Dr Oliver Mathematics
Mathematics: Advanced Higher
2011 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. Express (5)

$$\frac{13-x}{x^2+4x-5}$$

in partial fractions and hence obtain

$$\int \left(\frac{13-x}{x^2+4x-5}\right) \, \mathrm{d}x.$$

Solution

$$\begin{aligned}\frac{13-x}{x^2+4x-5} &\equiv \frac{13-x}{(x+5)(x-1)} \\ &\equiv \frac{A}{x+5} + \frac{B}{x-1} \\ &\equiv \frac{A(x-1)+B(x+5)}{(x+5)(x-1)}\end{aligned}$$

and hence

$$13-x \equiv A(x-1)+B(x+5).$$

$\underline{x=1}$: $12 = 6B \Rightarrow B = 2$.
$\underline{x=-5}$: $18 = -6A \Rightarrow A = -3$.
So,

$$\frac{13-x}{x^2+4x-5} \equiv \underline{\underline{-\frac{3}{x+5} + \frac{2}{x-1}}}$$

and

$$\begin{aligned}\int \left(\frac{13-x}{x^2+4x-5}\right) \mathrm{d}x &= \int \left(-\frac{3}{x+5} + \frac{2}{x-1}\right) \mathrm{d}x \\ &= \underline{\underline{-3\ln|x+5| + 2\ln|x-1| + c}}.\end{aligned}$$

2. Use the binomial theorem to expand (3)

$$\left(\tfrac{1}{2}x-3\right)^4$$

and simplify your answer.

Solution

$$\begin{aligned}\left(\tfrac{1}{2}x-3\right)^4 &= \left(\tfrac{1}{2}x\right)^4 + 4\left(\tfrac{1}{2}x\right)^3(-3) + +6\left(\tfrac{1}{2}x\right)^2(-3)^2 + 4\left(\tfrac{1}{2}x\right)(-3)^3 + (-3)^4\\ &= \underline{\underline{\tfrac{1}{16}x^4 - \tfrac{3}{2}x^2 + \tfrac{27}{2}x^2 - 54x + 81}}.\end{aligned}$$

3. (a) Obtain $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when y is defined as a function of x by the equation (3)

$$y + \mathrm{e}^y = x^2.$$

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x} = 2x &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}(1+\mathrm{e}^y) = 2x\\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{1+\mathrm{e}^y}}}.\end{aligned}$$

(b) Given (3)

$$\mathrm{f}(x) = \sin x\cos^3 x,$$

obtain $\mathrm{f}'(x)$.

Solution

$$\begin{aligned}\mathrm{f}'(x) &= \sin x\frac{\mathrm{d}}{\mathrm{d}x}(\cos^3 x) + \frac{\mathrm{d}}{\mathrm{d}x}(\sin x)\cos^3 x\\ &= \sin x(-3\cos^2 x\sin x) + (\cos x)\cos^3 x\\ &= \underline{\underline{-3\cos^2 x\sin^2 x + \cos^4 x}}.\end{aligned}$$

4. (a) For what value of λ is (3)

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ -1 & \lambda & 6 \end{pmatrix}$$

singular?

Solution

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ -1 & \lambda & 6 \end{vmatrix} = 0 \Rightarrow 1(0 - 2\lambda) - 2(18 + 2) + (-1)(3\lambda - 0) = 0$$
$$\Rightarrow -5\lambda = 40$$
$$\Rightarrow \underline{\underline{\lambda = -8}}.$$

(b) For (3)

$$\mathbf{A} = \begin{pmatrix} 2 & 2\alpha - \beta & -1 \\ 3\alpha + 2\beta & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix},$$

obtain values of α and β such that

$$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 2 & -5 & -1 \\ -1 & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}.$$

Solution

If

$$\mathbf{A} = \begin{pmatrix} 2 & 2\alpha - \beta & -1 \\ 3\alpha + 2\beta & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix},$$

then

$$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 2 & 3\alpha + 2\beta & -1 \\ 2\alpha - \beta & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}.$$

Hence

$$3\alpha + 2\beta = -5 \quad (1)$$
$$2\alpha - \beta = -1 \quad (2).$$

Now, do (1) + 2 × (2):

$$7\alpha = -7 \Rightarrow \underline{\underline{\alpha = -1}}.$$

$$\Rightarrow -2 - \beta = -1$$
$$\Rightarrow \underline{\underline{\beta = -1}}.$$

5. (a) Obtain the first four terms in the Maclaurin series of (4)

$$\sqrt{1+x},$$

and hence write down the first four terms in the Maclaurin series of

$$\sqrt{1+x^2}.$$

Solution

$$\begin{aligned}
y &= (1+x)^{\frac{1}{2}} \Rightarrow x = 0, y = 1 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \tfrac{1}{2}(1+x)^{-\frac{1}{2}} \Rightarrow x = 0, \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{2} \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\tfrac{1}{4}(1+x)^{-\frac{3}{2}} \Rightarrow x = 0, \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\tfrac{1}{4} \\
\frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= \tfrac{3}{8}(1+x)^{-\frac{5}{2}} \Rightarrow x = 0, \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \tfrac{3}{8}
\end{aligned}$$

and

$$\begin{aligned}
\sqrt{1+x} &= (1+x)^{\frac{1}{2}} \\
&= 1 + \tfrac{1}{2}x + (-\tfrac{1}{4})\left(\frac{1}{2!}\right)x^2 + (\tfrac{3}{8})\left(\frac{1}{3!}\right)x^3 + \ldots \\
&= \underline{\underline{1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 + \ldots}}.
\end{aligned}$$

So,

$$\begin{aligned}
\sqrt{1+x^2} &= \sqrt{1+(x^2)} \\
&= 1 + \tfrac{1}{2}(x^2) - \tfrac{1}{8}(x^2)^2 + \tfrac{1}{16}(x^2)^3 + \ldots \\
&= \underline{\underline{1 + \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 + \tfrac{1}{16}x^6 + \ldots}}.
\end{aligned}$$

(b) Hence obtain the first four terms in the Maclaurin series of (2)

$$\sqrt{(1+x)(1+x^2)}.$$

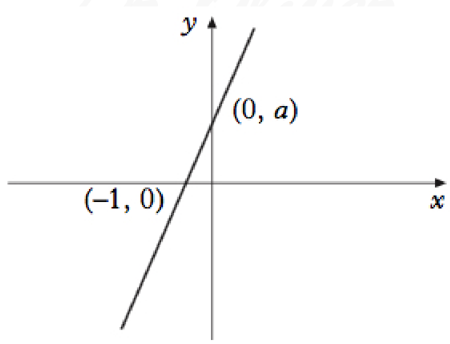
Solution

$$\begin{aligned}\sqrt{(1+x)(1+x^2)} &= \sqrt{1+x}\cdot\sqrt{1+x^2}\\ &= (1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3+\ldots)(1+\tfrac{1}{2}x^2+\ldots)\end{aligned}$$

$\times$	1	$+\frac{1}{2}x$	$-\frac{1}{8}x^2$	$+\frac{1}{16}x^3$
1	1	$+\frac{1}{2}x$	$-\frac{1}{8}x^2$	$+\frac{1}{16}x^3$
$+\frac{1}{2}x^2$	$+\frac{1}{2}x^2$	$+\frac{1}{4}x^3$	$\ldots$	$\ldots$

$$= \underline{\underline{1+\tfrac{1}{2}x+\tfrac{3}{8}x^2+\tfrac{5}{16}x^3+\ldots}}.$$

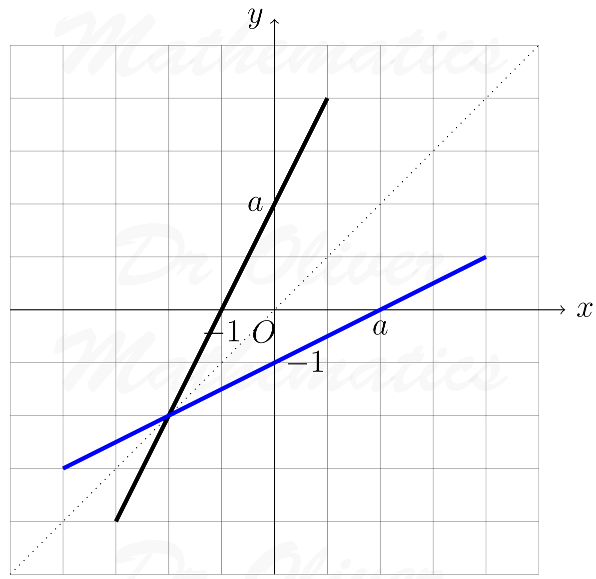
6. The diagram shows part of the graph of a function $\mathrm{f}(x)$. (4)



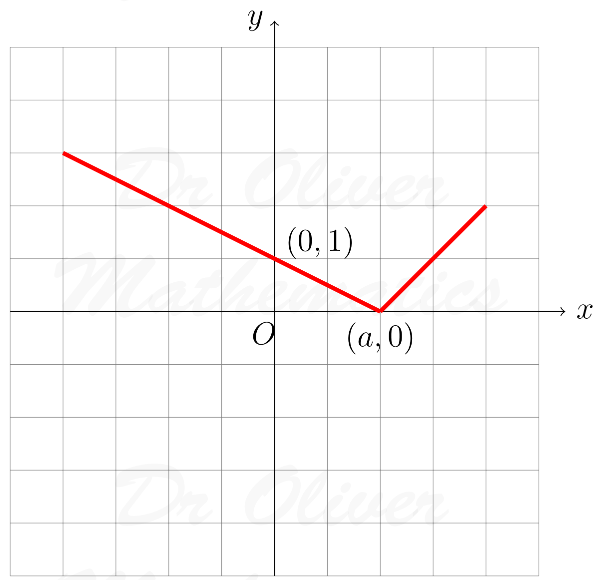
Sketch the graph of $|\,\mathrm{f}^{-1}(x)|$, showing the points of intersection with the axes.

Solution

Well, we reflect in the line $y=x$ to get ...



... and now apply the modulus function.



7. A curve is defined by the equation (4)

$$y = \frac{e^{\sin x}(2 + x)^3}{\sqrt{1 - x}} \text{ for } x < 1.$$

Calculate the gradient of the curve when $x = 0$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{1-x}\left[\mathrm{e}^{\sin x}\cdot 3(2+x)^2 + \cos x\,\mathrm{e}^{\sin x}(2+x)^3\right] - \left[-\frac{1}{2}(1-x)^{-\frac{1}{2}}\right]\mathrm{e}^{\sin x}(2+x)^3}{1-x}$$

and

$$\begin{aligned} x = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1[12+8-(-4)]}{1} \\ \Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 24}}. \end{aligned}$$

8. (a) Write down an expression for (1)

$$\sum_{r=1}^{n} r^3 - \left(\sum_{r=1}^{n} r\right)^2.$$

Solution

$$\begin{aligned} \sum_{r=1}^{n} r^3 - \left(\sum_{r=1}^{n} r\right)^2 &= \tfrac{1}{4}n^2(n+1)^2 - \left(\tfrac{1}{2}n(n+1)\right)^2 \\ &= \tfrac{1}{4}n^2(n+1)^2 - \tfrac{1}{4}n^2(n+1)^2 \\ &= \underline{\underline{0}}. \end{aligned}$$

(b) Write down an expression for (3)

$$\sum_{r=1}^{n} r^3 + \left(\sum_{r=1}^{n} r\right)^2.$$

Solution

$$\begin{aligned} \sum_{r=1}^{n} r^3 + \left(\sum_{r=1}^{n} r\right)^2 &= \tfrac{1}{4}n^2(n+1)^2 + \left(\tfrac{1}{2}n(n+1)\right)^2 \\ &= \tfrac{1}{4}n^2(n+1)^2 + \tfrac{1}{4}n^2(n+1)^2 \\ &= \underline{\underline{\tfrac{1}{2}n^2(n+1)^2}}. \end{aligned}$$

9. Given that $y > -1$ and $x > -1$, obtain the general solution of the differential equation (5)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3(1+y)\sqrt{1+x},$$

expressing your answer in the form $y = \mathrm{f}(x)$.

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 3(1+y)\sqrt{1+x} &\Rightarrow \frac{1}{1+y}\,\mathrm{d}y = 3(1+x)^{\frac{1}{2}}\,\mathrm{d}x \\ &\Rightarrow \ln(1+y) = 2(1+x)^{\frac{3}{2}} + c \\ &\Rightarrow 1+y = \mathrm{e}^{2(1+x)^{\frac{3}{2}}+c} \\ &\Rightarrow 1+y = A\mathrm{e}^{2(1+x)^{\frac{3}{2}}} \\ &\Rightarrow \underline{\underline{y = A\mathrm{e}^{2(1+x)^{\frac{3}{2}}} - 1}}.\end{aligned}$$

10. Identify the locus in the complex plane given by (5)

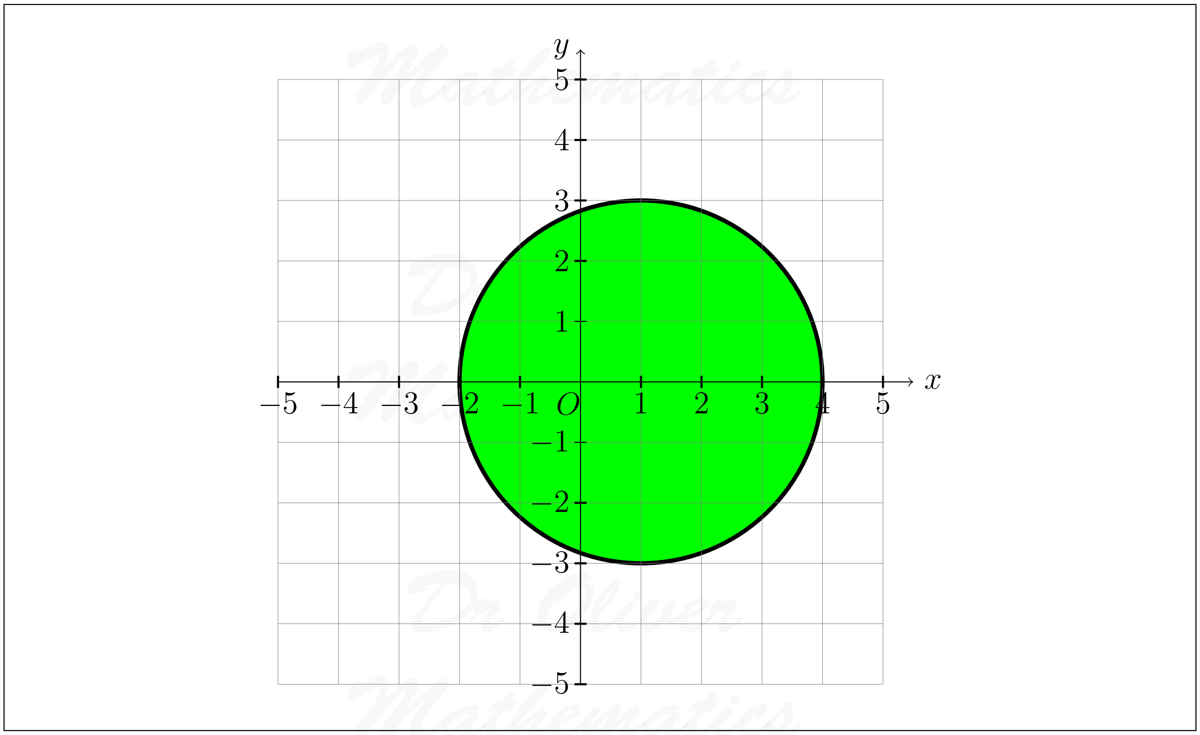
$$|z-1| = 3.$$

Show in a diagram the region given by

$$|z-1| = 3.$$

Solution

The locus is the <u><u>circle</u></u>, with centre $\underline{\underline{(1, 0)}}$, radius $\underline{\underline{3}}$.



11. (a) Obtain the exact value of (3)

$$\int_0^{\frac{1}{4}\pi} (\sec x - x)(\sec x + x)\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int_0^{\frac{1}{4}\pi} (\sec x - x)(\sec x + x)\,\mathrm{d}x &= \int_0^{\frac{1}{4}\pi} (\sec^2 x - x^2)\,\mathrm{d}x \\ &= \left[\tan x - \tfrac{1}{3}x^3\right]_{x=0}^{\frac{1}{4}\pi} \\ &= \left(1 - \tfrac{1}{192}\pi^3\right) - (0 - 0) \\ &= \underline{\underline{1 - \tfrac{1}{192}\pi^3}}.\end{aligned}$$

(b) Find (4)

$$\int \frac{x}{\sqrt{1 - 49x^4}}\,\mathrm{d}x.$$

Solution

$$u = 7x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 14x$$
$$\Rightarrow \mathrm{d}u = 14x\,\mathrm{d}x$$

and so

$$\begin{aligned}\int \frac{x}{\sqrt{1-49x^4}}\,\mathrm{d}x &= \tfrac{1}{14}\int \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u \\ &= \tfrac{1}{14}\sin^{-1} u + c \\ &= \underline{\underline{\tfrac{1}{14}\sin^{-1}(7x^2) + c}}.\end{aligned}$$

12. Prove by induction that (5)

$$8^n + 3^{n-2}$$

is divisible by 5 **for all integers** $n \geqslant 2$.

Solution

$\underline{n = 2}$: $8^2 + 3^0 = 65 = 5 \times 13$ and so the case $n = 2$ is true.

Suppose now that is is true for $n = k$, i.e., $8^k + 3^{k-2}$ is divisible by 5, i.e., $8^k + 3^{k-2} = 5p$ for some integer p. Now,

$$\begin{aligned}8^{k+1} + 3^{k-1} &= 8 \cdot 8^k + 3^{k-1} \\ &= 8(5p - 3^{k-2}) + 3^{k-1} \\ &= 40p - 8 \cdot 3^{k-2} + 3 \cdot 3^{k-2} \\ &= 40p + (3-8)3^{k-2} \\ &= 40p - 5 \cdot 3^{k-2} \\ &= 5(5p - 3^{k-2}),\end{aligned}$$

and so $n = k + 1$ is divisible by 5.

Hence, by mathematical induction, we have proved that induction that $8^n + 3^{n-2}$ is $\underline{\underline{\text{divisible by 5}}}$ for all integers $n \geqslant 2$.

13. (a) The first three terms of an arithmetic sequence are (5)

$$a, \frac{1}{a}, 1,$$

where $a < 0$.
Obtain the value of a and the common difference.

Solution

$$\begin{aligned} 1 - \frac{1}{a} = \frac{1}{a} - a &\Rightarrow a - 1 = 1 - a^2 \\ &\Rightarrow a^2 + a - 2 = 0 \\ &\Rightarrow (a+2)(a-1) = 0 \\ &\Rightarrow \underline{\underline{a = -2 \text{ (as } a < 0)}} \end{aligned}$$

and

$$d = -\tfrac{1}{2} - (-2) = \underline{\underline{1\tfrac{1}{2}}}.$$

(b) Obtain the smallest value of n for which the sum of the first n terms is greater than 1 000. (4)

Solution

$$\begin{aligned} S_n > 1\,000 &\Rightarrow \tfrac{1}{2}n[2(-2) + \tfrac{3}{2}(n-1)] > 1\,000 \\ &\Rightarrow n[-4 + \tfrac{3}{2}n - \tfrac{3}{2}] > 2\,000 \\ &\Rightarrow n[-\tfrac{11}{2} + \tfrac{3}{2}n] > 2\,000 \\ &\Rightarrow -\tfrac{11}{2}n + \tfrac{3}{2}n^2 > 2\,000 \\ &\Rightarrow -11n + 3n^2 > 4\,000 \\ &\Rightarrow 3n^2 - 11n - 4\,000 > 0. \end{aligned}$$

Now, we use the quadratic formula, $a = 3$, $b = -11$, $c = -4\,000$:

$$n < \frac{11 - \sqrt{48\,121}}{6} \text{ or } n > \frac{11 + \sqrt{48\,121}}{6};$$

we want the upper bound:

$$n > 38.394\,165\,544 \text{ (FCD)}$$

and so $\underline{\underline{n = 39}}$.

14. (a) Find the general solution of the differential equation (7)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = \mathrm{e}^x + 12.$$

Solution

Complementary function:

$$m^2 - m - 2 = 0 \Rightarrow (m-2)(m+1) = 0 \Rightarrow m = -1 \text{ or } m = 2$$

and hence the complementary function is

$$y = A\mathrm{e}^{-x} + B\mathrm{e}^{2x}.$$

Particular integral: try

$$y = C\mathrm{e}^{x} + D \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = C\mathrm{e}^{x} \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = C\mathrm{e}^{x}.$$

Substitute into the differential equation:

$$C\mathrm{e}^{x} - C\mathrm{e}^{x} - 2(C\mathrm{e}^{x} + D) = \mathrm{e}^{x} + 12 \Rightarrow C = -\tfrac{1}{2}, D = -6.$$

Hence the particular integral is $y = -\frac{1}{2}\mathrm{e}^{x} - 6$.

General solution: hence the general solution is

$$\underline{\underline{y = A\mathrm{e}^{-x} + B\mathrm{e}^{2x} - \tfrac{1}{2}\mathrm{e}^{x} - 6.}}$$

(b) Find the particular solution for which $y = -\frac{3}{2}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$ when $x = 0$. (3)

Solution

$$\begin{aligned} x = 0, y = -\tfrac{3}{2} &\Rightarrow A + B - \tfrac{1}{2} - 6 = -\tfrac{3}{2} \\ &\Rightarrow A + B = 5 \quad (1). \end{aligned}$$

Next,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -A\mathrm{e}^{-x} + 2B\mathrm{e}^{2x} - \tfrac{1}{2}\mathrm{e}^{x}$$

and

$$\begin{aligned} x = 0, \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{2} &\Rightarrow -A + 2B - \tfrac{1}{2} = \tfrac{1}{2} \\ &\Rightarrow -A + 2B = 1 \quad (2). \end{aligned}$$

Now, (1) + (2):

$$\begin{aligned} 3B = 6 \Rightarrow B &= 2 \\ \Rightarrow A &= 3 \end{aligned}$$

and, hence,

$$\underline{\underline{y = 3e^{-x} + 2e^{2x} - \tfrac{1}{2}e^{x} - 6.}}$$

15. The lines L_1 and L_2 are given by the equations

$$\frac{x-1}{k} = \frac{y}{-1} = \frac{z+3}{1} \text{ and } \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2},$$

respectively.

Find

(a) the value of k for which L_1 and L_2 intersect and the point of intersection, (6)

Solution

For L_1,

$$\begin{aligned} x &= kt + 1 \\ y &= -t \\ z &= t - 3 \end{aligned}$$

and, for L_2,

$$\begin{aligned} x &= s + 4 \\ y &= s - 3 \\ z &= 2s - 3. \end{aligned}$$

Hence,

$$\begin{aligned} kt + 1 = s + 4 &\Rightarrow kt - s = 3 \quad (1) \\ -t = s - 3 &\Rightarrow -t - s = -3 \quad (2) \\ t - 3 = 2s - 3 &\Rightarrow t - 2s = 0 \quad (3). \end{aligned}$$

(2) + (3):

$$\begin{aligned} -3s = -3 \Rightarrow s &= 1 \\ \Rightarrow t &= 2. \end{aligned}$$

Now, the first component:

$$2k + 1 = 4 + 1 \Rightarrow \underline{\underline{k = 2}}$$

and the point of intersection is $\underline{\underline{(5, -2, -1)}}$.

(b) the acute angle between L_1 and L_2. (4)

Solution
Let the angle between L_1 and L_2 be θ°. Then,

$$\begin{aligned} & (2\mathbf{i} - \mathbf{j} + \mathbf{k}).(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = |2\mathbf{i} - \mathbf{j} + \mathbf{k}| \cdot |\mathbf{i} + \mathbf{j} + 2\mathbf{k}| \cdot \cos\theta^\circ \\ \Rightarrow \quad & 2 - 1 + 2 = \sqrt{6} \cdot \sqrt{6} \cdot \cos\theta^\circ \\ \Rightarrow \quad & \cos\theta^\circ = \tfrac{1}{2} \\ \Rightarrow \quad & \underline{\underline{\theta = 60}}. \end{aligned}$$

16. Define

$$I_n = \int_0^1 \frac{1}{(1 + x^2)^n} \, \mathrm{d}x$$

for $n \geqslant 1$.

(a) Use integration by parts to show that (3)

$$I_n = \frac{1}{2^n} + 2n \int_0^1 \frac{x^2}{(1 + x^2)^{n+1}} \, \mathrm{d}x.$$

Solution

$$u = \frac{1}{(1 + x^2)^n} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{2nx}{(1 + x^2)^{n+1}}$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = 1 \Rightarrow v = x$$

and so

$$\begin{aligned} I_n &= \int_0^1 \frac{1}{(1+x^2)^n}\,\mathrm{d}x \\ &= \left[\frac{x}{(1+x^2)^n}\right]_{x=0}^1 + \int_0^1 \frac{2nx^2}{(1+x^2)^{n+1}}\,\mathrm{d}x \\ &= \left(\frac{1}{(1+1)^n} - 0\right) + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,\mathrm{d}x \\ &= \underline{\underline{\frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,\mathrm{d}x}}, \end{aligned}$$

as required.

(b) Find the values of A and B for which (5)

$$\frac{A}{(1+x^2)^n} + \frac{B}{(1+x^2)^{n+1}} \equiv \frac{x^2}{(1+x^2)^{n+1}},$$

and hence show that

$$I_{n+1} = \frac{1}{n\cdot 2^{n+1}} + \left(\frac{2n-1}{2n}\right) I_n.$$

Solution

$$\begin{aligned} \frac{x^2}{(1+x^2)^{n+1}} &\equiv \frac{A}{(1+x^2)^n} + \frac{B}{(1+x^2)^{n+1}} \\ &\equiv \frac{A(1+x^2)+B}{(1+x^2)^{n+1}} \end{aligned}$$

and so

$$x^2 \equiv A(1+x^2) + B.$$

$\underline{x=0}$: $0 = A + B \quad (1)$.
$\underline{x=1}$: $1 = 2A + B \quad (2)$.
Do (2) − (1):

$$A = 1 \text{ and } B = -1$$

and, hence,

$$\underline{\underline{\frac{1}{(1+x^2)^n} - \frac{1}{(1+x^2)^{n+1}} \equiv \frac{x^2}{(1+x^2)^{n+1}}}}.$$

Finally,

$$\begin{aligned}
& I_n = \frac{1}{2^n} + 2n \int_0^1 \frac{x^2}{(1+x^2)^{n+1}} \, \mathrm{d}x \\
\Rightarrow \quad & I_n = \frac{1}{2^n} + 2n \int_0^1 \left(\frac{1}{(1+x^2)^n} - \frac{1}{(1+x^2)^{n+1}} \right) \mathrm{d}x \\
\Rightarrow \quad & I_n = \frac{1}{2^n} + 2nI_n - 2nI_{n+1} \\
\Rightarrow \quad & 2nI_{n+1} = \frac{1}{2^n} + 2nI_n - I_n \\
\Rightarrow \quad & 2nI_{n+1} = \frac{1}{2^n} + (2n-1)I_n \\
\Rightarrow \quad & \underline{\underline{I_{n+1} = \frac{1}{n \cdot 2^{n+1}} + \left(\frac{2n-1}{2n} \right) I_n,}}
\end{aligned}$$

as required.

(c) Hence obtain the exact value of (3)

$$\int_0^1 \frac{1}{(1+x^2)^3} \, \mathrm{d}x.$$

Solution

$$\begin{aligned}
\int_0^1 \frac{1}{(1+x^2)^3} \, \mathrm{d}x &= I_3 \\
&= \tfrac{1}{16} + \tfrac{3}{4} I_2 \\
&= \tfrac{1}{16} + \tfrac{3}{4} \left(\tfrac{1}{4} + \tfrac{1}{2} I_1 \right) \\
&= \tfrac{1}{4} + \tfrac{3}{8} I_1 \\
&= \tfrac{1}{4} + \tfrac{3}{8} \int_0^1 \frac{1}{1+x^2} \, \mathrm{d}x \\
&= \tfrac{1}{4} + \tfrac{3}{8} \left[\tan^{-1} x \right]_{x=0}^{1} \\
&= \tfrac{1}{4} + \tfrac{3}{8} \left(\tfrac{1}{4}\pi - 0 \right) \\
&= \underline{\underline{\tfrac{1}{4} + \tfrac{3}{32}\pi}}.
\end{aligned}$$