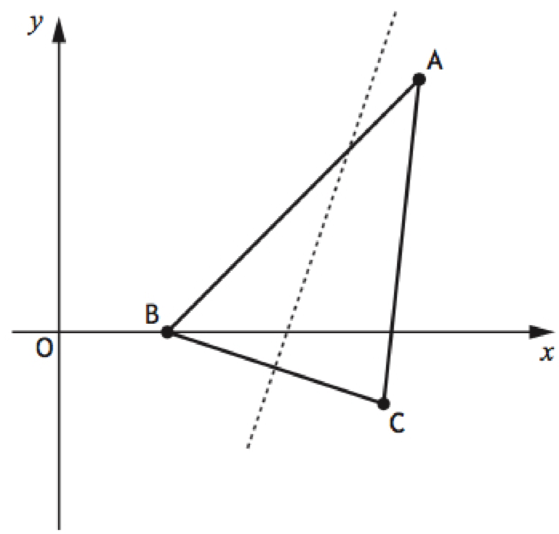
Dr Oliver Mathematics
Mathematics: Higher
2017 Paper 2: Calculator
1 hour 30 minutes

The total number of marks available is 70.
You must write down all the stages in your working.

1. Triangle ABC is shown in the diagram below.



The coordinates of B are $(3, 0)$ and the coordinates of C are $(9, -2)$.
The broken line is the perpendicular bisector of BC.

(a) Find the equation of the perpendicular bisector of BC. (4)

Solution

$$\text{Gradient} = \frac{-2-0}{9-3} = -\tfrac{1}{3}$$

and the gradient of the perpendicular bisector is

$$-\frac{1}{-\frac{1}{3}} = 3.$$

Now, the midpoint of BC is

$$\left(\frac{3+9}{2}, \frac{0+(-2)}{2}\right) = (6, -1).$$

Finally, the equation of the perpendicular bisector is

$$\begin{aligned} y-(-1)=3(x-6) &\Rightarrow y+1=3x-18 \\ &\Rightarrow \underline{\underline{y=3x-19}}. \end{aligned}$$

The line AB makes an angle of 45° with the positive direction of the x-axis.

(b) Find the equation of AB. (2)

Solution

The line AB has gradient

$$\tan 45^\circ = 1$$

and so the equation is

$$y-0=1(x-3) \Rightarrow \underline{\underline{y=x-3}}.$$

(c) Find the coordinates of the point of intersection of AB and the perpendicular bisector of BC. (2)

Solution

$$\begin{aligned} 3x-19=x-3 &\Rightarrow 2x=16 \\ &\Rightarrow x=8 \\ &\Rightarrow y=5; \end{aligned}$$

hence, the coordinates are $\underline{\underline{(8,5)}}$.

2. (a) Show that $(x-1)$ is a factor of (2)

$$\text{f}(x)=2x^3-5x^2+x+2.$$

Solution

1	2	−5	1	2
	↓	2	−3	−2
	2	−3	−2	0

Hence, because there is no remainder, $(x-1)$ is a factor of $2x^3 - 5x^2 + x + 2$.

(b) Hence, or otherwise, solve $f(x) = 0$. (3)

Solution

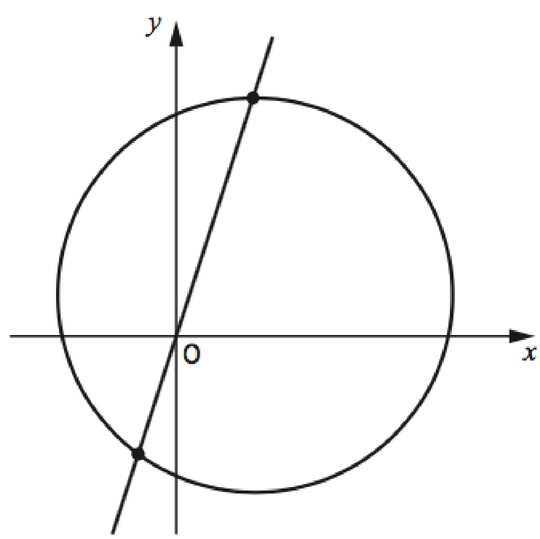
$$\begin{aligned} f(x) = 0 &\Rightarrow 2x^3 - 5x^2 + x + 2 = 0 \\ &\Rightarrow (x-1)(2x^2 - 3x - 2) = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -3 \\ \text{multiply to:} & (+2)\times(-2) = -4 \end{array}\right\} -4,\ +1$$

$$\begin{aligned} &\Rightarrow (x-1)[2x^2 - 4x + x - 2] = 0 \\ &\Rightarrow (x-1)[2x(x-2) + (x-2)] = 0 \\ &\Rightarrow (x-1)(2x+1)(x-2) = 0 \\ &\Rightarrow \underline{\underline{x = -\tfrac{1}{2},\ x = 1,\ \text{or } x = 2.}} \end{aligned}$$

3. The line $y = 3x$ intersects the circle with equation (5)

$$(x-2)^2 + (y-1)^2 = 25.$$



Find the coordinates of the points of intersection.

Solution

$$\begin{aligned} & (x-2)^2 + (y-1)^2 = 25 \\ \Rightarrow\quad & (x-2)^2 + (3x-1)^2 = 25 \\ \Rightarrow\quad & (x^2-4x+4) + (9x^2-6x+1) = 25 \\ \Rightarrow\quad & 10x^2 - 10x - 20 = 0 \\ \Rightarrow\quad & 10(x^2 - x - 2) = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -1 \\ \text{multiply to:} & -2 \end{array}\right\} -2,\ +1$$

$$\begin{aligned} \Rightarrow\quad & 10(x-2)(x+1) = 0 \\ \Rightarrow\quad & x = -1 \text{ or } x = 2 \\ \Rightarrow\quad & y = -3 \text{ or } y = 6; \end{aligned}$$

hence, the coordinates of the points of intersection are

$$\underline{\underline{(-1,-3) \text{ and } (2,6)}}.$$

4. (a) Express (3)

$$3x^2 + 24x + 50$$

in the form

$$a(x+b)^2 + c.$$

Solution

$$\begin{aligned} 3x^2 + 24x + 50 &= 3(x^2 + 8x) + 50 \\ &= 3[(x^2 + 8x + 16) - 16] + 50 \\ &= 3[(x+4)^2 - 16] + 50 \\ &= 3(x+4)^2 - 48 + 50 \\ &= \underline{\underline{3(x+4)^2 + 2}}; \end{aligned}$$

hence, $\underline{\underline{a = 3}}$, $\underline{\underline{b = 4}}$, and $\underline{\underline{c = 2}}$.

(b) Given that (2)

$$f(x) = x^3 + 12x^2 + 50x - 11,$$

find $f'(x)$.

Solution

$$f(x) = x^3 + 12x^2 + 50x - 11 \Rightarrow \underline{\underline{f'(x) = 3x^2 + 24x + 50}}.$$

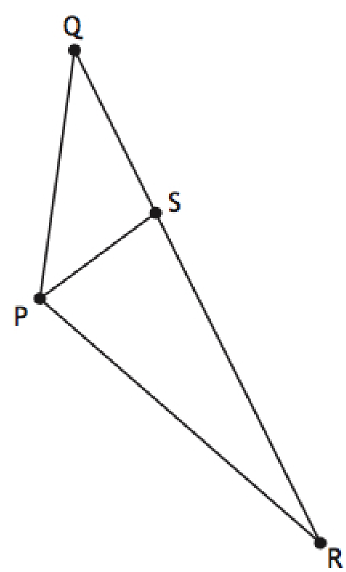
(c) Hence, or otherwise, explain why the curve with equation $y = f(x)$ is strictly increasing for all values of x. (2)

Solution

$$\begin{aligned} f'(x) &= 3x^2 + 24x + 50 \\ &= 3(x-4)^2 + 2 \\ &\geqslant 2 \end{aligned}$$

so the curve is strictly increasing for all values of x.

5. In the diagram, $\overrightarrow{PR} = 9\mathbf{i}+5\mathbf{j}+2\mathbf{k}$ and $\overrightarrow{RQ} = -12\mathbf{i}-9\mathbf{j}+3\mathbf{k}$.



(a) Express $\overrightarrow{PQ}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. (2)

Solution

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PR} + \overrightarrow{RQ} \\ &= (9\mathbf{i}+5\mathbf{j}+2\mathbf{k}) + (-12\mathbf{i}-9\mathbf{j}+3\mathbf{k}) \\ &= \underline{\underline{-3\mathbf{i}-4\mathbf{j}+5\mathbf{k}}}.\end{aligned}$$

The point S divides QR in the ratio $1:2$.

(b) Show that $\overrightarrow{PS} = \mathbf{i}-\mathbf{j}+4\mathbf{k}$. (2)

Solution

$$\begin{aligned}\overrightarrow{PS} &= \overrightarrow{PQ} + \overrightarrow{QS} \\ &= \overrightarrow{PQ} + \tfrac{1}{3}\overrightarrow{QR} \\ &= \overrightarrow{PQ} - \tfrac{1}{3}\overrightarrow{RQ} \\ &= (-3\mathbf{i}-4\mathbf{j}+5\mathbf{k}) - \tfrac{1}{3}(-12\mathbf{i}-9\mathbf{j}+3\mathbf{k}) \\ &= (-3\mathbf{i}-4\mathbf{j}+5\mathbf{k}) - (-4\mathbf{i}-3\mathbf{j}+\mathbf{k}) \\ &= \underline{\underline{\mathbf{i}-\mathbf{j}+4\mathbf{k}}},\end{aligned}$$

as required.

(c) Hence, find the size of angle QPS. (5)

Solution

Now,

$$\begin{aligned}|\overrightarrow{PQ}| &= \sqrt{(-3)^2 + (-4)^2 + 5^2} \\ &= \sqrt{9+16+25} \\ &= \sqrt{50} \\ &= 5\sqrt{2}\end{aligned}$$

and

$$\begin{aligned}|\overrightarrow{PS}| &= \sqrt{1^2 + (-1)^2 + 4^2} \\ &= \sqrt{1+1+16} \\ &= \sqrt{18} \\ &= 3\sqrt{2}.\end{aligned}$$

$$
\begin{aligned}
&\overrightarrow{PQ}.\overrightarrow{PS} = |\overrightarrow{PQ}||\overrightarrow{PR}|\cos QPS \\
\Rightarrow\quad &(-3\mathbf{i}-4\mathbf{j}+5\mathbf{k}).(\mathbf{i}-\mathbf{j}+4\mathbf{k}) = 5\sqrt{2}\cdot 3\sqrt{2}\cdot\cos QPS \\
\Rightarrow\quad &-3+4+20 = 5\sqrt{2}\cdot 3\sqrt{2}\cdot\cos QPS \\
\Rightarrow\quad &\cos QPS = \tfrac{7}{10} \\
\Rightarrow\quad &\angle QPS = 45.572\,996 \text{ (FCD)} \\
\Rightarrow\quad &\underline{\underline{\angle QPS = 45.6^\circ \text{ (3 sf)}}}.
\end{aligned}
$$

6. Solve (5)

$$5\sin x - 4 = 2\cos 2x$$

for $0 \leqslant x < 2\pi$.

Solution

$$
\begin{aligned}
5\sin x - 4 = 2\cos 2x &\Rightarrow 5\sin x - 4 = 2(1-2\sin^2 x) \\
&\Rightarrow 5\sin x - 4 = 2 - 4\sin^2 x \\
&\Rightarrow 4\sin^2 x + 5\sin x - 6 = 0
\end{aligned}
$$

$$
\left.\begin{array}{ll}
\text{add to:} & +5 \\
\text{multiply to:} & (+4)\times(-6) = -24
\end{array}\right\} -3,\ +8
$$

$$
\begin{aligned}
&\Rightarrow 4\sin^2 x + 8\sin x - 3\sin x - 6 = 0 \\
&\Rightarrow 4\sin x(\sin x + 2) - 3(\sin x + 2) = 0 \\
&\Rightarrow (4\sin x - 3)(\sin x + 2) = 0 \\
&\Rightarrow \sin x = \tfrac{3}{4} \\
&\Rightarrow x = 0.848\,062\,079,\ 2.293\,530\,575 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{x = 0.848,\ 2.29 \text{ (3 sf)}}}.
\end{aligned}
$$

7. (a) Find the x-coordinate of the stationary point on the curve with equation (4)

$$y = 6x - 2\sqrt{x^3}.$$

Solution

$$y = 6x - 2\sqrt{x^3} \Rightarrow y = 6x - 2x^{\frac{3}{2}}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 6 - 3x^{\frac{1}{2}}.$$

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 6 - 3x^{\frac{1}{2}} = 0$$
$$\Rightarrow 3x^{\frac{1}{2}} = 6$$
$$\Rightarrow x^{\frac{1}{2}} = 2$$
$$\Rightarrow \underline{\underline{x = 4}}.$$

(b) Hence, determine the greatest and least values of y in the interval $1 \leqslant x \leqslant 9$. (3)

Solution

$$x = 1 \Rightarrow y = 4$$
$$x = 4 \Rightarrow y = 8$$
$$x = 9 \Rightarrow y = 0;$$

hence, the greatest is $\underline{\underline{8}}$ and the least is $\underline{\underline{0}}$.

8. Sequences may be generated by recurrence relations of the form

$$u_{n+1} = ku_n - 20,\ u_0 = 5 \text{ where } k \in \mathbb{R}.$$

(a) Show that (2)

$$u_2 = 5k^2 - 20k - 20.$$

Solution

$$
\begin{aligned}
u_1 &= ku_0 - 20 \\
&= 5k - 20 \\
u_2 &= ku_1 - 20 \\
&= 5(5k - 20) - 20 \\
&= \underline{\underline{5k^2 - 20k - 20}},
\end{aligned}
$$

as required.

(b) Determine the range of values of k for which $u_2 < u_0$. (4)

Solution

$$
\begin{aligned}
u_2 < u_0 &\Rightarrow 5k^2 - 20k - 20 < 5 \\
&\Rightarrow 5k^2 - 20k - 25 < 0 \\
&\Rightarrow 5(k^2 - 4k - 5) < 0
\end{aligned}
$$

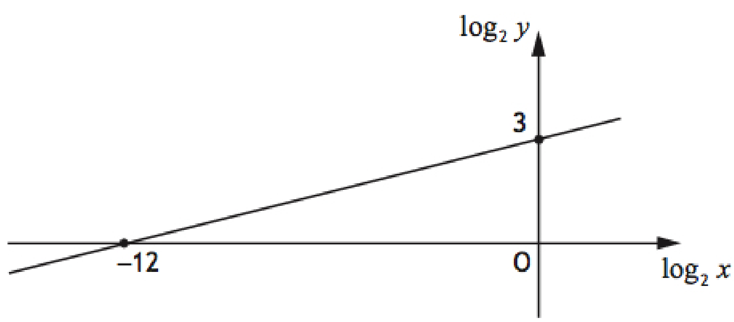
$$
\left.\begin{array}{ll} \text{add to:} & -4 \\ \text{multiply to:} & -5 \end{array}\right\} -5, +1
$$

$$
\begin{aligned}
&\Rightarrow 5(k - 5)(k + 1) < 0 \\
&\Rightarrow \underline{\underline{-1 < k < 5}}.
\end{aligned}
$$

9. Two variables, x and y, are connected by the equation (5)

$$y = kx^n.$$

The graph of $\log_2 y$ against $\log_2 x$ is a straight line as shown.



Find the values of k and n.

Solution

$$\text{Gradient} = \frac{3-0}{0-(-12)} = \tfrac{1}{4}$$

and the equation is

$$\begin{aligned}\log_2 y - 3 = \tfrac{1}{4}(\log_2 x - 0) &\Rightarrow \log_2 y - 3 = \tfrac{1}{4}x \\ &\Rightarrow \log_2 y - \tfrac{1}{4}\log_2 x = 3 \\ &\Rightarrow \log_2 y - \log_2 x^{\frac{1}{4}} = 3 \\ &\Rightarrow \log_2\left(\frac{y}{x^{\frac{1}{4}}}\right) = 3 \\ &\Rightarrow \frac{y}{x^{\frac{1}{4}}} = 2^3 \\ &\Rightarrow \underline{\underline{y = 8x^{\frac{1}{4}}}};\end{aligned}$$

hence, $\underline{\underline{k = 8}}$ and $\underline{\underline{n = \tfrac{1}{4}}}$.

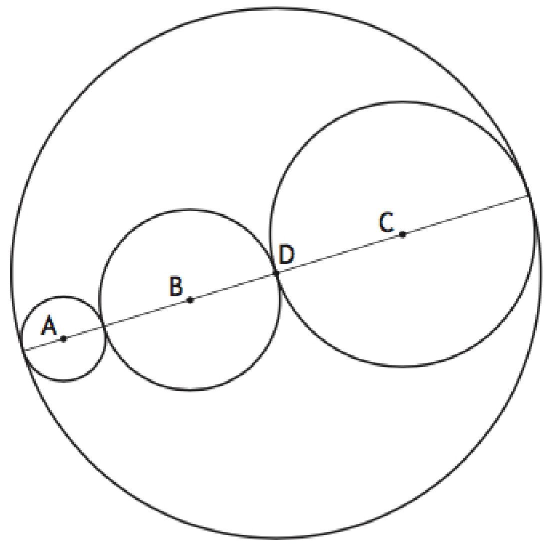
10. (a) Show that the points $A(-7,-2)$, $B(2,1)$, and $C(17,6)$ are collinear. (3)

Solution

$$\begin{aligned}\overrightarrow{AB} &= \begin{pmatrix} 9 \\ 3 \end{pmatrix} \\ \overrightarrow{AC} &= \begin{pmatrix} 24 \\ 8 \end{pmatrix} \\ &= \tfrac{8}{3}\begin{pmatrix} 9 \\ 3 \end{pmatrix} \\ &= \tfrac{8}{3}\overrightarrow{AB};\end{aligned}$$

as they have A in common, A, B, and C are $\underline{\underline{\text{collinear}}}$.

Three circles with centres A, B, and C are drawn inside a circle with centre D as shown.



The circles with centres A, B, and C have radii r_A, r_B, and r_C respectively.

- $r_A = \sqrt{10}$,
- $r_B = 2r_A$, and
- $r_C = r_A + r_B$.

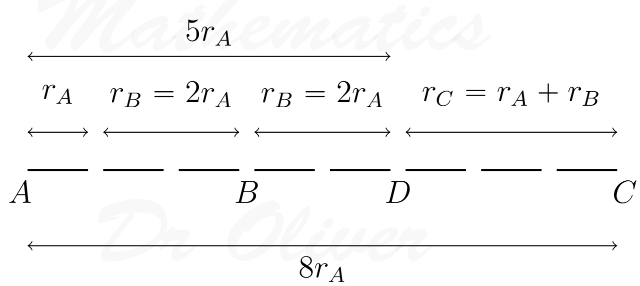
(b) Determine the equation of the circle with centre D. (4)

Solution

$$\begin{aligned} r_C &= r_A + r_B \\ &= r_A + 2r_A \\ &= \sqrt{10} + 2\sqrt{10} \\ &= 3\sqrt{10} \end{aligned}$$

and the radius is double this:

$$3\sqrt{10} \times 2 = 6\sqrt{10}.$$



Now,

$$\begin{aligned}
\overrightarrow{OD} &= \overrightarrow{OA} + \overrightarrow{AD} \\
&= \overrightarrow{OA} + \tfrac{5}{8}\overrightarrow{AC} \\
&= \begin{pmatrix} -7 \\ -2 \end{pmatrix} + \tfrac{5}{8}\begin{pmatrix} 24 \\ 8 \end{pmatrix} \\
&= \begin{pmatrix} -7 \\ -2 \end{pmatrix} + \begin{pmatrix} 15 \\ 5 \end{pmatrix} \\
&= \begin{pmatrix} 8 \\ 3 \end{pmatrix}.
\end{aligned}$$

Finally, the equation of the circle is

$$(x-8)^2 + (y-3)^2 = (6\sqrt{10})^2 \Rightarrow \underline{\underline{(x-8)^2 + (y-3)^2 = 640}}.$$

11. (a) Show that (3)

$$\frac{\sin 2x}{2\cos x} - \sin x \cos^2 x \equiv \sin^3 x,$$

where $0 < x < \frac{1}{2}\pi$.

Solution

$$\begin{aligned}
\frac{\sin 2x}{2\cos x} - \sin x \cos^2 x &\equiv \frac{2\sin x \cos x}{2\cos x} - \sin x(1 - \sin^2 x) \\
&\equiv \sin x - \sin x + \sin^3 x \\
&\equiv \underline{\underline{\sin^3 x}},
\end{aligned}$$

as required.

(b) Hence, differentiate (3)

$$\frac{\sin 2x}{2\cos x} - \sin x \cos^2 x,$$

where $0 < x < \frac{1}{2}\pi$.

Solution

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin 2x}{2\cos x} - \sin x \cos^2 x\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^3 x\right) \\
&= \underline{\underline{3\sin^2 x \cos x}}.
\end{aligned}$$