Dr Oliver Mathematics
Mathematics: Advanced Higher
2023 Paper 2: Calculator
2 hours

The total number of marks available is 65.
You must write down all the stages in your working.

1. The function f is defined by (2)

$$\mathrm{f}(x) = 2\sin^{-1} 3x.$$

Find $\mathrm{f}'(x)$.

Solution

$$\begin{aligned}\mathrm{f}'(x) &= 2 \times \frac{1}{\sqrt{1-(3x)^2}} \times 3 \\ &= \underline{\underline{\frac{6}{\sqrt{1-9x^2}}}}.\end{aligned}$$

2. Find (2)

$$\int \left(\frac{x^2}{x^3+10} \right) \mathrm{d}x.$$

Solution

$$\begin{aligned}\int \left(\frac{x^2}{x^3+10} \right) \mathrm{d}x &= \tfrac{1}{3} \int \left(\frac{3x^2}{x^3+10} \right) \mathrm{d}x \\ &= \underline{\underline{\tfrac{1}{3} \ln |x^3+10| + c}}.\end{aligned}$$

3. Matrix $\mathbf{A}$ is defined by

$$\mathbf{A} = \begin{pmatrix} 2 & 2x & 4 \\ x & -1 & 0 \\ 1 & 0 & -2 \end{pmatrix}, \text{ where } x \in \mathbb{R}.$$

(a) Find a simplified expression for the determinant of $\mathbf{A}$. (2)

Solution

$$\begin{aligned}\det \mathbf{A} &= 2(2-0) - 2x(-2x-0) + 4(0+1) \\ &= \underline{\underline{4x^2+8}}.\end{aligned}$$

(b) Hence, determine whether $\mathbf{A}^{-1}$ exists for all values of x. (1)

Solution

As $\det \mathbf{A} > 0$, $\mathbf{A}^{-1}$ $\underline{\underline{\text{exists}}}$ for all values of x

4. Calculate the gradient of the tangent to the curve with equation (3)

$$x^2y^2 - 2y = \sin 3x$$

at the point $(0, 0)$.

Solution

Implicit differentiation:

$$(2x)(y^2) - (x^2)\left(2y\frac{\mathrm{d}y}{\mathrm{d}x}\right) - 2\frac{\mathrm{d}y}{\mathrm{d}x} = 3\cos 3x.$$

Now,

$$\begin{aligned} x = 0, y = 0 &\Rightarrow -2\frac{\mathrm{d}y}{\mathrm{d}x} = 3 \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = -1\tfrac{1}{2}}}.\end{aligned}$$

5. (a) Write down and simplify the general term in the binomial expansion of (3)

$$\left(3x - \frac{2}{x^2}\right)^8.$$

Solution

The general term is

$$\binom{8}{r}(3x)^r\left(-\frac{2}{x^2}\right)^{8-r} = \binom{8}{n}(3^n)(x^n)(-2^{8-n})x^2(n-8)$$
$$= \binom{8}{r}(3^r)(-2)^{8-r}\,x^{r+2(r-8)}$$
$$= \binom{8}{r}(3^n)(-2)^{8-r}\,x^{r+2r-16}$$
$$= \underline{\underline{\binom{8}{r}(3^r)(-2)^{8-r}\,x^{3r-16}}}.$$

(b) Hence, or otherwise, determine the coefficient of x^{-1}. (2)

Solution

Well,

$$3n - 16 = -1 \Rightarrow 3n = 15$$
$$\Rightarrow n = 5$$

and the coefficient of x^{-1} is

$$\binom{8}{5}(3^5)(-2)^{8-5} = 56 \times 243 \times (-8)$$
$$= \underline{\underline{-108\,864}}.$$

6. (a) Use the Euclidean algorithm to find d, the greatest common divisor of 703 and 399. (1)

Solution

Well,

$$703 = 399 \times 1 + 304$$
$$399 = 304 \times 1 + 95$$
$$304 = 95 \times 3 + 19$$
$$95 = 19 \times 5;$$

hence, the greatest common divisor of 703 and 399 is $\underline{\underline{d = 19}}$.

(b) Find integers a and b such that (2)

$$d = 703a + 399b.$$

Solution

$$\begin{aligned} 19 &= 304 - (95 \times 3) \\ &= 304 - 3(399 - 304) \\ &= 4 \times 304 - 3 \times 399 \\ &= 4(703 - 304) - 3 \times 399 \\ &= \underline{\underline{4 \times 703 + (-7) \times 399}}; \end{aligned}$$

hence, $\underline{\underline{a = 4}}$ and $\underline{\underline{b = -7}}$.

(c) Hence find integers p and q such that (1)

$$76 = 703p + 399q.$$

Solution

Multiply each term by 4:

$$76 = \underline{\underline{703(16) + 399(-28)}};$$

hence, $\underline{\underline{p = 16}}$ and $\underline{\underline{q = -28}}$.

7. (a) Solve the differential equation (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 6\mathrm{e}^{5x},$$

given that when $x = 0$, $y = -1$.

Express y in terms of x.

Solution

Well,

$$\begin{aligned} \text{IF} &= \mathrm{e}^{\int (-2)\,\mathrm{d}x} \\ &= \mathrm{e}^{-2x} \end{aligned}$$

and so

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 6\mathrm{e}^{5x} &\Rightarrow \mathrm{e}^{-2x}\frac{\mathrm{d}y}{\mathrm{d}x} - 2y\mathrm{e}^{-2x}y = 6\mathrm{e}^{3x} \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-2x}y) = 6\mathrm{e}^{3x} \\ &\Rightarrow \mathrm{e}^{-2x}y = \int 6\mathrm{e}^{3x}\,\mathrm{d}x \\ &\Rightarrow \mathrm{e}^{-2x}y = 2\mathrm{e}^{3x} + c \\ &\Rightarrow y = 2\mathrm{e}^{5x} + c\mathrm{e}^{2x}. \end{aligned}$$

Now,

$$\begin{aligned} x = 0, y = -1 &\Rightarrow -1 = 2 + c \\ &\Rightarrow c = -3 \end{aligned}$$

and

$$\underline{\underline{y = 2\mathrm{e}^{5x} - 3\mathrm{e}^{2x}}}.$$

The solution of the differential equation in (a) is also a solution of

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 5\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = k\mathrm{e}^{2x},\ k \in \mathbb{R}.$$

(b) Find the value of k. (2)

Solution

Now,

$$\begin{aligned} y = 2\mathrm{e}^{5x} - 3\mathrm{e}^{2x} &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 10\mathrm{e}^{5x} - 6\mathrm{e}^{2x} \\ &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 50\mathrm{e}^{5x} - 12\mathrm{e}^{2x} \\ &\Rightarrow \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 250\mathrm{e}^{5x} - 24\mathrm{e}^{2x} \end{aligned}$$

and

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 5\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (250\mathrm{e}^{5x} - 24\mathrm{e}^{2x}) - 5(50\mathrm{e}^{5x} - 12\mathrm{e}^{2x})$$
$$= 36\mathrm{e}^{2x};$$

hence, $\underline{\underline{k = 36}}$.

8. The fourth and seventh terms of a geometric sequence are 9 and 243 respectively.

(a) Find the

(i) common ratio, (1)

Solution
Let a be the first term and r be the common ratio. Then

$$ar^3 = 9 \quad (1)$$
$$ar^6 = 243 \quad (2).$$

Do (2) ÷ (1):

$$\frac{ar^6}{ar^3} = \frac{243}{9} \Rightarrow r^3 = 27$$
$$\Rightarrow \underline{\underline{r = 3}}.$$

(ii) first term. (1)

Solution
Now,

$$a \times 3^3 = 9 \Rightarrow 27a = 9$$
$$\Rightarrow \underline{\underline{a = \tfrac{1}{3}}}.$$

(b) Show that (2)

$$\frac{S_{2n}}{S_n} = 1 + 3^n,$$

where S_n represents the sum of the first n terms of this geometric sequence.

Solution

$$S_n = \frac{\frac{1}{3}(3^n - 1)}{3 - 1} = \tfrac{1}{6}(3^n - 1)$$

and, clearly,

$$S_{2n} = \tfrac{1}{6}(3^{2n} - 1).$$

Finally,

$$\frac{S_{2n}}{S_n} = \frac{\frac{1}{6}(3^{2n} - 1)}{\frac{1}{6}(3^n - 1)} = \frac{3^{2n} - 1}{3^n - 1}$$

difference of two squares:

$$= \frac{(3^n - 1)(3^n + 1)}{3^n - 1} = \underline{\underline{1 + 3^n}},$$

as required.

9. Express 572_{10} in base 9. (2)

Solution

$$572_{10} = \left[(7 \times 9^2) + (0 \times 9) + (5 \times 1)\right]_9 = \underline{\underline{705_9}}.$$

10. A curve is defined by (5)

$$y = x^{5x^2}, \text{ where } x > 0.$$

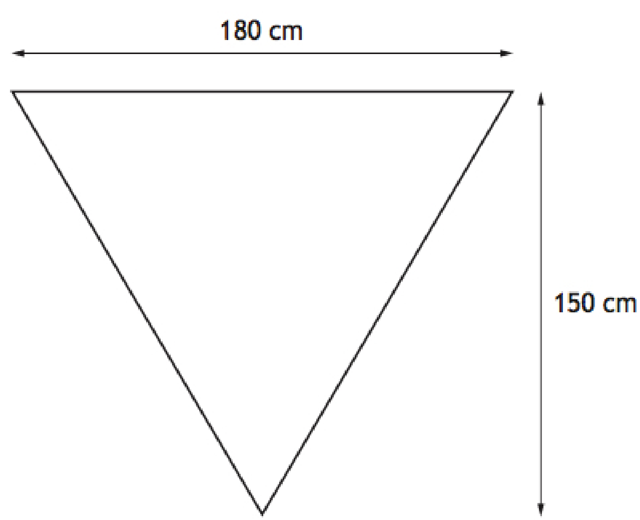
Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x.

Solution

Logarthmic

$$\begin{aligned} y = x^{5x^2} &\Rightarrow \ln y = \ln x^{5x^2} \\ &\Rightarrow \ln y = 5x^2 \ln x \\ &\Rightarrow \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 10x\ln x + 5x \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = y(10x\ln x + 5x) \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = (10x\ln x + 5x)x^{5x^2}}}. \end{aligned}$$

11. On a building site, water is stored in a container.



The container is a cone with diameter 180 cm at its widest point and height of 150 cm. A cross section of the cone is shown below.

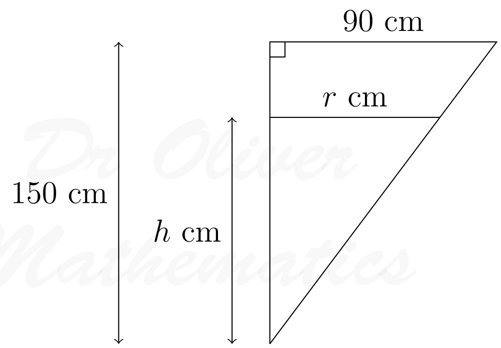
(a) Show that when the water level is at a height of h cm, $0 \leqslant h \leqslant 150$, the volume of water in the container can be written as (1)

$$V = \frac{3\pi h^3}{25}.$$

Water is pumped into the container at a constant rate of 10 litres per second.

Solution

Let the radius of the water be r cm.



Similar triangles:

$$\frac{r}{h} = \frac{90}{150} \Rightarrow r = \tfrac{3}{5}h$$

so

$$\begin{aligned} V &= \tfrac{1}{3}\pi r^2 h \\ &= \tfrac{1}{3}\pi(\tfrac{3}{5}h)^2 h \\ &= \tfrac{1}{3}\pi(\tfrac{9}{25}h^2)h \\ &= \underline{\underline{\frac{3\pi h^3}{25}}}, \end{aligned}$$

as required.

(b) Find the rate at which the height is increasing when $h = 125$. (5)

Solution

Well,

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \tfrac{9}{25}\pi h^2$$

and

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}h} \times \frac{\mathrm{d}h}{\mathrm{d}t}.$$

Now,

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{\frac{\mathrm{d}V}{\mathrm{d}t}}{\frac{\mathrm{d}V}{\mathrm{d}h}} \\ &= \frac{10\,000}{\frac{9}{25}\pi(125^2)} \\ &= \underline{\underline{\left(\frac{16}{9\pi}\right)\ \mathrm{cm\,s^{-1}}}}.\end{aligned}$$

12. Prove by induction that, for all positive integers n, (5)

$$\sum_{r=1}^{n} 2^{r-1} r = 2^n(n-1) + 1.$$

Solution

$\underline{n = 1}$:

$$\text{LHS} = 2^0(1) = 1$$

and

$$\text{RHS} = 2^0(1-1) + 1 = 1.$$

So, $n = 1$ is true.

Suppose now that it is true for $n = k$, i.e.,

$$\sum_{r=1}^{k} 2^{r-1} r = 2^k(k-1) + 1.$$

Then

$$\begin{aligned}\sum_{r=1}^{k+1} 2^{r-1} r &= \left(\sum_{r=1}^{k} 2^{r-1} r\right) + 2^{(k+1)-1}(k+1) \\ &= \big(2^k(k-1)+1\big) + 2^k(k+1) \\ &= 2^k[(k-1)(k+1)] + 1 \\ &= 2^k(2k) + 1 \\ &= 2^{k+1}k + 1,\end{aligned}$$

and we have proved that it is true for $n = k + 1$.

Hence, by mathematical induction, it is $\underline{\underline{\text{true}}}$ for all positive integers n.

13. Points scored in the long jump element of the decathlon can be calculated using a solution of the differential equation (6)

$$(m - 220)\frac{\mathrm{d}P}{\mathrm{d}m} = 1.4P,\ m > 220,$$

where m is the distance jumped in centimetres and P the points scored.

Given that a jump of 807 centimetres scores 1 079 points, find an expression for P in terms of m.

Solution

$$\begin{aligned}(m - 220)\frac{\mathrm{d}P}{\mathrm{d}m} = 1.4P &\Rightarrow \frac{1}{P}\,\mathrm{d}P = \frac{1.4}{(m-220)}\,\mathrm{d}m \\ &\Rightarrow \int \frac{1}{P}\,\mathrm{d}P = \int \frac{1.4}{(m-220)}\,\mathrm{d}m \\ &\Rightarrow \ln P = 1.4\ln(m-220) + c,\end{aligned}$$

for some constant c. Now,

$$\begin{aligned}m = 807, P = 1\,079 &\Rightarrow \ln 1\,079 = 1.4\ln(807 - 220) + c \\ &\Rightarrow \ln 1\,079 = 1.4\ln 587 + c \\ &\Rightarrow c = \ln 1\,079 - 1.4\ln 587 \\ &\Rightarrow c = -1.941\,244\,783 \text{ (FCD)}\end{aligned}$$

and

$$\begin{aligned}\ln P = 1.4\ln(m-220) - 1.941\ldots &\Rightarrow \ln P = \ln(m-220)^{1.4} - 1.941\ldots \\ &\Rightarrow \ln P - \ln(m-220)^{1.4} = -1.941\ldots \\ &\Rightarrow \ln\left(\frac{P}{(m-220)^{1.4}}\right) = -1.941\ldots \\ &\Rightarrow \frac{P}{(m-220)^{1.4}} = \mathrm{e}^{-1.941\ldots} \\ &\Rightarrow \underline{\underline{P = 0.143\,525\,180\,9(m-220)^{1.4} \text{ (FCD)}}}.\end{aligned}$$

14. A complex number is defined by (4)

$$w = a + b\mathrm{i},$$

where a and b are positive real numbers.

Given

$$w^2 = 8 + 6\text{i},$$

determine the values of a and b.

Solution

Well,

$\times$	a	$+b\text{i}$
a	a^2	$+ab\text{i}$
$+b\text{i}$	$+ab\text{i}$	$-b^2$

and

$$w^2 = (a^2 - b^2) + 2ab\text{i}.$$

Now,

$$\begin{aligned} a^2 - b^2 &= 8 \quad (1) \\ 2ab &= 6 \quad (2). \end{aligned}$$

Next,

$$2ab = 6 \Rightarrow b = \frac{3}{a}$$

and we will insert (2) into (1):

$$\begin{aligned} a^2 - \left(\frac{3}{a}\right)^2 = 8 &\Rightarrow a^2 - \frac{9}{a^2} = 8 \\ &\Rightarrow a^4 - 9 = 8a^2 \\ &\Rightarrow a^4 - 8a^2 - 9 = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -8 \\ \text{multiply to:} & -9 \end{array}\right\} -9,\ +1$$

$$\begin{aligned} &\Rightarrow (a^2 - 9)(a^2 + 1) = 0 \\ &\Rightarrow a^2 - 9 = 0 \text{ (only)} \\ &\Rightarrow a^2 = 9 \\ &\Rightarrow \underline{\underline{a = 3}} \text{ (positive real numbers)} \\ &\Rightarrow \underline{\underline{b = 1}}. \end{aligned}$$

15. A function $\mathrm{f}(x)$ has the following properties:

- $f'(x) = \dfrac{x+1}{1+(x+1)^4}$ and
- the first term in the Maclaurin expansion of $\mathrm{f}(x)$ is 1.

(a) Find the Maclaurin expansion of $\mathrm{f}(x)$ up to and including the term in x^2. (3)

Solution

Well,

$$u = x + 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$v = 1 + (x+1)^4 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 4(x+1)^3$$

and

$$\begin{aligned} f''(x) &= \frac{[1+(x+1)^4]\cdot 1 - (x+1)\cdot 4(x+1)^3}{(1+(x+1)^4)^2} \\ &= \frac{1+(x+1)^4 - (x+1)\cdot 4(x+1)^3}{(1+(x+1)^4)^2} \\ &= \frac{1+(x+1)^4 - 4(x+1)^4}{(1+(x+1)^4)^2} \\ &= -\frac{1-3(x+1)^4}{(1+(x+1)^4)^2}. \end{aligned}$$

Now,

$$\begin{aligned} \mathrm{f}(0) &= 1 \\ \mathrm{f}'(0) &= \tfrac{1}{2} \\ \mathrm{f}''(0) &= -\tfrac{1}{2} \end{aligned}$$

and the Maclaurin expansion of $\mathrm{f}(x)$ is

$$\begin{aligned} \mathrm{f}(x) &= 1 + \tfrac{1}{2}x + \tfrac{1}{2!}(-1)x^2 + \ldots \\ &= \underline{\underline{1 + \tfrac{1}{2}x - \tfrac{1}{4}x^2 + \ldots}}. \end{aligned}$$

(b) Use the substitution (3)

$$u = (x+1)^2$$

to find

$$\int \frac{x+1}{1+(x+1)^4}\,\mathrm{d}x.$$

Solution

$$u = (x+1)^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2(x+1)$$
$$\Rightarrow \mathrm{d}u = 2(x+1)\,\mathrm{d}x$$

and

$$\begin{aligned}\int \frac{x+1}{1+(x+1)^4}\,\mathrm{d}x &= \tfrac{1}{2}\int \frac{2(x+1)}{1+[(x+1)^2]^2}\,\mathrm{d}x \\ &= \tfrac{1}{2}\int \frac{1}{1+u^2}\,\mathrm{d}x \\ &= \tfrac{1}{2}\arctan u + c \\ &= \underline{\underline{\tfrac{1}{2}\arctan[(x+1)^2] + c}}.\end{aligned}$$

(c) Determine an expression for $\mathrm{f}(x)$. (2)

Solution

Well,

$$\begin{aligned}\mathrm{f}(0) = 1 &\Rightarrow \tfrac{1}{2}\arctan 1 + c = 1 \\ &\Rightarrow c = 1 - \tfrac{1}{2}(\tfrac{1}{4}\pi) \\ &\Rightarrow c = 1 - \tfrac{1}{8}\pi;\end{aligned}$$

hence,

$$\underline{\underline{\mathrm{f}(x) = \tfrac{1}{2}\arctan[(x+1)^2] + 1 - \tfrac{1}{8}\pi.}}$$