Dr Oliver Mathematics
Mathematics: Advanced Higher
2008 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. The first term of an arithmetic sequence is 2 and the 20th term is 97. (4)
   Obtain the sum of the first 50 terms.

Solution

$$a = 2 \text{ and } 97 = a + 19d.$$

Now,

$$19d = 95 \Rightarrow d = 19.$$

Finally,

$$\begin{aligned} S_{50} &= \tfrac{50}{2}[2 \times 2 + 49 \times 5] \\ &= \underline{\underline{6\,225}}. \end{aligned}$$

2. (a) Differentiate (2)

$$\mathrm{f}(x) = \cos^{-1}(3x)$$

where $-\frac{1}{3} < x < \frac{1}{3}$.

Solution

$$\begin{aligned} \mathrm{f}'(x) &= -\frac{1}{\sqrt{1-(3x)^2}} \times 3 \\ &= \underline{\underline{-\frac{3}{\sqrt{1-9x^2}}}}. \end{aligned}$$

(b) Given (3)

$$x = 2\sec\theta, y = 3\sin\theta,$$

use parametric differentiation to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of θ.

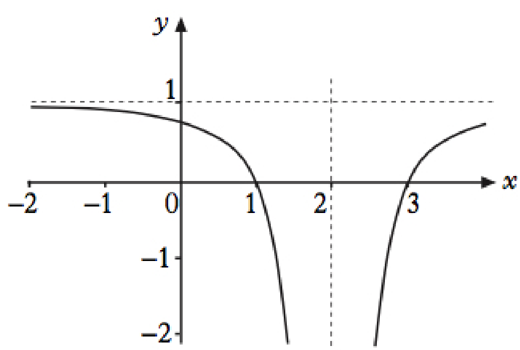
Solution

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sec\theta\tan\theta \text{ and } \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3\cos\theta.$$

Now,

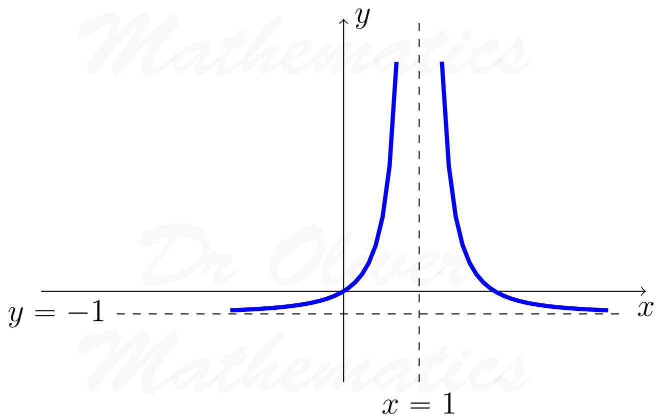
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} \\ &= \frac{3\cos\theta}{2\sec\theta\tan\theta} \\ &= \frac{3\cos^2\theta}{\frac{2\sin\theta}{\cos\theta}} \\ &= \underline{\underline{\frac{3\cos^3\theta}{2\sin\theta}}}.\end{aligned}$$

3. Part of the graph $y = \mathrm{f}(x)$ is shown below, where the dotted lines indicate asymptotes. (4)



Sketch the graph $y = -\,\mathrm{f}(x+1)$ showing its asymptotes.
Write down the equations of the asymptotes.

Solution



Note: the curve *must* pass through $(0,0)$.
Asymptotes: $\underline{\underline{x = 1}}$ and $\underline{\underline{y = -1}}$.

4. (a) Express (3)

$$\frac{12x^2 + 20}{x(x^2 + 5)}$$

in partial fractions.

Solution

$$\begin{aligned}\frac{12x^2 + 20}{x(x^2 + 5)} &\equiv \frac{A}{x} + \frac{Bx + C}{x^2 + 5} \\ &\equiv \frac{A(x^2 + 5) + x(Bx + C)}{x(x^2 + 5)}\end{aligned}$$

which means

$$12x^2 + 20 \equiv A(x^2 + 5) + x(Bx + C).$$

$\underline{x = 0}$: $20 = 5A \Rightarrow A = 4$.
$\underline{x = 1}$: $32 = 6A + B + C \Rightarrow B + C = 8$.
$\underline{x = -1}$: $32 = 6A + B - C \Rightarrow B - C = 8$.
Solve: $B = 8$ and $C = 0$.
Hence,

$$\frac{12x^2 + 20}{x(x^2 + 5)} \equiv \underline{\underline{\frac{4}{x} + \frac{8x}{x^2 + 5}}}.$$

(b) Hence evaluate (3)

$$\int_1^2 \frac{12x^2+20}{x(x^2+5)}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int_1^2 \frac{12x^2+20}{x(x^2+5)}\,\mathrm{d}x &= \int_1^2 \left(\frac{4}{x}+\frac{8x}{x^2+5}\right)\mathrm{d}x\\ &= \left[4\ln|x| + 4\ln|x^2+5|\right]_{x=1}^2\\ &= (4\ln 2 + 4\ln 9) - (0 + 4\ln 6)\\ &= 4(\ln 18 - \ln 6)\\ &= 4\ln\left(\tfrac{18}{6}\right)\\ &= \underline{\underline{4\ln 3}}.\end{aligned}$$

5. A curve is defined by the equation

$$xy^2 + 3x^2y = 4$$

for $x > 0$ and $y > 0$.

(a) Use implicit differentiation to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. (3)

Solution

$$\begin{aligned}(1\times y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x}) + (6xy + 3x^2\frac{\mathrm{d}y}{\mathrm{d}x}) = 0 &\Rightarrow (2xy+3x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = -6xy - y^2\\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y(6x+y)}{x(2y+3x)}}}.\end{aligned}$$

(b) Hence find an equation of the tangent to the curve where $x = 1$. (3)

Solution

$$\begin{aligned}x = 1 &\Rightarrow y^2 + 3y = 4\\ &\Rightarrow y^2+3y-4=0\\ &\Rightarrow (y+4)(y-1)=0\\ &\Rightarrow y=-4 \text{ or } y=1.\end{aligned}$$

Since $y \neq -4$ (why?), $y = 1$. Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\tfrac{7}{5}$$

and the equation of the tangent is

$$\begin{aligned} y - 1 = -\tfrac{7}{5}(x-1) &\Rightarrow 5y - 5 = -7(x-1) \\ &\Rightarrow 5y - 5 = -7x + 7 \\ &\Rightarrow \underline{\underline{7x + 5y - 12 = 0}}. \end{aligned}$$

6. Let the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & x \\ x & 4 \end{pmatrix}.$$

(a) Obtain the value(s) of x for which $\mathbf{A}$ is singular. (2)

Solution

$$\begin{aligned} \det \mathbf{A} = 0 &\Rightarrow 4 - x^2 = 0 \\ &\Rightarrow x^2 = 4 \\ &\Rightarrow \underline{\underline{x = \pm 2}}. \end{aligned}$$

(b) When $x = 2$, show that (1)

$$\mathbf{A}^2 = p\mathbf{A}$$

for some constant p.

Solution

$$\begin{aligned} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} &= \begin{pmatrix} 5 & 10 \\ 10 & 20 \end{pmatrix} \\ &= \underline{\underline{5\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}}}; \end{aligned}$$

hence, $\underline{\underline{p = 5}}$.

(c) Determine the value of q such that (2)

$$\mathbf{A}^4 = q\mathbf{A}.$$

Solution

$$\begin{aligned}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}^4 &= \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}^2 \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}^2 \\ &= 25 \times 5 \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \\ &= \underline{\underline{125 \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}}};\end{aligned}$$

hence, $\underline{\underline{q = 125}}$.

7. Use integration by parts to obtain (5)

$$\int 8x^2 \sin 4x \, \mathrm{d}x.$$

Solution

$$u = 8x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 16x \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \sin 4x \Rightarrow v = -\tfrac{1}{4}\cos 4x.$$

$$\int 8x^2 \sin 4x \, \mathrm{d}x = -2x^2 \cos 4x + 4 \int x \cos 4x \, \mathrm{d}x$$

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 \text{ and } \frac{\mathrm{d}v}{\mathrm{d}x} = \cos 4x \Rightarrow v = \tfrac{1}{4}\sin 4x.$$

$$\begin{aligned} &= -2x^2 \cos 4x + 4\left(\tfrac{1}{4}x \sin 4x - \int \tfrac{1}{4}\sin 4x \, \mathrm{d}x\right) \\ &= -2x^2 \cos 4x + x \sin 4x - \int \sin 4x \, \mathrm{d}x \\ &= \underline{\underline{-2x^2 \cos 4x + x \sin 4x + \tfrac{1}{4}\cos 4x + c}}.\end{aligned}$$

8. (a) Write down and simplify the general term in the expansion of (3)

$$\left(x^2 + \frac{1}{x}\right)^{10}.$$

Solution

$$\begin{aligned}\text{General term} &= \begin{pmatrix} 10 \\ r \end{pmatrix} (x^2)^r \left(\frac{1}{x}\right)^{10-r} \\ &= \begin{pmatrix} 10 \\ r \end{pmatrix} x^{2r} x^{r-10} \\ &= \underline{\underline{\begin{pmatrix} 10 \\ r \end{pmatrix} x^{3r-10}}}.\end{aligned}$$

(b) Hence, or otherwise, obtain the term in x^{14}. (2)

Solution

$$3r - 10 = 14 \Rightarrow 3r = 24 \Rightarrow r = 8$$

and the term in x^{14} is

$$\begin{pmatrix} 10 \\ 8 \end{pmatrix} x^{14} = \underline{\underline{45x^{14}}}.$$

9. (a) Write down the derivative of $\tan x$. (1)

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \underline{\underline{\sec^2 x}}.$$

(b) Show that (1)

$$1 + \tan^2 x = \sec^2 x.$$

Solution

$$\begin{aligned}1 + \tan^2 x &\equiv 1 + \frac{\sin^2 x}{\cos^2 x} \\ &\equiv \frac{\cos^x + \sin^2 x}{\cos^2 x} \\ &\equiv \frac{1}{\cos^2 x} \\ &\equiv \underline{\underline{\sec^2 x}},\end{aligned}$$

as required.

(c) Hence obtain (2)

$$\int \tan^2 x \, \mathrm{d}x.$$

Solution

$$\int \tan^2 x \, \mathrm{d}x = \int (\sec^2 x - 1) \, \mathrm{d}x$$
$$= \underline{\underline{\tan x - x + c}}.$$

10. A body moves along a straight line with velocity

$$v = t^3 - 12t^2 + 32t$$

at time t.

(a) Obtain the value of its acceleration when $t = 0$. (1)

Solution

$$v = t^3 - 12t^2 + 32t \Rightarrow a = 3t^2 - 24t + 32$$

and

$$t = 0 \Rightarrow \underline{\underline{a = 32}}.$$

(b) At time $t = 0$, the body is at the origin O. (2)
Obtain a formula for the displacement of the body at time t.

Solution

$$v = t^3 - 12t^2 + 32t \Rightarrow s = \tfrac{1}{4}t^4 - 4t^3 + 16t^2 + c.$$

Now,

$$s = 0, t = 0 \Rightarrow 0 = 0 + 0 + 0 + c$$

and

$$\underline{\underline{s = \tfrac{1}{4}t^4 - 4t^3 + 16t^2}}.$$

(c) Show that the body returns to O, and obtain the time, T, when this happens. (2)

Solution

$$\begin{aligned} \tfrac{1}{4}t^4 - 4t^3 + 16t^2 = 0 &\Rightarrow \tfrac{1}{4}t^2(t^2 - 16t + 64) = 0 \\ &\Rightarrow \tfrac{1}{4}t^2(t-8)^2 = 0 \\ &\Rightarrow t = 0 \text{ or } \underline{\underline{t = 8}}. \end{aligned}$$

11. For each of the following statements, decide whether it is true or false and prove your conclusion.

(a) For all natural numbers m, if m^2 is divisible by 4, then m is divisible by 4. (2)

Solution

It is <u>false</u>: 4 is divisible by 4, but m is not divisible by 4.

(b) The cube of any odd integer p plus the square of any even integer q is always odd. (3)

Solution

It is <u>true</u>: p^3 is clearly odd (odd × odd × odd = odd), q^2 is clearly even (even × even = even), and the sum of an odd an even pair of numbers is odd (even + odd = odd).

12. Throughout this question, it can be assumed that $-2 < x < 2$.

(a) Obtain the first three non-zero terms in the Maclaurin expansion of (3)

$$x \ln(2 + x).$$

Solution

Let $f(x) = \ln(2 + x)$. Then

$$f'(x) = \frac{1}{2+x} \text{ and } f''(x) = -\frac{1}{(2+x)^2}.$$

Next,

$$f(x) = \ln 2, f'(x) = \tfrac{1}{2}, \text{ and } f''(x) = -\tfrac{1}{4}.$$

Now,

$$\begin{aligned} \text{Maclaurin expansion} &= x\left[\ln 2 + (\tfrac{1}{2} \times x) + \tfrac{1}{2!}x^2(-\tfrac{1}{4}) + \ldots\right] \\ &= \underline{\underline{x \ln 2 + \tfrac{1}{2}x^2 - \tfrac{1}{8}x^3 + \ldots}}. \end{aligned}$$

(b) Hence, or otherwise, deduce the first three non-zero terms in the Maclaurin expansion of (2)

$$x \ln(2 - x).$$

Solution

$$\begin{aligned}
\text{Maclaurin expansion} &= x \ln[2 + (-x)] \\
&= x\left[\ln 2 + \tfrac{1}{2} \times (-x) + \tfrac{1}{2!}(-x)^2(-\tfrac{1}{4}) + \ldots\right] \\
&= x\left[\ln 2 - \tfrac{1}{2} - \tfrac{1}{8}x^2 + \ldots\right] \\
&= \underline{\underline{x \ln 2 - \tfrac{1}{2}x^2 - \tfrac{1}{8}x^3 + \ldots}}.
\end{aligned}$$

(c) Hence obtain the first **two** non-zero terms in the Maclaurin expansion of (2)

$$x \ln(4 - x^2).$$

Solution

$$\begin{aligned}
x \ln(4 - x^2) &= x \ln[(2 + x)(2 - x)] \\
&= x \ln(2 + x) + x \ln(2 - x) \\
&= (x \ln 2 + \tfrac{1}{2}x^2 - \tfrac{1}{8}x^3 + \ldots) + (x \ln 2 - \tfrac{1}{2}x^2 - \tfrac{1}{8}x^3 + \ldots) \\
&= \underline{\underline{2x \ln 2 - -\tfrac{1}{4}x^3 + \ldots}}.
\end{aligned}$$

13. (a) Obtain the general solution of the differential equation (7)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 2x^2.$$

Solution

Complementary function:

$$m^2 - 3m + 2 = 0 \Rightarrow (m - 2)(m - 1) = 0 \Rightarrow m = 1, 2$$

and hence the complementary function is

$$y = A\mathrm{e}^x + Be^{2x}.$$

Particular integral: try

$$y = Cx^2 + Dx + E \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2Cx + D \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2C.$$

Substitute into the differential equation:

$$2C - 3(2Cx + D) + 2(Cx^2 + Dx + E) \equiv 2x^2.$$

x^2: $2C = 2 \Rightarrow C = 1$.
x: $-6C + 2D = 0 \Rightarrow 2D = 6 \Rightarrow D = 3$.
Constant term: $2C - 3D + 2E = 0 \Rightarrow 2 - 9 + 2E = 0 \Rightarrow E = \frac{7}{2}$.
Hence the particular integral is $y = x^2 + 3x + \frac{7}{2}$.

General solution: hence the general solution is

$$\underline{\underline{y = A\mathrm{e}^x + Be^{2x} + x^2 + 3x + \tfrac{7}{2}}}.$$

(b) Given that $y = \frac{1}{2}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$, when $x = 0$, find the particular solution. (3)

Solution

$$\begin{aligned} x = 0, y = \tfrac{1}{2} &\Rightarrow A + B + 0 + 0 + \tfrac{7}{2} = \tfrac{1}{2} \\ &\Rightarrow A + B = -3. \end{aligned}$$

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = A\mathrm{e}^x + 2Be^{2x} + 2x + 3$$

and

$$\begin{aligned} x = 0, \frac{\mathrm{d}y}{\mathrm{d}x} = 1 &\Rightarrow A + 2B + 0 + 3 = 1 \\ &\Rightarrow A + 2B = -2. \end{aligned}$$

Solve:

$$B = 1 \text{ and } A = -4.$$

Hence,

$$\underline{\underline{y = -4\mathrm{e}^x + e^{2x} + x^2 + 3x + \tfrac{7}{2}}}.$$

14. (a) Find an equation of the plane π_1 through the points $A(1, 1, 1)$, $B(2, -1, 1)$, and (3)

$C(0,3,3)$.

Solution
$\overrightarrow{AB} = \mathbf{i} - 2\mathbf{j}$ and $\overrightarrow{AC} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. Now,

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 0 \\ -1 & 2 & 2 \end{vmatrix} \\ &= -4\mathbf{i} - 2\mathbf{j}.\end{aligned}$$

The equation is

$$\begin{aligned}-4x - 2y = [(-4) \times 1] + [(-2) \times 1] &\Rightarrow -4x - 2y = -6 \\ &\Rightarrow 4x + 2y = 6 \\ &\Rightarrow \underline{\underline{2x + y = 3}}.\end{aligned}$$

The plane π_2 has equation $x + 3y - z = 2$.

(b) Given that the point $(0, a, b)$ lies on both the planes π_1 and π_2, find the values of a and b. (3)

Solution
Make $(0, a, b)$ the point on the line: from $2x + y = 3$ we get $\underline{\underline{a = 3}}$ and from $x + 3y - z = 2$ we get

$$9 - b = 2 \Rightarrow \underline{\underline{b = 7}}.$$

(c) Hence find an equation of the line of intersection of the planes π_1 and π_2. (1)

Solution
The line of intersection is parallel to

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 1 & 3 & -1 \end{vmatrix} = -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$$

and an equation of the line of intersection is

$$\underline{\underline{\frac{x}{-1} = \frac{y-3}{2} = \frac{z-7}{5}}}.$$

(d) Find the size of the acute angle between the planes π_1 and π_2. (3)

Solution

$$\cos\theta = \left|\frac{(2\mathbf{i}+\mathbf{j}).(\mathbf{i}+3\mathbf{j}-\mathbf{k})}{\sqrt{2^2+1^2}\cdot\sqrt{1+2+3^2+1^2}}\right| \Rightarrow \cos\theta = \left|\frac{5}{\sqrt{5}\cdot\sqrt{11}}\right|$$
$$\Rightarrow \theta = 47.607\,954\,29 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\theta = 47.6^\circ \text{ (1 dp)}}}.$$

15. Let

$$f(x) = \frac{x}{\ln x}$$

for $x > 1$.

(a) Derive expressions for $f'(x)$ and $f''(x)$, simplifying your answers. (4)

Solution

$$\begin{aligned} f'(x) &= \frac{(\ln x \times 1) - (\frac{1}{x} \times x)}{(\ln x)^2} \\ &= \underline{\underline{\frac{\ln x - 1}{(\ln x)^2}}} \end{aligned}$$

and

$$\begin{aligned} f''(x) &= \frac{[(\ln x)^2 \times \frac{1}{x}] - [(\ln x - 1) \times \frac{2\ln x}{x}]}{(\ln x)^4} \\ &= \frac{\ln x - 2(\ln x - 1)}{x(\ln x)^3} \\ &= \underline{\underline{\frac{2 - \ln x}{x(\ln x)^3}}}. \end{aligned}$$

(b) Obtain the coordinates and nature of the stationary point of the curve $y = f(x)$. (3)

Solution

$$\begin{aligned} f'(x) = 0 &\Rightarrow \frac{\ln x - 1}{(\ln x)^2} = 0 \\ &\Rightarrow \ln x - 1 = 0 \\ &\Rightarrow \ln x = 1 \\ &\Rightarrow x = e \\ &\Rightarrow y = e. \end{aligned}$$

Now,

$$\begin{aligned} f''(x) &= \frac{2 - \ln e}{e(\ln e)^3} \\ &= \frac{2-1}{e} \\ &> 0. \end{aligned}$$

So, $\underline{\underline{(e, e)}}$ is a $\underline{\underline{\text{minimum}}}$ point.

(c) Obtain the coordinates of the point of inflexion. (2)

Solution

$$\begin{aligned} f''(x) = 0 &\Rightarrow \frac{2 - \ln x}{x(\ln x)^3} = 0 \\ &\Rightarrow \ln x = 2 \\ &\Rightarrow x = e^2 \\ &\Rightarrow y = e^2; \end{aligned}$$

hence, $\underline{\underline{(e^2, e^2)}}$ is the point of inflexion.

16. (a) Given $z = \cos\theta + i\sin\theta$, use de Moivre's theorem to write down an expression for z^k in terms of θ, where k is a positive integer. (1)

Solution

$$z^k = \underline{\underline{\cos k\theta + i\sin k\theta}}.$$

(b) Hence show that (2)

$$\frac{1}{z^k} = \cos k\theta - i\sin k\theta.$$

Solution

$$\begin{aligned}\frac{1}{z^k} &= \frac{1}{\cos k\theta + \mathrm{i}\sin k\theta}\\ &= \frac{1}{\cos k\theta + \mathrm{i}\sin k\theta} \times \frac{\cos k\theta - \mathrm{i}\sin k\theta}{\cos k\theta - \mathrm{i}\sin k\theta}\\ &= \frac{\cos k\theta - \mathrm{i}\sin k\theta}{\cos^2 k\theta + \sin^2 k\theta}\\ &= \underline{\underline{\cos k\theta - \mathrm{i}\sin k\theta}},\end{aligned}$$

as required.

(c) Deduce expressions for $\cos k\theta$ and $\sin k\theta$ in terms of z. (2)

Solution

Add:

$$z^k + \frac{1}{z^k} = 2\cos k\theta \Rightarrow \cos k\theta = \underline{\underline{\tfrac{1}{2}\left(z^k + \frac{1}{z^k}\right)}}.$$

Subtract:

$$z^k - \frac{1}{z^k} = 2i\sin k\theta \Rightarrow \sin k\theta = \underline{\underline{\tfrac{1}{2\mathrm{i}}\left(z^k - \frac{1}{z^k}\right)}}.$$

(d) Show that (3)

$$\cos^2\theta\sin^2\theta = -\tfrac{1}{16}\left(z^2 - \frac{1}{z^2}\right)^2.$$

Solution

$$\begin{aligned}\cos^2\theta\sin^2\theta &= (\cos\theta\sin\theta)^2\\ &= \left[\tfrac{1}{2}\left(z + \frac{1}{z}\right)\cdot\tfrac{1}{2\mathrm{i}}\left(z - \frac{1}{z}\right)\right]^2\\ &= \left[\tfrac{1}{4\mathrm{i}}\left(z^2 - \frac{1}{z^2}\right)\right]^2\\ &= \underline{\underline{-\tfrac{1}{16}\left(z^2 - \frac{1}{z^2}\right)^2}},\end{aligned}$$

as required.

(e) Hence show that (2)

$$\cos^2\theta\sin^2\theta = a + b\cos 4\theta,$$

for suitable constants a and b.

Solution

$$\begin{aligned}\cos^2\theta\sin^2\theta &= -\tfrac{1}{16}\left(z^2 - \frac{1}{z^2}\right)^2 \\ &= -\tfrac{1}{16}\left(z^4 - 2 + \frac{1}{z^4}\right) \\ &= -\tfrac{1}{8}\left[\tfrac{1}{2}\left(z^4 + \frac{1}{z^4}\right) - 1\right] \\ &= \underline{\underline{\tfrac{1}{8} - \tfrac{1}{8}\cos 4\theta}};\end{aligned}$$

hence, $\underline{\underline{a = \tfrac{1}{8}}}$ and $\underline{\underline{b = -\tfrac{1}{8}}}$.