Dr Oliver Mathematics
Mathematics: Advanced Higher
2023 Paper 1: Non-Calculator
1 hour

The total number of marks available is 35.
You must write down all the stages in your working.

1. Given (2)

$$y = 7x \tan 2x,$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

Well,

$$u = 7x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 7$$
$$v = \tan 2x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2\sec^2 2x$$

Now,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (7x)(2\sec^2 2x) + (\tan 2x)(7) \\ &= \underline{\underline{14x\sec^2 2x + 7\tan 2x}}.\end{aligned}$$

2. Express (3)

$$\frac{3x^2 - x - 14}{(x+3)(x-1)^2}$$

in partial fractions.

Solution

Now,

$$\begin{aligned}\frac{3x^2 - x - 14}{(x+3)(x-1)^2} &\equiv \frac{A}{(x+3)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} \\ &\equiv \frac{A(x-1)^2 + B(x+3)(x-1) + C(x+3)}{(x+3)(x-1)^2}\end{aligned}$$

and so

$$3x^2 - x - 14 \equiv A(x-1)^2 + B(x+3)(x-1) + C(x+3).$$

Next,

$\underline{x = -3}$:

$$\begin{aligned} 3[(-3)^2] - (-3) - 14 = A(-3-1)^2 &\Rightarrow 27 + 3 - 14 = 16A \\ &\Rightarrow 16 = 16A \\ &\Rightarrow A = 1. \end{aligned}$$

$\underline{x = 1}$:

$$\begin{aligned} 3[1^2] - (1) - 14 = C(1+3) &\Rightarrow 3 - 1 - 14 = 4C \\ &\Rightarrow -12 = 4C \\ &\Rightarrow C = -3. \end{aligned}$$

$\underline{x = 0}$:

$$\begin{aligned} -14 = A(-1)^2 + B(3)(-1) + C(3) &\Rightarrow -14 = 1 - 3B - 9 \\ &\Rightarrow 3B = 6 \\ &\Rightarrow B = 2. \end{aligned}$$

Hence,

$$\frac{3x^2 - x - 14}{(x+3)(x-1)^2} \equiv \underline{\underline{\frac{1}{(x+3)} + \frac{2}{(x-1)} - \frac{3}{(x-1)^2}}}.$$

3. A system of equations is defined by (3)

$$\begin{aligned} x - 3y + z &= -1 \\ 3x - 2y + 4z &= 11 \\ x + 4y + 2z &= 15. \end{aligned}$$

Use Gaussian elimination to determine whether the system shows redundancy, inconsistency, or has a unique solution.

Solution

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & -1 \\ 3 & -2 & 4 & 11 \\ 1 & 4 & 2 & 15 \end{array}\right)$$

Do $R_2 - 3R_1$ and $R_3 - R_1$:

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & -1 \\ 0 & 7 & 1 & 14 \\ 0 & 7 & 1 & 16 \end{array}\right)$$

Do $R_3 - R_2$:

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & -1 \\ 0 & 7 & 1 & 14 \\ 0 & 0 & 0 & 2 \end{array}\right)$$

Hence, Gaussian elimination shows $\underline{\underline{\text{inconsistency}}}$.

4. Use integration by parts to find (3)

$$\int x^4 \ln x \, \mathrm{d}x, \; x > 0.$$

Solution

Well,

$$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = x^4 \Rightarrow v = \tfrac{1}{5}x^5$$

so

$$\begin{aligned} \int x^4 \ln x \, \mathrm{d}x &= \tfrac{1}{5}x^5 \ln x - \int \left(\frac{1}{x}\right) \left(\tfrac{1}{5}x^5\right) \mathrm{d}x \\ &= \tfrac{1}{5}x^5 \ln x - \int \tfrac{1}{5}x^4 \, \mathrm{d}x \\ &= \underline{\underline{\tfrac{1}{5}x^5 \ln x - \tfrac{1}{25}x^5 + c}}. \end{aligned}$$

5. Find the particular solution of the differential equation (9)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = 10x^2 + 11x - 23,$$

given that $y = 2$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 14$ when $x = 0$.

Solution

<u>Complementary function</u>:

$$\begin{aligned} m^2 - 4m - 5 = 0 &\Rightarrow (m-5)(m+1) = 0 \\ &\Rightarrow m = 5 \text{ or } m = -1 \end{aligned}$$

and hence the complementary function is

$$y = A\mathrm{e}^{5x} + B\mathrm{e}^{-x}.$$

<u>Particular integral</u>: try

$$\begin{aligned} y = Cx^2 + Dx + Ex &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2Cx + D \\ &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2C \end{aligned}$$

and

$$\begin{aligned} 10x^2 + 11x - 23 &= 2C - 4(2Cx + D) - 5(Cx^2 + Dx + E) \\ &= -5Cx^2 + (-8C - 5D)x + (2C - 4D - 5E). \end{aligned}$$

Now,

$$10 = -5C \Rightarrow C = -2;$$

$$\begin{aligned} 11 = -8(-2) - 5D &\Rightarrow 11 = 16 - 5D \\ &\Rightarrow 5D = 5 \\ &\Rightarrow D = 1; \end{aligned}$$

$$\begin{aligned} -23 = 2(-2) - 4(1) - 5E &\Rightarrow -23 = 8 - 5E \\ &\Rightarrow 5E = 15 \\ &\Rightarrow E = 3; \end{aligned}$$

the particular integral is $y = -2x^2 + x + 3$.

Hence, the general solution is

$$y = A\mathrm{e}^{5x} + B\mathrm{e}^{-x} - 2x^2 + x + 3.$$

Now,

$$\begin{aligned} x = 0, y = 2 &\Rightarrow 2 = A + B - 0 + 0 + 3 \\ &\Rightarrow A + B = -1. \quad (1) \end{aligned}$$

Next,

$$y = Ae^{5x} + Be^{-x} - 2x^2 + x + 3 \Rightarrow \frac{dy}{dx} = 5Ae^{5x} - Be^{-x} - 4x + 1$$

and

$$\begin{aligned} x = 0, y = 2 &\Rightarrow 14 = 5A - B - 0 + 1 \\ &\Rightarrow 5A - B = -13. \quad (2) \end{aligned}$$

Do (1) + (2):

$$\begin{aligned} 6A = 12 &\Rightarrow A = 2 \\ &\Rightarrow B = -3. \end{aligned}$$

Finally,

$$\underline{\underline{y = 2e^{5x} - 3e^{-x} - 2x^2 + x + 3}}.$$

6. (a) Express (2)

$$z = 1 + i\sqrt{3}$$

in polar form.

Solution
Well,

$$\begin{aligned} r &= \sqrt{1^2 + (\sqrt{3})^2} \\ &= \sqrt{1 + 3} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

and

$$\begin{aligned} z &= 2\left[\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right] \\ &= \underline{\underline{2\left[\cos(\tfrac{1}{3}\pi) + i\sin(\tfrac{1}{3}\pi)\right]}}. \end{aligned}$$

(b) Hence, or otherwise, show that z^3 is real. (2)

Solution

Now,

$$\begin{aligned} z^3 &= \left(2\left[\cos(\tfrac{1}{3}\pi) + \mathrm{i}\sin(\tfrac{1}{3}\pi)\right]\right)^3 \\ &= 8\left[\cos(\pi) + \mathrm{i}\sin(\pi)\right] \\ &= 8(-1) \\ &= \underline{\underline{-8}}; \end{aligned}$$

hence, z^3 is <u>real</u>.

7. (a) Find an expression for (2)

$$\sum_{r=1}^{n}(r^2+3r)$$

in terms of n.

Express your answer in the form

$$\tfrac{1}{3}n(n+a)(n+b).$$

Solution

$$\begin{aligned} \sum_{r=1}^{n}(r^2+3r) &= \sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r \\ &= \tfrac{1}{6}n(n+1)(2n+1) + 3\left[\tfrac{1}{2}n(n+1)\right] \\ &= \tfrac{1}{6}n(n+1)(2n+1) + \tfrac{3}{2}n(n+1) \\ &= \tfrac{1}{6}n(n+1)\left[(2n+1)+9\right] \\ &= \tfrac{1}{6}n(n+1)(2n+10) \\ &= \underline{\underline{\tfrac{1}{3}n(n+1)(n+5)}}; \end{aligned}$$

hence, $\underline{\underline{a=1}}$ and $\underline{\underline{b=5}}$.

(b) Hence, or otherwise, find (2)

$$\sum_{r=11}^{20}(r^2+3r).$$

Solution

$$\begin{aligned}\sum_{r=11}^{20}(r^2+3r) &= \sum_{r=1}^{20}(r^2+3r) - \sum_{r=1}^{10}(r^2+3r) \\ &= \tfrac{1}{3}(20)(20+1)(20+5) - \tfrac{1}{3}(10)(10+1)(10+5) \\ &= \tfrac{1}{3}(20)(21)(25) - \tfrac{1}{3}(10)(11)(15) \\ &= (20)(7)(25) - (10)(11)(5) \\ &= 3\,500 - 550 \\ &= \underline{\underline{2\,950}}.\end{aligned}$$

8. (a) Consider the statement: (1)

For all integers a and b, if $a < b$ then $a^2 < b^2$.

Find a counterexample to show that the statement is false.

Solution

E.g., $a = -2$ and $b = -1$. Then $a < b$ <u>but</u> $a^2 > b^2$.

Let n be an odd integer.

(b) Prove directly that $(n^2 - 1)$ is divisible by 4. (2)

Solution

Let $n = 2m + 1$ where $m \in \mathbb{Z}$. Then

$$n^2 - 1 = (2m+1)^2 - 1$$

×	$2m$	$+1$
$2m$	$4m^2$	$+2m$
$+1$	$+2m$	$+1$

$$\begin{aligned} &= (4m^2 + 4m + 1) - 1 \\ &= 4(m^2 + m) \\ &= 4 \times \text{some integer;}\end{aligned}$$

hence, $(n^2 - 1)$ is <u>divisible by 4</u>.

9. (a) State the matrix $\mathbf{A}$, associated with an anti-clockwise rotation of $\frac{1}{2}\pi$ radians about the origin. (1)

Solution

$$\mathbf{A} = \underline{\underline{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}}.$$

The matrix $\mathbf{B}$ is given by

$$\mathbf{B} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}.$$

The matrix given by $\mathbf{AB}$ is associated with an anti-clockwise rotation of α radians about the origin.

(b) (i) Determine $\mathbf{AB}$. (1)

Solution

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}}}. \end{aligned}$$

(ii) Find the value of α. (1)

Solution

$\underline{\underline{\alpha = \frac{5}{3}\pi}}.$

(c) Determine the least positive integer value of n such that $(\mathbf{AB})^n = \mathbf{I}$, where $\mathbf{AB}$ is the 2×2 identity matrix. (1)

Solution

$$\begin{aligned} \text{1st turn}: &\quad \tfrac{5}{3}\pi \\ \text{2nd turn}: &\quad \tfrac{10}{3}\pi \to \tfrac{4}{3}\pi \\ \text{3rd turn}: &\quad \tfrac{15}{3}\pi \to \pi \end{aligned}$$

so $n = 6$ — but if you haven't there yet ...

$$\begin{aligned} \text{4th turn} : &\quad \tfrac{20}{3}\pi \to \tfrac{2}{3}\pi \\ \text{5th turn} : &\quad \tfrac{25}{3}\pi \to \tfrac{1}{3}\pi \\ \text{6th turn} : &\quad \tfrac{30}{3}\pi \to 0; \end{aligned}$$

hence, $\underline{\underline{n = 6}}$.