal to the rectangular hyperbola $xy = c^2$, at the point $P(ct, \frac{c}{t})$, $t \neq 0$, is (5)

$$y = t^2x + \frac{c}{t} - ct^3.$$

Solution

$$y = \frac{c^2}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{x^2}.$$

At $x = ct$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{(ct)^2} = -\frac{1}{t^2}$$

and the normal to the gradient is t^2. Now,

$$\begin{aligned} y - \frac{c}{t} = t^2(x - ct) &\Rightarrow y - \frac{c}{t} = t^2x - ct^3 \\ &\Rightarrow \underline{\underline{y = t^2x + \frac{c}{t} - ct^3}}. \end{aligned}$$

The normal to the hyperbola at P meets the hyperbola again at Q.

(b) Find, in terms of t, the coordinates of the point Q. (5)

Solution

Substitute the equation of the hyperbola into the line (or vice versa):

$$\begin{aligned} xy = c^2 &\Rightarrow x(t^2x + \frac{c}{t} - ct^3) = c^2 \\ &\Rightarrow t^2x^2 + \left(\frac{c}{t} - ct^3\right)x = c^2 \\ &\Rightarrow t^2x^2 + \left(\frac{c}{t} - ct^3\right)x - c^2 = 0 \\ &\Rightarrow (x - ct)\left(t^2x + \frac{c}{t}\right) = 0 \\ &\Rightarrow x = ct \text{ or } x = -\frac{c}{t^3}; \end{aligned}$$

it is $\underline{\underline{Q(-\frac{c}{t^3}, -ct^3)}}$.

Given that the mid-point of PQ is (X, Y) and that $t \neq \pm 1$,

(c) show that $\dfrac{X}{Y} = -\dfrac{1}{t^2}$, (2)

Solution

Well,

$$X = \frac{ct - \frac{c}{t^3}}{2} = \frac{ct^4 - c}{2t^3},$$

$$Y = \frac{-ct^3 + \frac{c}{t}}{2} = \frac{c - ct^4}{2t} = -\frac{(ct^4 - c)}{2t},$$

and hence

$$\frac{X}{Y} = \underline{\underline{-\frac{1}{t^2}}}.$$

(d) show that, at t varies, the locus of the mid-point of PQ is given by the equation (2)

$$4xy + c^2\left(\frac{y}{x} - \frac{x}{y}\right)^2 = 0.$$

Solution

$$\begin{aligned} 4XY &= 4\left(\frac{ct^4 - c}{2t^3}\right)\left(\frac{c - ct^4}{2t}\right) \\ &= -\frac{c^2(t^4 - 1)^2}{t^4} \\ &= -c^2\left(-t^2 + \frac{1}{t^2}\right)^2 \\ &= -c^2\left(\frac{Y}{X} - \frac{X}{Y}\right) \end{aligned}$$

and hence

$$\underline{\underline{4xy + c^2\left(\frac{y}{x} - \frac{x}{y}\right)^2 = 0.}}$$

8. The rectangular hyperbola, H, has parametric equations $x = 5t, y = \frac{5}{t}, t \neq 0$.

(a) Write the cartesian equation of H in the form $xy = c^2$. (1)

Solution

$$xy = 5t \times \frac{5}{t} = \underline{\underline{25}}.$$

Points A and B on the hyperbola have parameters $t = 1$ and $t = 5$ respectively.

(b) Find the coordinates of the mid-point of AB. (3)

Solution

$A(5, 5)$ and $B(25, 1)$:

$$\begin{aligned}\frac{y-1}{x-25} = \frac{5-1}{5-25} &\Rightarrow \frac{y-1}{x-25} = -\frac{1}{5} \\ &\Rightarrow y - 1 = -\tfrac{1}{5}(x - 25) \\ &\Rightarrow y - 1 = -\tfrac{1}{5}x + 5 \\ &\Rightarrow y = -\tfrac{1}{5}x + 6\end{aligned}$$

and we have

$$x = 15 \Rightarrow y = 3;$$

the centre of AB is $\underline{\underline{(15, 3)}}$.

9. The rectangular hyperbola H has equation $xy = c^2$, where c is a constant. The point $P\left(ct, \dfrac{c}{t}\right)$ is a general point on H.

(a) Show that the tangent to H at the point P has equation (4)

$$t^2 y + x = 2ct.$$

Solution

$$\begin{aligned}xy = c^2 &\Rightarrow y = \frac{c^2}{x} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{x^2},\end{aligned}$$

and, at the $P\left(ct, \dfrac{c}{t}\right)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{c^2 t^2} = -\frac{1}{t^2}.$$

Now,

$$\begin{aligned}y - \frac{c}{t} = -\frac{1}{t^2}(x - ct) &\Rightarrow t^2 y - ct = -x + ct \\ &\Rightarrow \underline{\underline{t^2 y + x = 2ct}}.\end{aligned}$$

The tangents to H at the points A and B meets at the point $(15c, -c)$.

(b) Find, in terms of c, the coordinates of A and B. (5)

Solution

We substitute $(15c, -c)$ into $t^2y + x = 2ct$:

$$\begin{aligned} t^2(-c) + (15c) = 2ct &\Rightarrow t^2 - 15 = -2t \\ &\Rightarrow t^2 + 2t - 15 = 0 \\ &\Rightarrow (t+5)(t-3) = 0 \\ &\Rightarrow t = -5 \text{ or } t = 3; \end{aligned}$$

hence, $\underline{\underline{\left(-5c, -\frac{c}{5}\right)}}$ and $\underline{\underline{\left(3c, \frac{c}{3}\right)}}$.

10. The rectangular hyperbola H has equation $xy = c^2$, where c is a positive constant. The point A on H has x-coordinate $3c$.

(a) Write down the y-coordinate of A. (1)

Solution

$\underline{\underline{y = \frac{c}{3}}}$.

(b) Show that an equation of the normal to H at A is (5)

$$3y = 27x - 80c.$$

Solution

$$y = \frac{c^2}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{x^2}.$$

At $x = ct$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{(3c)^2} = -\frac{1}{9}$$

and the normal to the gradient is 9. Now,

$$\begin{aligned} y - \frac{c}{3} = 9(x - 3c) &\Rightarrow 3y - c = 27x - 81c \\ &\Rightarrow \underline{\underline{3y = 27x - 80c}}. \end{aligned}$$

The normal to H at A meets H again at the point B.

(c) Find, in terms of c, the coordinates of B. (5)

Solution

We substitute $\left(ct, \frac{c}{t}\right)$ into $3y = 27x - 80c$:

$$3\left(\frac{c}{t}\right) = 27(ct) - 80c \Rightarrow 3 = 27t^2 - 80t$$
$$\Rightarrow 27t^2 - 80t - 3 = 0$$
$$\Rightarrow (27t + 1)(t - 3) = 0$$
$$\Rightarrow t = -\tfrac{1}{27} \text{ or } t = 3;$$

hence, $\underline{\underline{\left(-\frac{c}{27}, -27c\right)}}$.

11. The point $P\left(6t, \frac{6}{t}\right)$, $t \neq 0$, lies on the rectangular hyperbola H has equation $xy = 36$.

(a) Show that an equation of the tangent to H at P is (5)

$$y = -\frac{1}{t^2}x + \frac{12}{t}.$$

Solution

$$xy = 36 \Rightarrow y = \frac{36}{x}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{36}{x^2},$$

and, at the $P\left(6t, \frac{6}{t}\right)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{36}{36t^2} = -\frac{1}{t^2}.$$

Now,

$$y - \frac{6}{t} = -\frac{1}{t^2}(x - 6t) \Rightarrow y - \frac{6}{t} = -\frac{1}{t^2}x + \frac{6}{t}$$
$$\Rightarrow \underline{\underline{y = -\frac{1}{t^2}x + \frac{12}{t}}}.$$

The tangent to H at the point A and the tangent to H at the point B meet the point $(-9, 12)$.

(b) Find the coordinates of A and B. (7)

Solution
We substitute $(-9, 12)$ into $y = -\frac{1}{t^2}x + \frac{12}{t}$:

$$\begin{aligned} 12 = \frac{9}{t^2} + \frac{12}{t} &\Rightarrow 12t^2 = 9 + 12t \\ &\Rightarrow 12t^2 - 12t - 9 = 0 \\ &\Rightarrow 4t^2 - 4t - 3 = 0 \\ &\Rightarrow (2t-3)(2t+1) = 0 \\ &\Rightarrow t = \tfrac{3}{2} \text{ or } t = -\tfrac{1}{2}; \end{aligned}$$

hence, $\underline{\underline{(9, 4)}}$ and $\underline{\underline{(-3, -12)}}$.

12. The rectangular hyperbola H has cartesian equation $xy = 9$. The points $P\left(3p, \dfrac{3}{p}\right)$ and $Q\left(3q, \dfrac{3}{q}\right)$ lie on H, where $p = \pm\, q$.

(a) Show that the equation of the tangent at P is (4)

$$x + p^2 y = 6p.$$

Solution

$$\begin{aligned} xy = 9 \Rightarrow y &= \frac{9}{x} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{9}{x^2}, \end{aligned}$$

and, at the $P\left(3p, \dfrac{3}{p}\right)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{9}{9p^2} = -\frac{1}{p^2}.$$

Now,

$$\begin{aligned} y - \frac{3}{p} = -\frac{1}{p^2}(x - 3p) &\Rightarrow p^2 y - 3p = -x + 3p \\ &\Rightarrow \underline{\underline{x + p^2 y = 6p}}. \end{aligned}$$

(b) Write down the equation of the tangent at Q. (1)

Solution

$\underline{\underline{x + q^2 y = 6q}}$.

The tangent at the point P and the tangent at the point Q meet the point R.

(c) Find, as single fractions in their simplest form, the coordinates of R in terms of p and q. (4)

Solution

Subtract:

$$\begin{aligned} y(p^2 - q^2) = 6(p - q) &\Rightarrow y(p + q)(p - q) = 6(p - q) \\ &\Rightarrow y(p + q) = 6 \\ &\Rightarrow y = \frac{6}{p + q} \\ &\Rightarrow x = 6p - \frac{6p^2}{p + q} \\ &\Rightarrow x = \frac{6p(p + q) - 6p^2}{p + q} \\ &\Rightarrow x = \frac{6pq}{p + q}; \end{aligned}$$

hence, $\underline{\underline{\left(\frac{6pq}{p + q}, \frac{6}{p + q}\right)}}$.

13. The rectangular hyperbola H has equation $xy = c^2$, where c is a positive constant. The point $P\left(ct, \frac{c}{t}\right)$, $t \neq 0$, is a general point on H.

(a) Show that an equation of the tangent to H at P is (4)

$$t^2 y + x = 2ct.$$

Solution

$$\begin{aligned} xy = c^2 &\Rightarrow y = \frac{c^2}{x} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{x^2}, \end{aligned}$$

and, at the $P\left(ct, \frac{c}{t}\right)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2}.$$

Now,

$$\begin{aligned} y - \frac{c}{t} = -\frac{1}{t^2}(x - ct) &\Rightarrow t^2y - ct = -x + ct \\ &\Rightarrow \underline{\underline{x + t^2y = 2ct}}. \end{aligned}$$

The tangent to H at the point P meets the x-axis at the point A and the y-axis at the point B. Given that the area of the triangle OAB, where O is the origin, is 36,

(b) find the exact value of c, expressing your answer in the form $k\sqrt{2}$, where k is an integer. (4)

Solution

$A(2ct, 0)$ and $B(0, \frac{2c}{t})$:

$$\begin{aligned} \text{area of the triangle } OAB = 36 &\Rightarrow \frac{1}{2} \times 2ct \times \frac{2c}{t} = 36 \\ &\Rightarrow 2c^2 = 36 \\ &\Rightarrow c^2 = 18 \\ &\Rightarrow \underline{\underline{c = 3\sqrt{2}}}. \end{aligned}$$

14. The rectangular hyperbola H has cartesian equation $xy = 9$. The point $P\left(3p, \frac{3}{p}\right)$, and $Q\left(3q, \frac{3}{q}\right)$, where $p \neq 0$, $q \neq 0$, $p \neq q$, are points on the rectangular hyperbola H.

(a) Show that an equation of the tangent to H at the point P is (4)

$$p^2y + x = 10p.$$

Solution

$$\begin{aligned} xy = 25 \Rightarrow y &= \frac{259}{x} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{25}{x^2}, \end{aligned}$$

and, at the $P\left(5p, \dfrac{5}{p}\right)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{25}{25p^2} = -\frac{1}{p^2}.$$

Now,

$$\begin{aligned} y - \frac{5}{p} = -\frac{1}{p^2}(x - 5p) &\Rightarrow p^2y - 5p = -x + 10p \\ &\Rightarrow \underline{\underline{p^2y + x = 10p}}. \end{aligned}$$

(b) Write down the equation of the tangent at Q. (1)

Solution

$\underline{\underline{q^2y + x = 10q}}$.

The tangents at P and Q meet at the point N. Given that $p + q \neq 0$,

(c) show that the point N has coordinates $\left(\dfrac{10pq}{p+q}, \dfrac{10}{p+q}\right)$. (4)

Solution

Subtract:

$$\begin{aligned} y(p^2 - q^2) = 10(p - q) &\Rightarrow y(p + q)(p - q) = 10(p - q) \\ &\Rightarrow y(p + q) = 10 \\ &\Rightarrow y = \frac{10}{p + q} \\ &\Rightarrow x = 10p - \frac{10p^2}{p + q} \\ &\Rightarrow x = \frac{10p(p + q) - 10p^2}{p + q} \\ &\Rightarrow x = \frac{10pq}{p + q}; \end{aligned}$$

hence, $\underline{\underline{\left(\dfrac{10pq}{p+q}, \dfrac{10}{p+q}\right)}}$.

The line joining N to the origin is perpendicular to the line PQ.

(d) Find the value of p^2q^2. (5)

Solution

PQ:

$$\begin{aligned} m_{PQ} &= \frac{\frac{5}{p}-\frac{5}{q}}{5p-5q} \\ &= \frac{\frac{5(q-p)}{pq}}{5(p-q)} \\ &= -\frac{1}{pq}. \end{aligned}$$

ON:

$$\begin{aligned} m_{ON} &= \frac{\frac{10}{p+q}-0}{\frac{10pq}{p+q}-0} \\ &= \frac{1}{pq}. \end{aligned}$$

Now,

$$\begin{aligned} m_{PQ} \times m_{ON} = -1 &\Rightarrow -\frac{1}{p^2q^2} = -1 \\ &\Rightarrow \underline{\underline{p^2q^2 = 1}}. \end{aligned}$$

15. The rectangular hyperbola H has Cartesian equation $xy = 4$. The point $P\left(2t, \frac{2}{t}\right)$ lies on H, where $t \neq 0$.

(a) Show that an equation of the normal to H at the point P is (5)

$$ty - t^3x = 2 - 2t^4.$$

Solution

$$y = \frac{4}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{x^2}.$$

At $x = 2t$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{(2t)^2} = -\frac{1}{t^2}$$

and the normal to the gradient is t^2. Now,

$$y - \frac{2}{t} = t^2(x - 2t) \Rightarrow ty - 2 = t^3x - 2t^4$$
$$\Rightarrow \underline{\underline{ty - t^3x = 2 - 2t^4}}.$$

The normal to H at the point where $t = -\frac{1}{2}$ meets H again at the point Q.

(b) Find the coordinates of the point Q. (4)

Solution

$$t = -\tfrac{1}{2} \Rightarrow -\tfrac{1}{2}y + \tfrac{1}{8}x = \tfrac{15}{8} \Rightarrow 4y - x = -15.$$

Substitute:

$$4\left(\frac{2}{t}\right) - (2t) = -15 \Rightarrow 8 - 2t^2 = -15t$$
$$\Rightarrow 2t^2 - 15t - 8 = 0$$
$$\Rightarrow (2t+1)(t-8) = 0$$
$$\Rightarrow t = -\tfrac{1}{2} \text{ or } t = 8;$$

hence, $\underline{\underline{(16, \tfrac{1}{4})}}$.

16. Figure 1 shows a rectangular hyperbola H with parametric equations

$$x = 3t, y = \frac{3}{t}, t \neq 0.$$

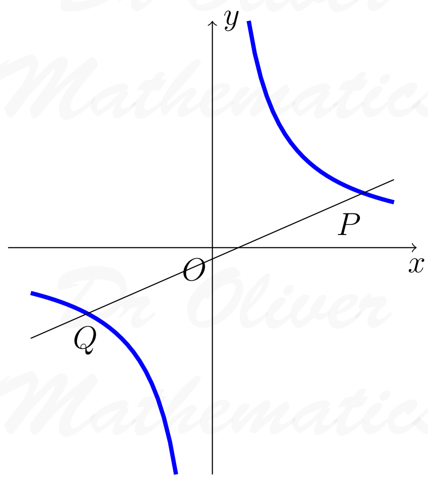

Figure 1: $x = 3t, y = \frac{3}{t}, t \neq 0$

The line L with equation $6y = 4x - 15$ intersects H at the point P and at the point Q, as shown in Figure 1.

(a) Show that L intersects H where $4t^2 - 5t - 6 = 0$. (3)

Solution

$$\begin{aligned} 6y = 4x - 15 &\Rightarrow 6\left(\frac{3}{t}\right) = 4(3t) - 15 \\ &\Rightarrow \frac{18}{t} = 12t - 15 \\ &\Rightarrow 18 = 12t^2 - 15t \\ &\Rightarrow 12t^2 - 15t - 18 = 0 \\ &\Rightarrow \underline{\underline{4t^2 - 5t - 6 = 0}}. \end{aligned}$$

(b) Hence, or otherwise, find the coordinates of points P and Q. (5)

Solution

$$\begin{aligned} 4t^2 - 5t - 6 = 0 &\Rightarrow (4t + 3)(t - 2) = 0 \\ &\Rightarrow t = -\tfrac{3}{4} \text{ or } t = 2; \end{aligned}$$

hence, $\underline{\underline{P(6, \frac{3}{2})}}$ and $\underline{\underline{Q(-\frac{9}{4}, -4)}}$

17. The rectangular hyperbola H has Cartesian equation $xy = c^2$. The point $P\left(2t, \dfrac{2}{t}\right)$,

$t > 0$, is a general point on H.

(a) Show that an equation of the tangent to H at the point P is (4)

$$t^2 y + x = 2ct.$$

Solution

$$xy = c^2 \Rightarrow y = \frac{c^2}{x}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{x^2},$$

and, at the $P\left(ct, \frac{c}{t}\right)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{c^2 t^2} = -\frac{1}{t^2}.$$

Now,

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct) \Rightarrow t^2 y - ct = -x + ct$$
$$\Rightarrow \underline{\underline{t^2 y + x = 2ct}}.$$

An equation of the normal to H at the point P is

$$t^3 x - ty = ct^4 - c.$$

Given that the normal to H at P meets the x-axis at the point A and the tangent to H at P meets the x-axis at the point B,

(b) find, in terms of c and t, the coordinates of A and the coordinates of B. (2)

Solution
$\underline{\underline{A\left(ct - \frac{c}{t^3}, 0\right)}}$ and $\underline{\underline{B(2ct, 0)}}$.

Given that $c = 4$,

(c) find, in terms of t, the area of the triangle APB. Give your answer in its simplest form. (3)

Solution

$A\left(4t - \dfrac{4}{t^3}, 0\right)$ and $B(8t, 0)$. Now,

$$\begin{aligned}\text{area} &= \frac{1}{2} \times \left[8t - \left(4t - \frac{4}{t^3}\right)\right] \times \frac{4}{t} \\ &= \frac{1}{2} \times \left(4t + \frac{4}{t^3}\right) \times \frac{4}{t} \\ &= \frac{1}{2} \times \frac{4(t^4+1)}{t^3} \times \frac{4}{t} \\ &= \underline{\underline{\frac{8(t^4+1)}{t^4}}}.\end{aligned}$$

18. The rectangular hyperbola H has Cartesian equation $xy = c^2$, where c is a positive constant. (7)
The point $P\left(ct, \dfrac{c}{t}\right)$, $t \neq 0$, is a general point on H.
An equation of the tangent to H at P is

$$y = -\frac{1}{t^2}x + \frac{2c}{t}.$$

The points A and B lie on H.
The tangent to H at A and the tangent to H at B meet at the point $\left(-\dfrac{6c}{7}, \dfrac{12c}{7}\right)$.
Find, in terms of c, the coordinates of A and the coordinates of B.

Solution

We substitute $\left(-\dfrac{6c}{7}, \dfrac{12c}{7}\right)$ into $y = -\dfrac{1}{t^2}x + \dfrac{2c}{t}$:

$$\begin{aligned}\frac{12c}{7} = \frac{6c}{7t^2} + \frac{2c}{t} &\Rightarrow 12ct^2 = -6c + 14ct \\ &\Rightarrow 12ct^2 - 14ct - 6c = 0 \\ &\Rightarrow 6t^2 - 7t - 3 = 0 \\ &\Rightarrow (3t+1)(2t-3) = 0 \\ &\Rightarrow t = -\tfrac{1}{3} \text{ or } t = \tfrac{3}{2};\end{aligned}$$

hence, $\underline{\underline{\left(-\dfrac{c}{3}, -3c\right)}}$ and $\underline{\underline{\left(\dfrac{3c}{2}, \dfrac{2c}{3}\right)}}$.

19. The rectangular hyperbola H has equation $xy = 9$. The point A on H has coordinates $(6, \frac{3}{2})$.

(a) Show that the normal to H at the point A has equation (5)

$$2y - 8x + 45 = 0.$$

Solution

$$y = \frac{9}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{9}{x^2}.$$

So the gradient of the tangent at A is $-\frac{1}{4}$ and hence the gradient of the normal is 4. So the equation of the normal is

$$y - \tfrac{3}{2} = 4(x-6) \Rightarrow 2y - 3 = 8x - 48 \Rightarrow \underline{\underline{2y - 8x + 45 = 0}},$$

as required.

The normal at A meets H again at the point B.

(b) Find the coordinates of B. (4)

Solution

Substitute the equation of the hyperbola into the line (or vice versa):

$$\begin{aligned} 2\left(\frac{9}{x}\right) - 8x + 45 = 0 &\Rightarrow 18 - 8x^2 + 45x = 0 \\ &\Rightarrow 8x^2 - 45x - 18 = 0 \\ &\Rightarrow (x-6)(8x+3) = 0, \end{aligned}$$

and so $\underline{\underline{B(-\frac{3}{8}, -24)}}$.

20. The rectangular hyperbola, H, has cartesian equation $xy = 25$.

(a) Show that an equation of the normal to H at the point $P(5p, \frac{5}{p})$, $p \neq 0$, is (5)

$$y - p^2x = \frac{5}{p} - 5p^3.$$

Solution

$$y = \frac{25}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{25}{x^2}.$$

At $x = 5p$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{25}{(5p)^2} = -\frac{1}{p^2}$$

and the normal to the gradient is p^2. Now,

$$\begin{aligned} y - \frac{5}{p} = p^2(x - 5p) &\Rightarrow y - \frac{5}{p} = p^2x - 5p^3 \\ &\Rightarrow \underline{\underline{y - p^2x = \frac{5}{p} - 5p^3}}. \end{aligned}$$

This normal meets the line with equation $y = -x$ at the A.

(b) Show that the coordinates of A are (5)

$$\left(-\frac{5}{p} + 5p, \frac{5}{p} - 5p\right).$$

Solution

$$\begin{aligned} -x - p^2x = \frac{5}{p} - 5p^3 &\Rightarrow -(1 + p^2)x = \frac{5}{p} - 5p^3 \\ &\Rightarrow -(1 + p^2)x = \frac{5}{p}(1 - p^4) \\ &\Rightarrow x = -\frac{5}{p(1 + p^2)}(1 - p^4) \\ &\Rightarrow x = -\frac{5}{p}(1 - p^2) \\ &\Rightarrow x = -\frac{5}{p} + 5p, \end{aligned}$$

and, since $y = -x$, we have

$$\underline{\underline{\left(-\frac{5}{p} + 5p, \frac{5}{p} - 5p\right)}}.$$

The point M is the midpoint of the line segment AP. Given that M lies on the positive x-axis,

(c) find exact value of the x-coordinate of point M. (3)

Solution

M has coordinates of

$$\left(-\frac{5}{2p}+5p, \frac{5}{p}-\frac{5p}{2}\right)'$$

So we make $y=0$:

$$-\frac{5}{2p}+5p=0 \Rightarrow 5p=\frac{5}{2p} \Rightarrow p^2=2 \Rightarrow p=\sqrt{2},$$

giving the x-coordinate of M is

$$-\frac{5}{2\sqrt{2}}+5\sqrt{2}=\underline{\underline{\frac{15}{4}\sqrt{2}}}.$$

21. The rectangular hyperbola H has parametric equations

$$x=4t,\ y=\frac{4}{t},\ t\neq 0.$$

The points P and Q on this hyperbola have parameters $t=\frac{1}{4}$ and $t=2$ respectively. The line l passes through the origin O and is perpendicular to the line PQ.

(a) Find an equation for l. (3)

Solution

$t=\frac{1}{4} \Rightarrow P(1,16)$ and $t=2 \Rightarrow Q(8,2)$. Now,

$$m_{PQ}=\frac{2-16}{8-1}=\frac{-14}{7}=-2$$

which means

$$m=\tfrac{1}{2}$$

and this means

$$\underline{\underline{y=\tfrac{1}{2}x}}$$

(b) Find a cartesian equation for H. (1)

Solution

$\underline{\underline{xy = 16}}$.

(c) Find the exact coordinates of the two points where l intersects H. Give your answers in their simplest form. (3)

Solution

Substitute the equation of the hyperbola into the line (or vice versa):

$$\begin{aligned} \tfrac{1}{2}x = \frac{16}{x} &\Rightarrow x^2 = 32 \\ &\Rightarrow x = \pm 4\sqrt{2} \\ &\Rightarrow \underline{\underline{(4\sqrt{2}, 2\sqrt{2})}} \text{ and } \underline{\underline{(-4\sqrt{2}, -2\sqrt{2})}}. \end{aligned}$$