Dr Oliver Mathematics

The Standard Normal Distribution

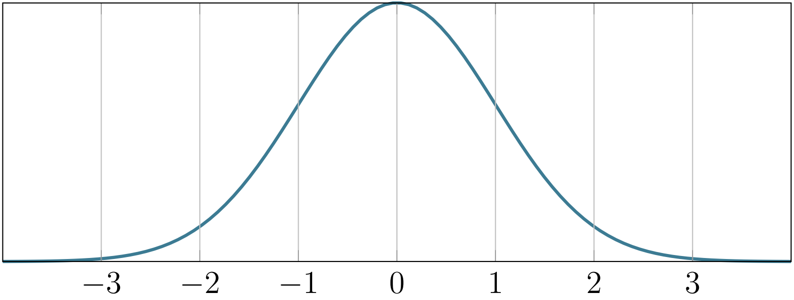
April 2014

The Equation of the Standard Normal Distribution

Let

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2},$$

where $e = 2.718\,281\ldots$. Then the function $f(z)$ has the following graph:



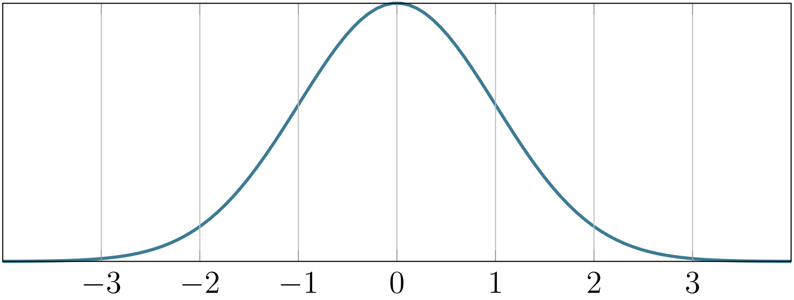
The Notation for the Standard Normal Distribution

We use

$$Z \sim \mathrm{N}(0, 1)$$

to represent the standard normal distribution. The standard normal distribution has a mean of 0 and a standard deviation of 1.

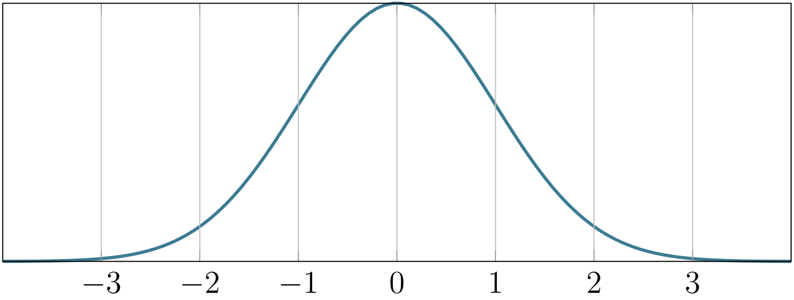
Properties of the Standard Normal Distribution



Property 1

The mean, median, and mode of the standard normal distribution are all 0.

Properties of the Standard Normal Distribution



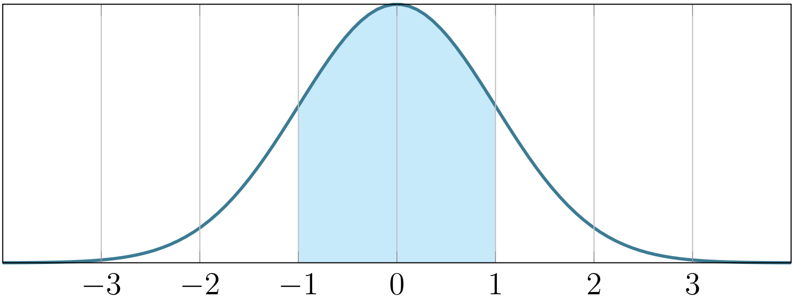
Property 1

The mean, median, and mode of the standard normal distribution are all 0.

Property 2

The standard normal distributions is symmetrical about its mean, median, and mode.

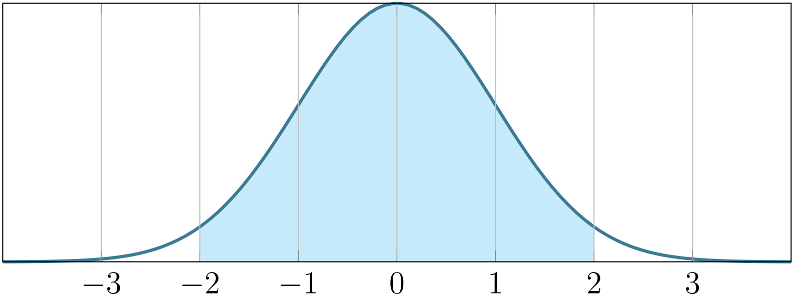
Properties of the Standard Normal Distribution



Property 3

Approximately 68.3% of the distribution lies within one standard deviation of the mean, median, and mode.

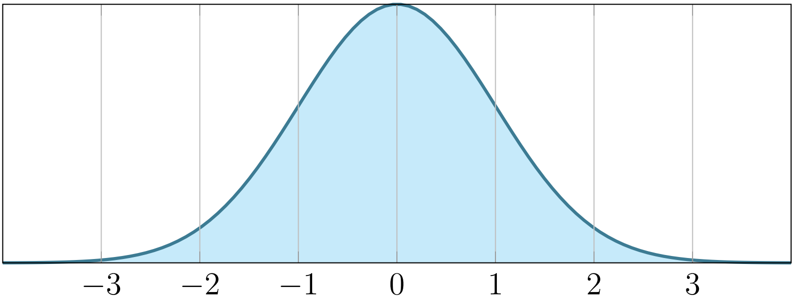
Properties of the Standard Normal Distribution



Property 4

Approximately 95.4% of the distribution lies within two standard deviations of the mean, median, and mode.

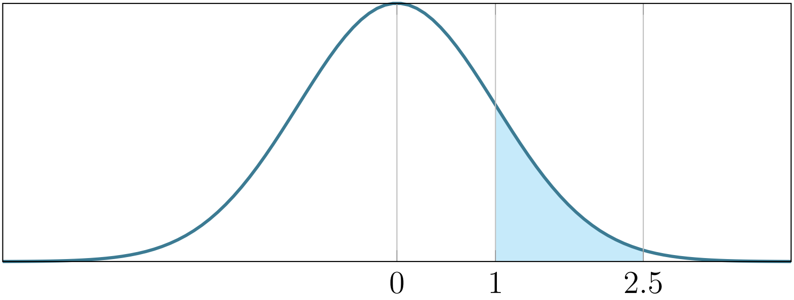
Properties of the Standard Normal Distribution



Property 5

Approximately 99.7% of the distribution lies within three standard deviations of the mean, median, and mode.

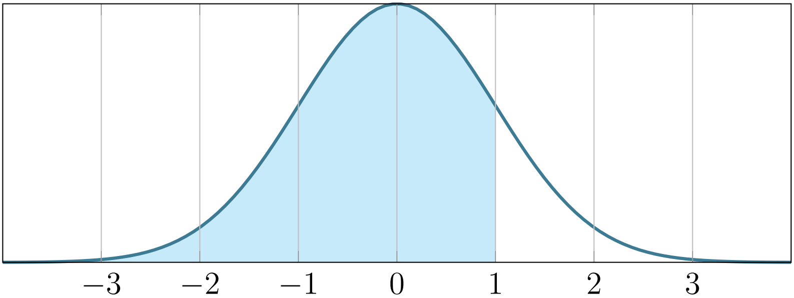
Properties of the Standard Normal Distribution



Property 6

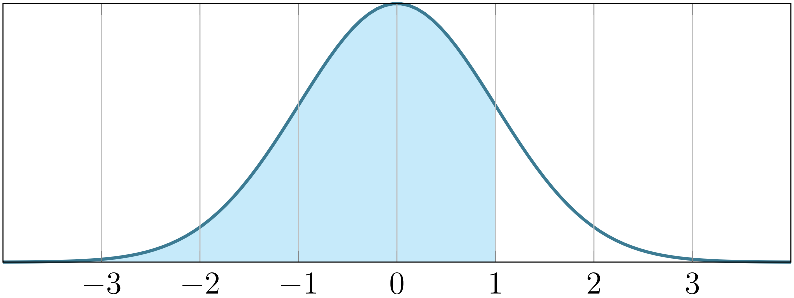
The total area under the graph is exactly 1. In the above picture, the shaded area represents $P(1 < Z < 2.5)$.

Calculating the Probability



We use tables of values: see page 201 of the textbook.

Calculating the Probability



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1. The area under the graph represents the probability.

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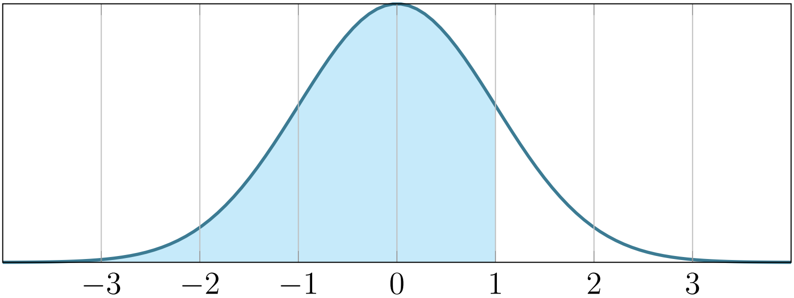
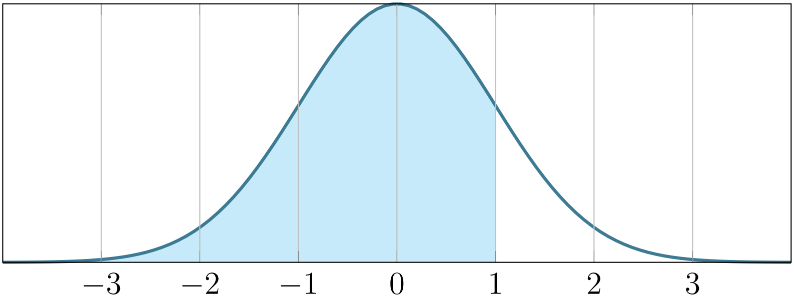


We use tables of values: see page 201 of the textbook.

1. The area under the graph represents the probability.
2. The function that we use is $\Phi(z) = \mathrm{P}(Z < z)$.

Calculating the Probability



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1. The area under the graph represents the probability.
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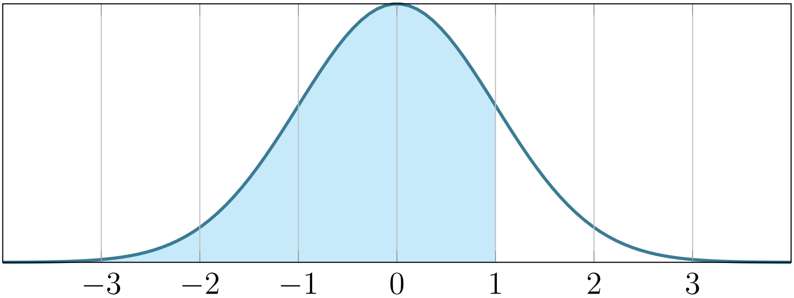
3. Using the tables, $\Phi(1) = 0.8413$.

Calculating the Probability



We use tables of values: see page 201 of the textbook.

1. The area under the graph represents the probability.
2. The function that we use is $\Phi(z) = \mathrm{P}(Z < z)$.
3. Using the tables, $\Phi(1) = 0.8413$.
4. Hence $\mathrm{P}(Z < 1) = 0.8413$.

Examples of the Φ Function

$$\Phi(1.59) =$$

$$\Phi(0.86) =$$

$$\Phi(0.33) =$$

$$\Phi(2.70) =$$

$$\Phi(0.52) =$$

$$\Phi(0.19) =$$

$$\Phi(-1.26) =$$

Examples of the Φ Function

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Examples of the Φ Function

$$\Phi(1.59) = 0.9941 \qquad \Phi(0.86) = 0.8051$$

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Examples of the Φ Function

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The Φ Function for Negative Values

$$\Phi(-1.26) = \mathrm{P}(Z < -1.26).$$

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and so

$$\begin{aligned}\Phi(-1.26) &= 1 - \Phi(1.26) \\ &= 1 - 0.8962 \\ &= \underline{\underline{0.1038}}.\end{aligned}$$

The Φ Function for Negative Values

Using the same idea as in the previous example we can show that

$$\Phi(-z) = 1 - \Phi(z),$$

for all values of z.

The Inverse Φ Function

$$\Phi(z) = 0.7967 \Rightarrow z =$$

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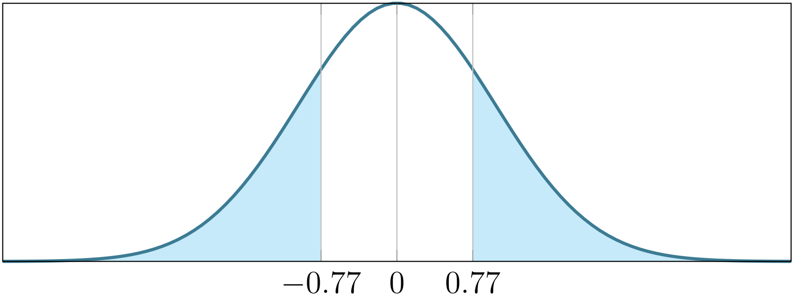
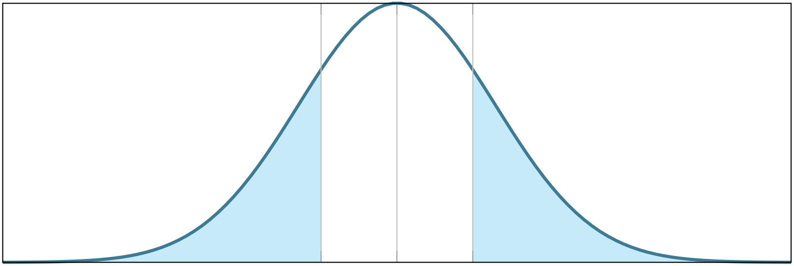
$\Phi(z) = 0.2206 \Rightarrow z = -0.77$



$$\Phi(z) = 0.2206$$

$\Phi(z) = 0.2206 \Rightarrow z = -0.77$



$$\Phi(z) = 0.2206 \Rightarrow \mathrm{P}(Z < z) = 0.2206$$

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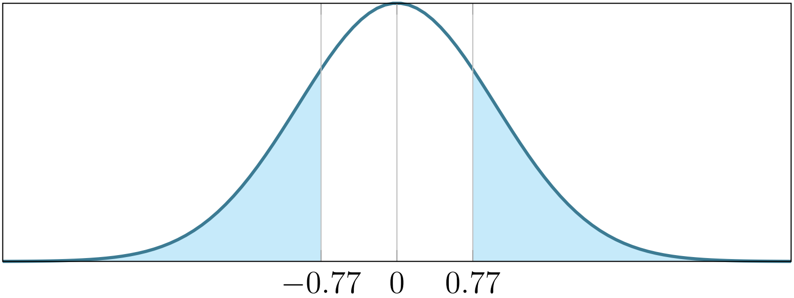

$$\Phi(z) = 0.2206 \Rightarrow \mathrm{P}(Z < z) = 0.2206$$
$$\Rightarrow \mathrm{P}(Z > -z) = 0.2206 \text{ (by symmetry)}$$

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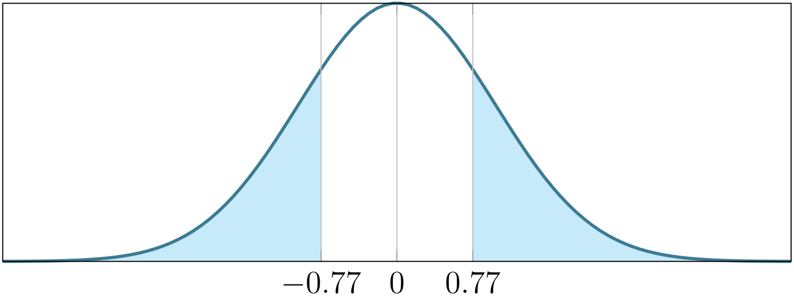

$$\begin{aligned}\Phi(z) = 0.2206 &\Rightarrow \mathrm{P}(Z < z) = 0.2206 \\ &\Rightarrow \mathrm{P}(Z > -z) = 0.2206 \text{ (by symmetry)} \\ &\Rightarrow \mathrm{P}(Z < -z) = 0.7794\end{aligned}$$

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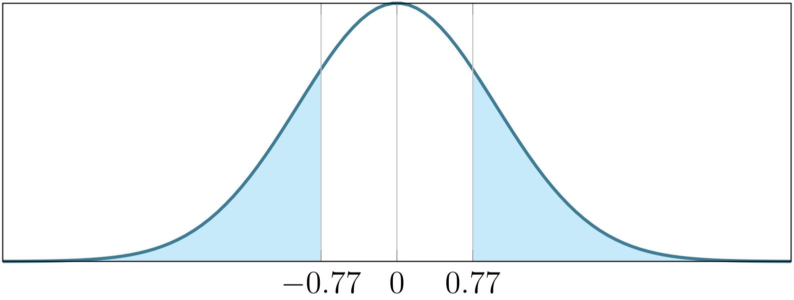

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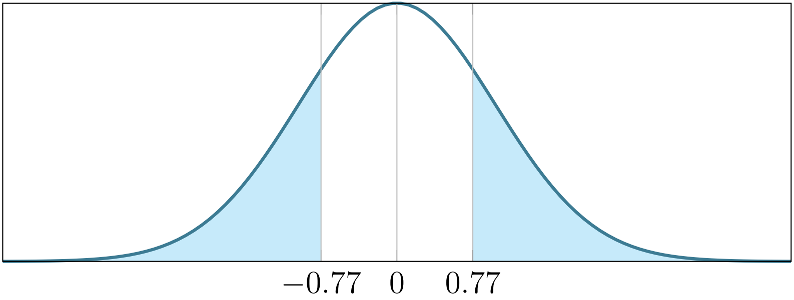

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&\Rightarrow -z = 0.77 \\
&\Rightarrow \underline{\underline{z = -0.77}}.
\end{aligned}
$$

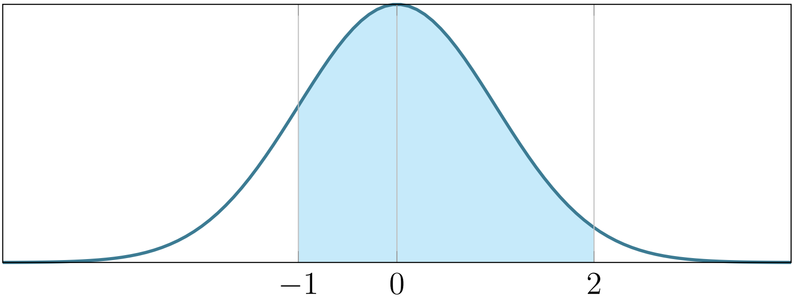
Since the probability is connected with an area, this means that

$$P(Z = z) = 0$$

for any particular value of z. In addition, it does not matter if inequalities are strict or inclusive:

$$\begin{aligned} P(a < Z < b) &= P(a \le Z < b) \\ &= P(a < Z \le b) \\ &= P(a \le Z \le b). \end{aligned}$$

Example: Find $P(-1 < Z < 2)$



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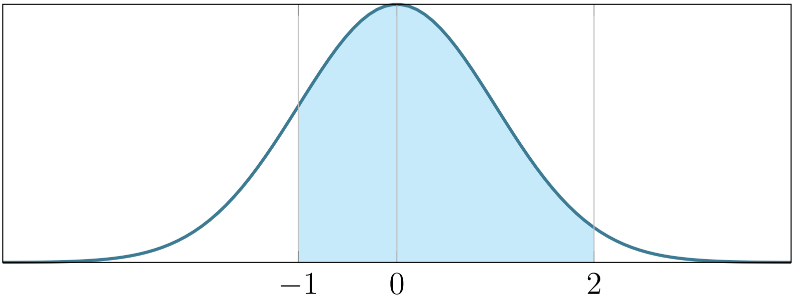
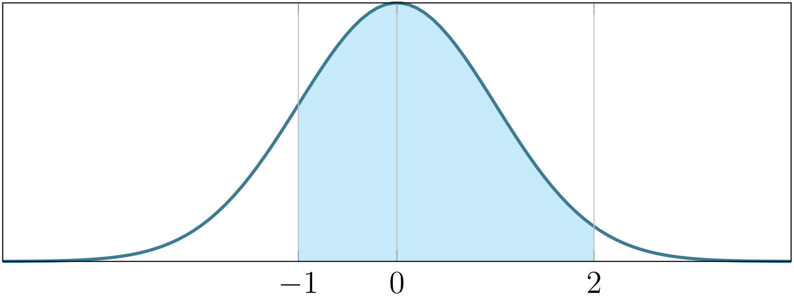


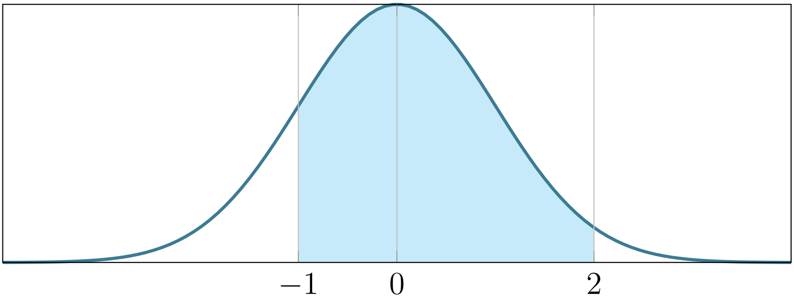
$$P(-1 < Z < 2) = \Phi(2) - \Phi(-1)$$

Example: Find P$(-1 < Z < 2)$



$$\begin{aligned} \mathrm{P}(-1 < Z < 2) &= \Phi(2) - \Phi(-1) \\ &= \Phi(2) - [1 - \Phi(1)] \end{aligned}$$

Example: Find $P(-1 < Z < 2)$



$$
\begin{aligned}
P(-1 < Z < 2) &= \Phi(2) - \Phi(-1) \\
&= \Phi(2) - [1 - \Phi(1)] \\
&= 0.9772 - [1 - 0.8413]
\end{aligned}
$$

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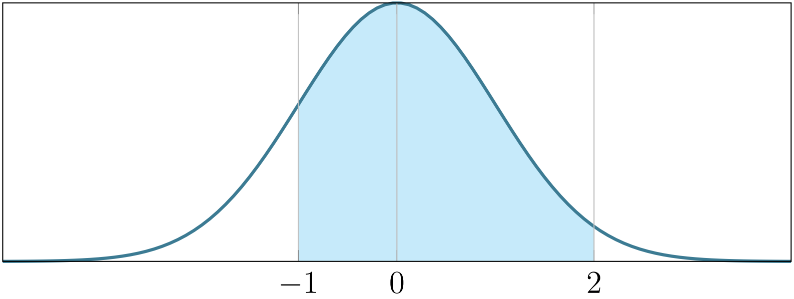
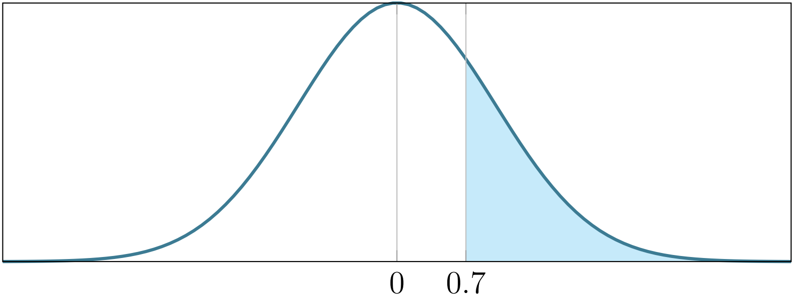


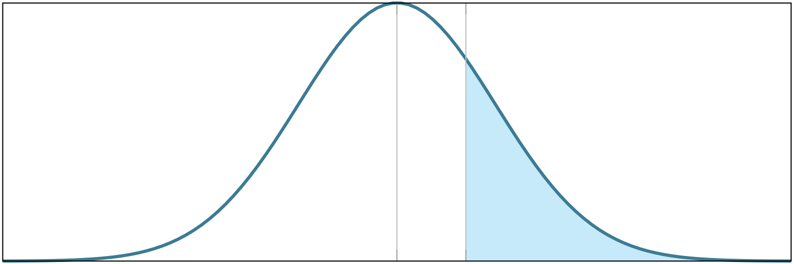
$$\begin{aligned} \mathrm{P}(-1 < Z < 2) &= \Phi(2) - \Phi(-1) \\ &= \Phi(2) - [1 - \Phi(1)] \\ &= 0.9772 - [1 - 0.8413] \\ &= \underline{\underline{0.8185}}. \end{aligned}$$

Example: Find $\mathrm{P}(Z > 0.7)$



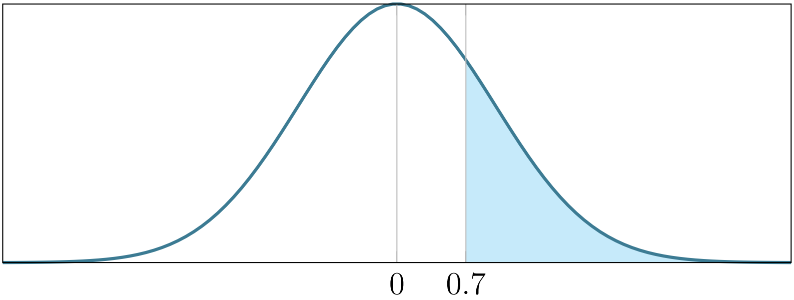
$$\mathrm{P}(Z > 0.7) =$$

Example: Find $P(Z > 0.7)$



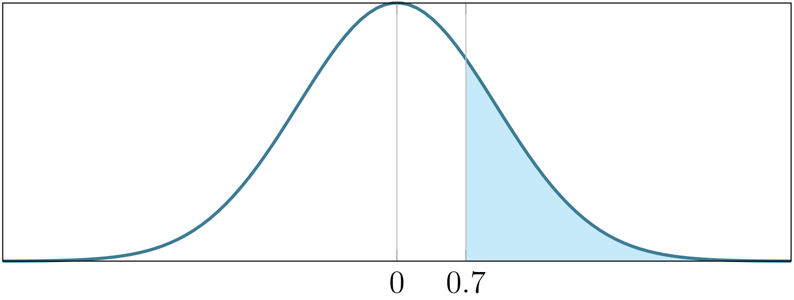
$$P(Z > 0.7) = 1 - P(Z \leq 0.7)$$

Example: Find $P(Z > 0.7)$



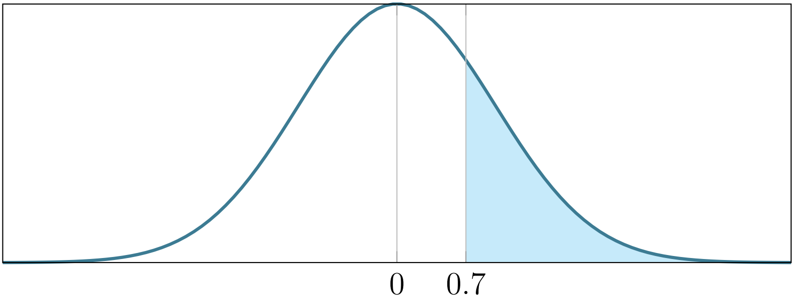
$$\begin{aligned} P(Z > 0.7) &= 1 - P(Z \leq 0.7) \\ &= 1 - \Phi(0.7) \end{aligned}$$

Example: Find $\mathrm{P}(Z > 0.7)$



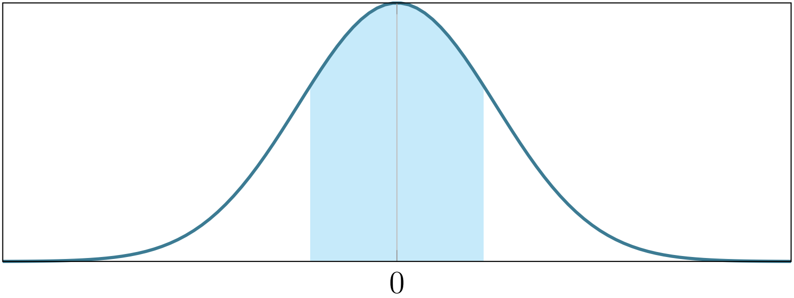
$$
\begin{aligned}
\mathrm{P}(Z > 0.7) &= 1 - \mathrm{P}(Z \leq 0.7) \\
&= 1 - \Phi(0.7) \\
&= 1 - 0.7580
\end{aligned}
$$

Example: Find $\mathrm{P}(Z > 0.7)$



$$\begin{aligned}\mathrm{P}(Z > 0.7) &= 1 - \mathrm{P}(Z \leq 0.7) \\ &= 1 - \Phi(0.7) \\ &= 1 - 0.7580 \\ &= \underline{\underline{0.2420}}.\end{aligned}$$

Example: If $\mathrm{P}(-z < Z < z) = 0.6212$, find z



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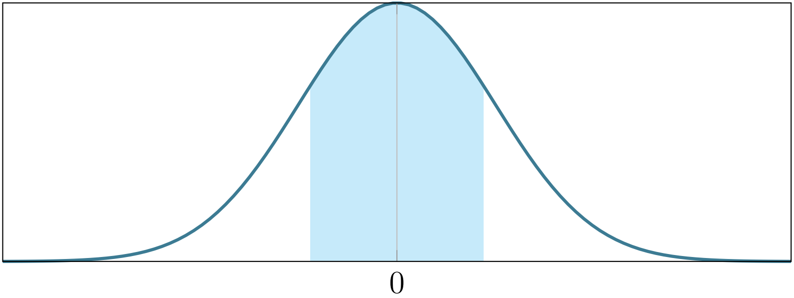

$$\mathrm{P}(-z < Z < z) = 0.6212 \Rightarrow \mathrm{P}(0 < Z < z) = 0.3106$$

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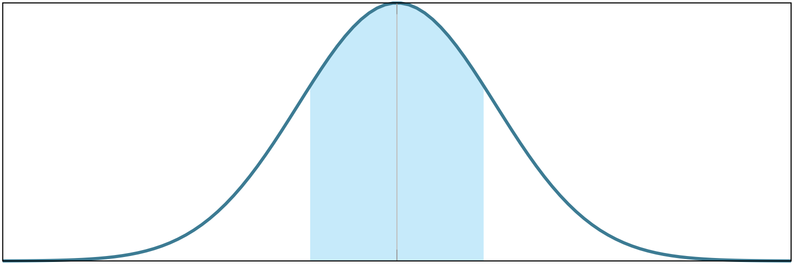
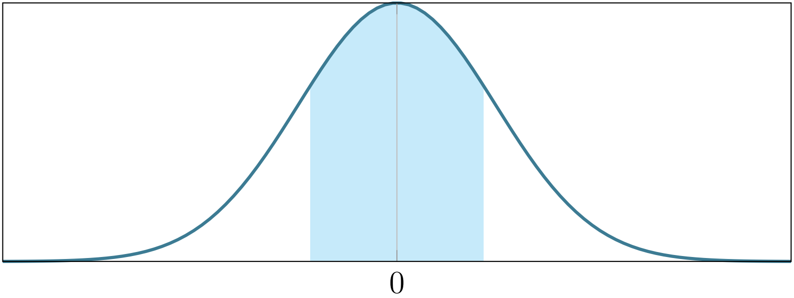


$$\mathrm{P}(-z < Z < z) = 0.6212 \Rightarrow \mathrm{P}(0 < Z < z) = 0.3106$$
$$\Rightarrow \mathrm{P}(Z < z) = 0.8106$$

Example: If $\mathrm{P}(-z < Z < z) = 0.6212$, find z



$$\begin{aligned}
\mathrm{P}(-z < Z < z) = 0.6212 &\Rightarrow \mathrm{P}(0 < Z < z) = 0.3106 \\
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&\Rightarrow \Phi(z) = 0.8106
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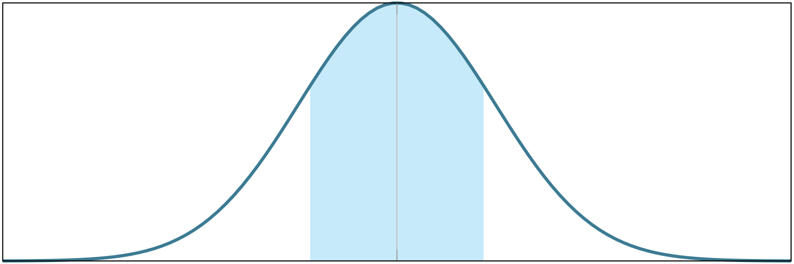
Example: If $\mathrm{P}(-z < Z < z) = 0.6212$, find z



$$
\begin{aligned}
\mathrm{P}(-z < Z < z) = 0.6212 &\Rightarrow \mathrm{P}(0 < Z < z) = 0.3106 \\
&\Rightarrow \mathrm{P}(Z < z) = 0.8106 \\
&\Rightarrow \Phi(z) = 0.8106 \\
&\Rightarrow \underline{\underline{z = 0.88}}.
\end{aligned}
$$

Percentage Points of the Standard Normal Distribution

$P(Z > z) = p$	z	$P(Z > z) = p$	z
0.5000	0.0000	0.0500	1.6449
0.4000	0.2533	0.0250	1.9600
0.3000	0.5244	0.0100	2.3263
0.2000	0.8416	0.0050	2.5758
0.1500	1.0364	0.0010	3.0902
0.1000	1.2816	0.0005	3.2905

This table will allow us to calculate the key z values where certain percentages of the distribution lie either above or below a certain value.

Percentage Points of the Standard Normal Distribution

$P(Z > z) = p$	z	$P(Z > z) = p$	z
0.5000	0.0000	0.0500	1.6449
0.4000	0.2533	0.0250	1.9600
0.3000	0.5244	0.0100	2.3263
0.2000	0.8416	0.0050	2.5758
0.1500	1.0364	0.0010	3.0902
0.1000	1.2816	0.0005	3.2905

So, for example, the highlighted entries tell us that

$$P(Z > 0.8416) = 0.2000,$$

in other words, there is only a 20% change of getting an outcome at least as great as 0.8416 from the standard normal distribution.

Percentage Points of the Standard Normal Distribution

$\text{P}(Z > z) = p$	z	$\text{P}(Z > z) = p$	z
0.5000	0.0000	0.0500	1.6449
0.4000	0.2533	0.0250	1.9600
0.3000	0.5244	0.0100	2.3263
0.2000	0.8416	0.0050	2.5758
0.1500	1.0364	0.0010	3.0902
0.1000	1.2816	0.0005	3.2905

What do these highlighted values mean?

Percentage Points of the Standard Normal Distribution

$P(Z > z) = p$	z	$P(Z > z) = p$	z
0.5000	0.0000	0.0500	1.6449
0.4000	0.2533	0.0250	1.9600
0.3000	0.5244	0.0100	2.3263
0.2000	0.8416	0.0050	2.5758
0.1500	1.0364	0.0010	3.0902
0.1000	1.2816	0.0005	3.2905

What do these highlighted values mean?

$$P(Z > 1.9600) = 0.0250,$$

in other words, there is only a 2.5% change of getting an outcome at least as great as 1.9600 from the standard normal distribution.

And over to you . . .

Exercise 9A (page 179)

Q1-4

Exercise 9B (page 181)

Q1-5

- Show all of your working.
- If it helps, draw a sketch of a normal distribution.
- Use the tables on pages 201 and 202.