Dr Oliver Mathematics
GCSE Mathematics
2023 June Paper 3H: Calculator
1 hour 30 minutes

The total number of marks available is 80.
You must write down all the stages in your working.

1. (a) Simplify (1)

$$(m^2)^3.$$

Solution

$$(m^2)^3 = m^{2\times 3} = \underline{\underline{m^6}}.$$

(b) Simplify (1)

$$x^5 \times x^8.$$

Solution

$$x^5 \times x^8 = x^{5+8} = \underline{\underline{x^{13}}}.$$

(c) Expand (2)

$$4p(p^2 + 3p).$$

Solution

$$4p(p^2 + 3p) = \underline{\underline{4p^3 + 12p^2}}.$$

2. Jonny wants to know how much coffee he will need for 800 people at a meeting.

Each person who drinks coffee will drink 2 cups of coffee.
10.6 g of coffee is needed for each cup of coffee.

Jonny assumes 68% of the people will drink coffee.

(a) Using this assumption, work out the amount of coffee Jonny needs. Give your answer correct to the nearest gram. (4)

Solution
He needs

$$800 \times 0.68 \times 2 \times 10.6 = 11\,532.8 \text{ (FCD)}$$
$$= \underline{\underline{11\,533 \text{ g (nearest gram)}}}.$$

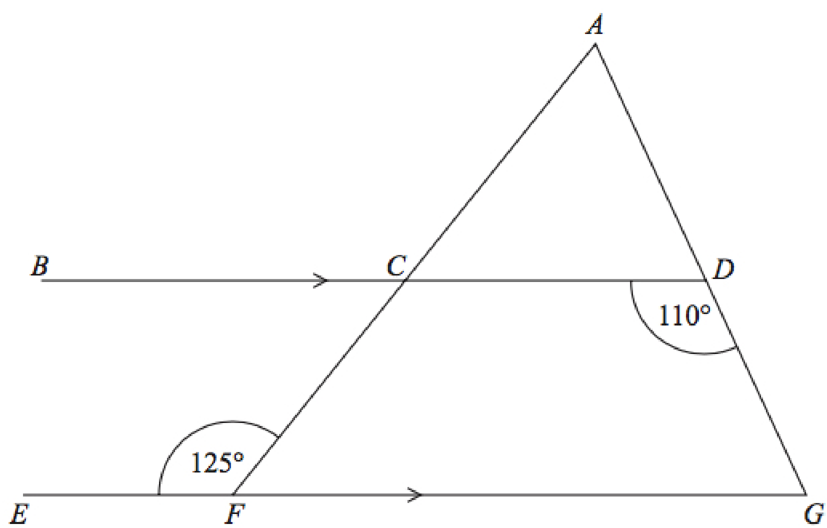
Jonny's assumption is wrong.
72% of the people will drink coffee.

(b) How does this affect your answer to part (a)? (1)

Solution
He needs <u>more coffee</u>:

$$800 \times 0.72 \times 2 \times 10.6 = 12\,211.5 \text{ g}.$$

3. ACF and ADG are straight lines. (5)
BCD and EFG are parallel lines.



Show that triangle ACD is isosceles.
Give a reason for each stage of your working.

Solution

$\angle FCD = 125°$ (alternate angles)
$\angle ACD = 180 - 125 = 55°$ (supplementary angles)
$\angle ADC = 180 - 110 = 70°$ (supplementary angles)
$\angle CAD = 180 - 70 - 55 = 55°$ (completing the triangle).
So $\angle CAD = \angle ACD$ and, hence, the triangle ACD is isosceles.

4. It takes 14 hours for 5 identical pumps to fill a water tank. (2)

How many hours would it take 4 of these pumps to fill another water tank of the same size?

Solution

Well,

$$\begin{aligned} 5 \text{ pumps} \leftrightarrow 14 \text{ hours} \Leftrightarrow 1 \text{ pump} &\leftrightarrow 70 \text{ hours} \\ \Leftrightarrow 4 \text{ pumps} &\leftrightarrow \underline{\underline{17\tfrac{1}{2} \text{ hours}}}. \end{aligned}$$

5. A and B are numbers such that

$$\begin{aligned} A &= 2^2 \times 3^4 \times 7 \\ B &= 3^2 \times 7^2. \end{aligned}$$

(a) Find the highest common factor (HCF) of A and B. (1)

Solution

$$\begin{aligned} \text{HCF} &= 3^2 \times 7 \\ &= \underline{\underline{63}}. \end{aligned}$$

(b) Find the lowest common multiple (LCM) of A and B. (2)

Solution

$$\begin{aligned} \text{LCM} &= 2^2 \times 3^4 \times 7^2 \\ &= \underline{\underline{15\,876}}. \end{aligned}$$

6. Lava flows from a volcano at a constant rate of 11.9 m^3/s. (3)

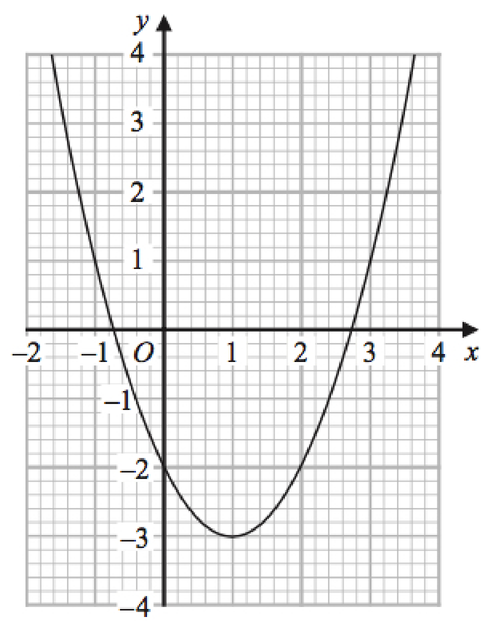
How many days does it take for 67 205 600 m^3 of lava to flow from the volcano?
Give your answer correct to the nearest day.

Solution

$$\begin{aligned}\text{Time} &= \frac{67\,205\,600}{11.9}\\ &= 5\,647\,529.412 \text{ s}\\ &= 94\,125.490\,2 \text{ mins}\\ &= 1\,568.758\,17 \text{ hours}\\ &= 65\tfrac{335}{918} \text{ days}\\ &= \underline{\underline{65 \text{ days (nearest day)}}}.\end{aligned}$$

7. Here is the graph of

$$y = x^2 - 2x - 2.$$



(a) Write down the coordinates of the turning point on the graph of (1)

$$y = x^2 - 2x - 2.$$

Solution

$\underline{\underline{(1, -3)}}$.

(b) Write down an estimate for one of the roots of (1)

$$x^2 - 2x - 2 = 0.$$

Solution

Correct read-off: approximately $\underline{\underline{x = -0.7}}$ or $\underline{\underline{x = 2.7}}$.

8. A solid cuboid is made of metal. (2)

The metal has a density of 9 g/cm^3.
The volume of the cuboid is 72 cm^3.

Work out the mass of the cuboid.

Solution

$$\begin{aligned}\text{Mass} &= \text{density} \times \text{volume} \\ &= 9 \times 72 \\ &= \underline{\underline{648\text{ g}}}.\end{aligned}$$

9. Some people were asked if they wanted a new television. (2)

70% of the people said yes.
80% of the people who said yes wanted a television with a large screen.

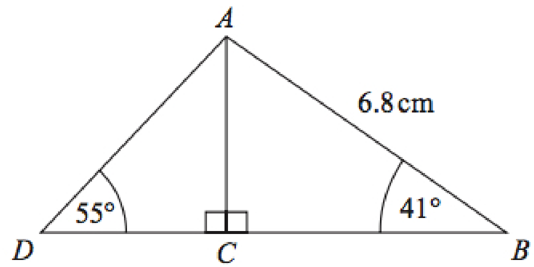
What percentage of the people asked said they wanted a television with a large screen?

Solution

$$0.7 \times 0.8 = 0.56,$$

so the percentage is $\underline{\underline{56\%}}$.

10. ABD is a triangle. (3)
C is a point on BD.



Work out the length of DC.
Give your answer correct to 1 decimal place.

Solution

Well,

$$\sin = \frac{\text{opp}}{\text{hyp}} \Rightarrow \sin 41^\circ = \frac{AC}{6.8}$$
$$\Rightarrow AC = 6.8 \sin 41^\circ$$

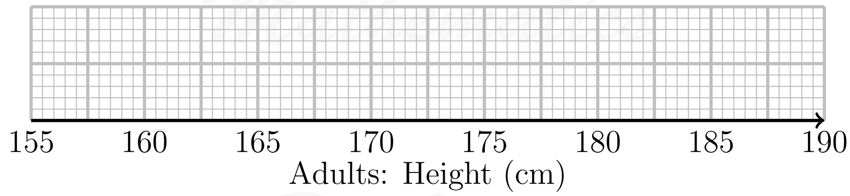
and

$$\tan = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan 55^\circ = \frac{6.8 \sin 41^\circ}{DC}$$
$$\Rightarrow DC = \frac{6.8 \sin 41^\circ}{\tan 55^\circ}$$
$$\Rightarrow DC = 3.123\,766\,848 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{DC = 3.1 \text{ cm (1 dp)}}}.$$

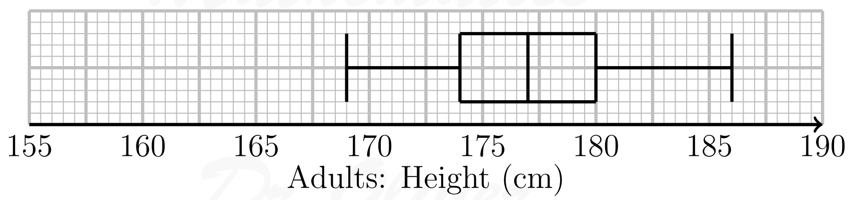
11. The table shows some information about the heights of a group of adults.

Least height	169 cm
Greatest height	186 cm
Median	177 cm
Lower quartile	174 cm
Upper quartile	180 cm

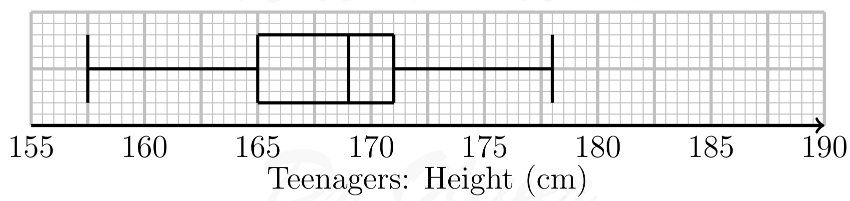
(a) On the grid, draw a box plot for the information in the table. (3)



Solution



The box plot below shows the distribution of the heights of a group of teenagers.



(b) Compare the distribution of the heights of the adults with the distribution of the heights of the teenagers. (2)

Solution

	Median	IQR	Range
Adults	177	$180 - 174 = 6$	$186 - 169 = 17$
Teenagers	169	$171 - 165 = 6$	$178 - 157.5 = 20.5$

E.g.,

- the median for the heights of adults is greater than the median for the heights of teenagers,
- the IQR is identical for the adults and teenagers, and

- the range of the heights of the teenagers is greater than the range of heights of the adults.

12. Show that (3)

$$(x-1)(x+3)(x-5)$$

can be written in the form

$$ax^3 + bx^2 + cx + d,$$

where a, b, c, and d are integers.

Solution

Now,

$\times$	x	-1
x	x^2	$-x$
$+3$	$+3x$	-3

and so

$$(x-1)(x+3) = x^2 + 2x - 3.$$

Next,

$\times$	x^2	$+2x$	-3
x	x^3	$+2x^2$	$-3x$
-5	$-5x^2$	$-10x$	$+15$

so

$$(x-1)(x+3)(x-5) = \underline{\underline{x^3 - 3x^2 - 13x + 15}}.$$

13. An expression for the nth term of the sequence of triangular numbers is (3)

$$\tfrac{1}{2}n(n+1).$$

Prove that the sum of any two consecutive triangular numbers is a square number.

Solution

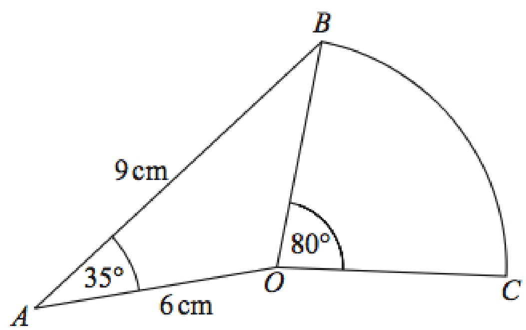
Well,

$$\begin{aligned}(n+1)\text{th number} &= \tfrac{1}{2}(n+1)[(n+1)+1] \\ &= \tfrac{1}{2}(n+1)(n+2)\end{aligned}$$

and the sum is

$$\begin{aligned}\tfrac{1}{2}n(n+1)+\tfrac{1}{2}(n+1)(n+2) &= \tfrac{1}{2}(n+1)[n+(n+2)] \\ &= \tfrac{1}{2}(n+1)(2n+2) \\ &= \tfrac{1}{2}(n+1)\cdot 2(n+1) \\ &= \underline{\underline{(n+1)^2}}.\end{aligned}$$

14. OAB is a triangle. (4)
OBC is a sector of a circle, centre O.



Calculate the area of OBC.
Give your answer correct to 3 significant figures.

Solution

Cosine rule:

$$\begin{aligned} & OB^2 = AO^2 + AB^2 - 2\times AO\times AB\times\cos OAB \\ \Rightarrow\quad & OB^2 = 6^2+9^2-2\times 6\times 9\times\cos 35^\circ \\ \Rightarrow\quad & OB^2 = 28.531\,579\,22 \text{ (FCD)} \\ \Rightarrow\quad & OB = 5.341\,495\,972 \text{ (FCD)}.\end{aligned}$$

Finally,

$$\begin{aligned}\text{area of } OBC &= \tfrac{80}{360} \times \pi \times 5.341\ldots^2 \\ &= 19.918\,799\,93 \text{ (FCD)} \\ &= \underline{\underline{19.9 \text{ cm}^2 \text{ (3 sf)}}}.\end{aligned}$$

15. (a) Factorise (1)

$$a^2 - b^2.$$

Solution
Difference of two squares:

$$a^2 - b^2 = \underline{\underline{(a-b)(a+b)}}.$$

(b) Show that (2)

$$2^{40} - 1$$

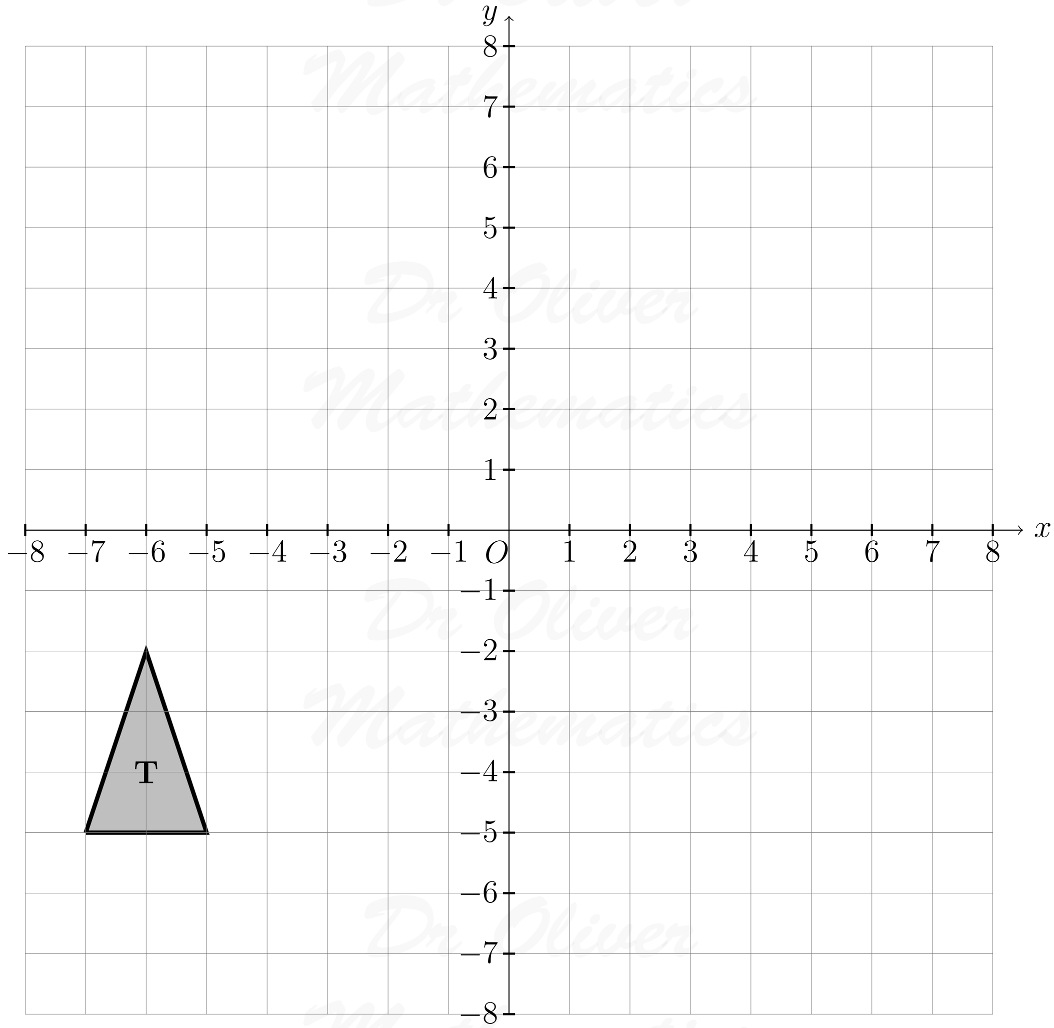
is the product of two consecutive odd numbers.

Solution
Well,

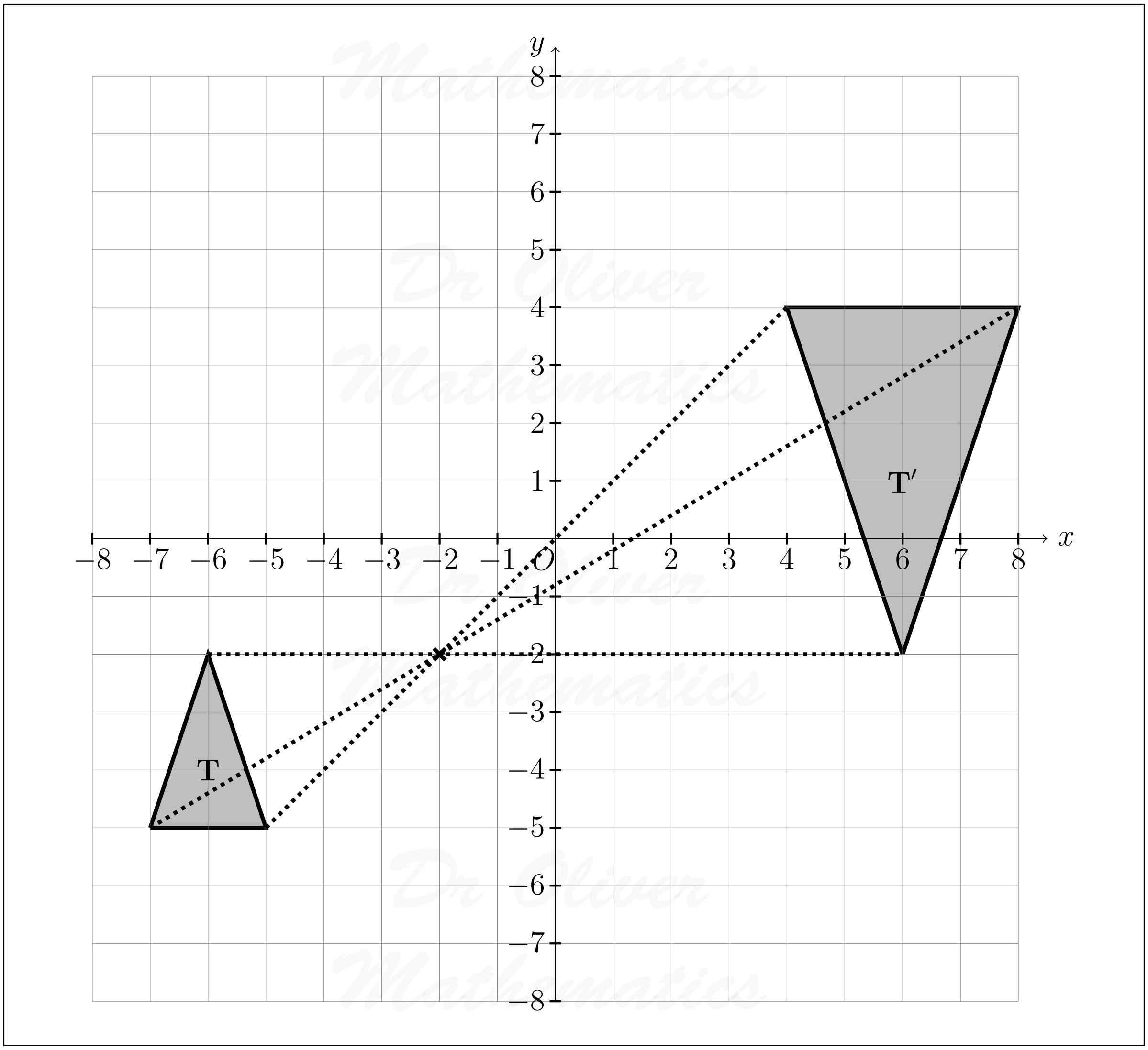
$$\begin{aligned}2^{40} - 1 &= (2^{20})^2 - 1^2 \\ &= (2^{20} - 1)(2^{20} + 1).\end{aligned}$$

Clearly, the expression is of consecutive odd numbers and it is a product.

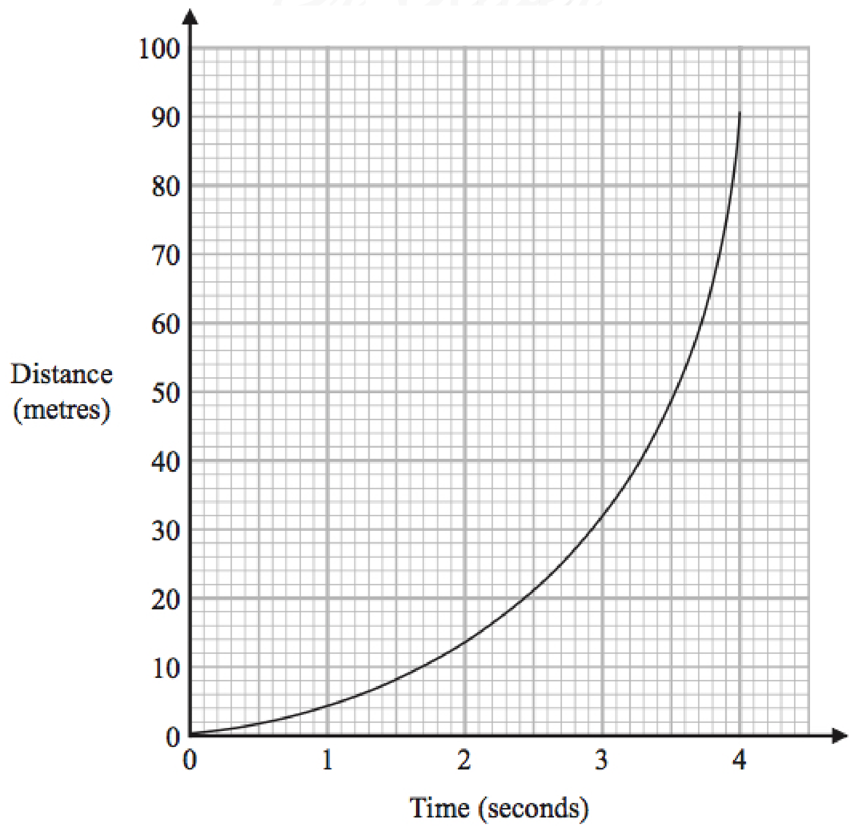
16. On the grid, enlarge triangle **T** by scale factor -2 with centre of enlargement $(-2, -2)$. (2)



Solution



17. Here is a distance-time graph.



(a) Find an estimate of the gradient of the graph at time 2.5 seconds. You must show how you get your answer. (3)

Solution

The gradient goes through $(1.3, 0)$ and $(4, 46)$:

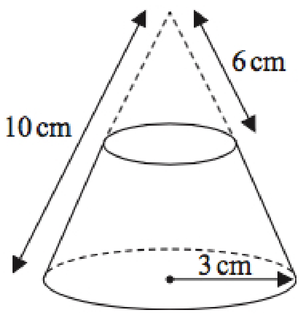
$$\begin{aligned}\text{gradient} &= \frac{46-0}{4-1.3}\\ &= \underline{\underline{17\tfrac{1}{27}\text{ m/s}}}.\end{aligned}$$

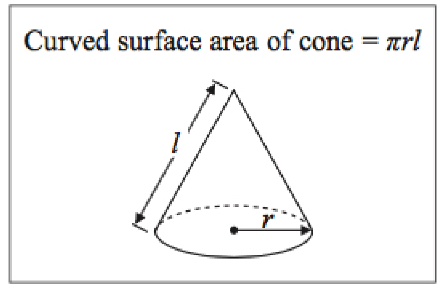
(b) What does the gradient of the graph represent? (1)

Solution

E.g., <u>speed</u> or <u>velocity</u>.

18. A solid frustum is made by removing a small cone from a large cone as shown in the diagram. (5)



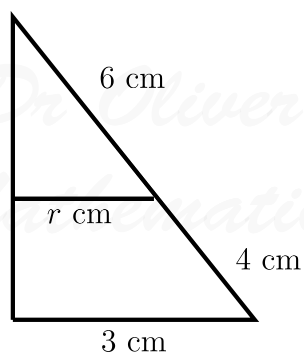


- The slant height of the small cone is 6 cm.
- The slant height of the large cone is 10 cm.
- The radius of the base of the large cone is 3 cm.

Calculate the total surface area of the frustum.
Give your answer correct to 3 significant figures.

Solution

Similar triangles:



Now,

$$\frac{r}{6} = \frac{3}{10} \Rightarrow r = \tfrac{3}{10} \times 6$$
$$\Rightarrow r = 1.8.$$

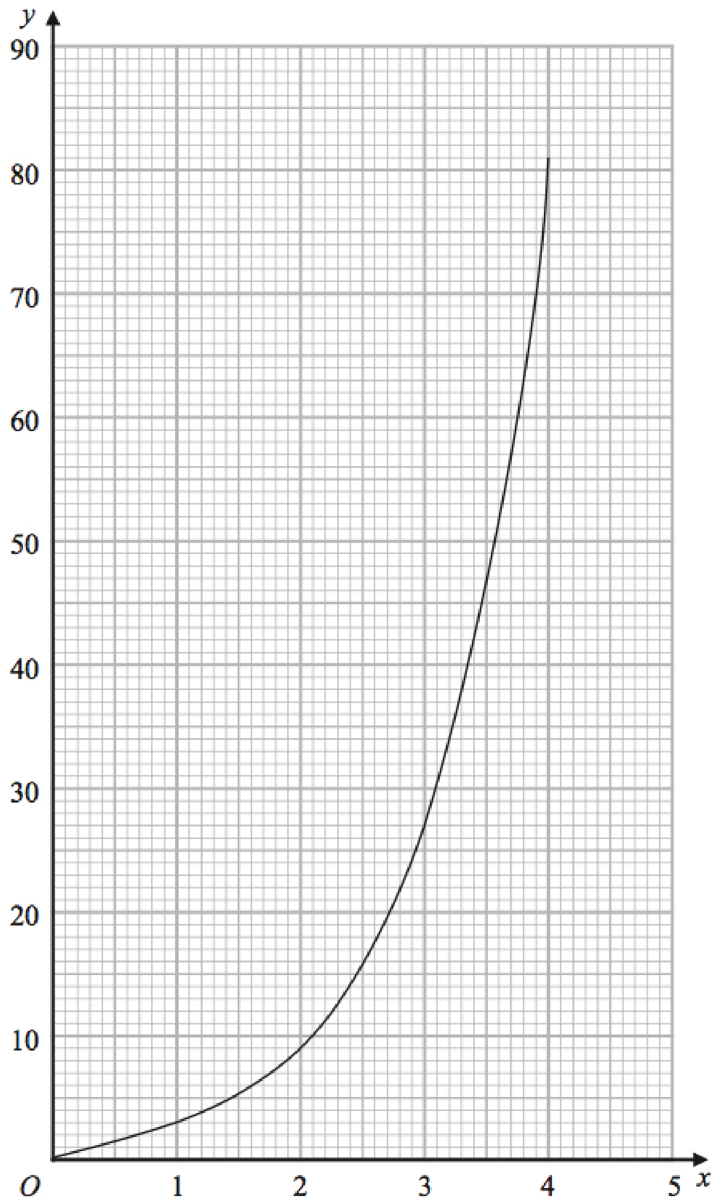
Finally,

$$\begin{aligned}\text{total surface area} &= (\text{big cone} - \text{little cone}) + \text{upper face} + \text{lower face}\\ &= (\pi \times 3 \times 10) - (\pi \times 1.8 \times 6) + (\pi \times 1.8^2) + (\pi \times 3^2)\\ &= 30\pi - 10.8\pi + 3.24\pi + 9\pi\\ &= 31.44\pi\\ &= \underline{\underline{98.8 \text{ cm}^2 \text{ (3 sf)}}}.\end{aligned}$$

19. Sana needs to draw the graph of (1)

$$y = 3^x \text{ for } 0 \leqslant x \leqslant 4.$$

She draws the graph shown on the grid.



Write down one thing Sana has done wrong.

> **Solution**
> E.g., she has drawn the graph through the origin — but it goes through $(0, 1)$.

20. Prove algebraically that (3)

$$0.1\dot{2}\dot{3}$$

can be written as

$$\tfrac{61}{495}.$$

Solution

Let $x = 0.1\dot{2}\dot{3}$. Then,

$$\begin{aligned} 10x &= 1.\dot{2}\dot{3} \quad (1) \\ 1\,000x &= 123.\dot{2}\dot{3} \quad (2). \end{aligned}$$

Do $(2) - (1)$:

$$\begin{aligned} 990x = 122 &\Rightarrow x = \tfrac{122}{990} \\ &\Rightarrow x = \tfrac{61 \times 2}{495 \times 2} \\ &\Rightarrow \underline{\underline{x = \tfrac{61}{495}}}, \end{aligned}$$

as required.

21. Solve (4)

$$\frac{1}{x+4} + \frac{3}{2-2x} = 1.$$

Solution

Now,

$\times$	x	$+4$
2	$2x$	$+8$
$-2x$	$-2x^2$	$-8x$

and we multiply by $(x+4)(2-2x)$:

$$\begin{aligned} \frac{1}{x+4} + \frac{3}{2-2x} = 1 &\Rightarrow (2-2x) + 3(x+4) = (x+4)(2-2x) \\ &\Rightarrow 2 - 2x + 3x + 12 = -2x^2 - 6x + 8 \\ &\Rightarrow 2x^2 + 7x + 6 = 0 \end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & +7 \\ \text{multiply to:} & (+2)\times(+6) = +12\end{array}\right\} +3,\ +4$$

e.g.,

$$\begin{aligned}
&\Rightarrow 2x^2 + 3x + 4x + 6 = 0 \\
&\Rightarrow x(2x+3) + 2(2x+3) = 0 \\
&\Rightarrow (x+2)(2x+3) = 0 \\
&\Rightarrow x + 2 = 0 \text{ or } 2x + 3 = 0 \\
&\Rightarrow \underline{\underline{x = -2 \text{ or } x = -1\tfrac{1}{2}}}.
\end{aligned}$$

22. Given that the vector (3)

$$a\begin{pmatrix} 2 \\ 6 \end{pmatrix} + b\begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

is parallel to the vector

$$\begin{pmatrix} 13 \\ 6 \end{pmatrix},$$

find an expression for b in terms of a.

Solution

Well,

$$a\begin{pmatrix} 2 \\ 6 \end{pmatrix} + b\begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 2a + 8b \\ 6a + 2b \end{pmatrix}$$

and

$$\begin{aligned}
\frac{2a+8b}{6a+2b} = \frac{13}{6} &\Rightarrow 6(2a+8b) = 13(6a+2b) \\
&\Rightarrow 12a + 48b = 78a + 26b \\
&\Rightarrow 22b = 66a \\
&\Rightarrow \underline{\underline{b = 3a}}.
\end{aligned}$$

23. A circle has equation (4)

$$x^2 + y^2 = 25.$$

The point P with coordinates $(-3, 4)$ lies on the circle.

Alex says that the tangent to the circle at P crosses the x-axis at the point $(-8, 0)$.

Is Alex correct?
You must show how you get your answer.

Solution

Now,

$$\begin{aligned} OP &= \frac{4-0}{-3-0} \\ &= -\tfrac{4}{3} \end{aligned}$$

which means

$$m_{\text{normal}} = -\frac{1}{-\frac{4}{3}} = \tfrac{3}{4}.$$

Next, the equation of the normal is

$$\begin{aligned} y - 4 = \tfrac{3}{4}(x+3) &\Rightarrow y - 4 = \tfrac{3}{4}x + \tfrac{9}{4} \\ &\Rightarrow y = \tfrac{3}{4}x + \tfrac{25}{4}. \end{aligned}$$

Finally,

$$\begin{aligned} y = 0 &\Rightarrow \tfrac{3}{4}x + \tfrac{25}{4} = 0 \\ &\Rightarrow \tfrac{3}{4}x = -\tfrac{25}{4} \\ &\Rightarrow \underline{\underline{x = -8\tfrac{1}{3}}} \end{aligned}$$

so, <u>no</u>, he is incorrect.

24. There is a total of y counters in a box. (5)

There are x pink counters and 5 blue counters in the box.
The rest of the counters are green.

$$x : y = 1 : 3.$$

Freda takes at random two counters from the box.

Find, in terms of x, an expression for the probability that Freda takes two counters of the same colour.

Give your answer as a fraction in the form

$$\frac{ax^2 + bx + c}{dx^2 + ex},$$

where a, b, c, d, and e are integers.

Solution

Well,

$$x : y = 1 : 3 \Rightarrow 3x = y$$

and

$$\begin{aligned}
\text{P(same colour)} &= \text{P}(PP) + \text{P}(BB) + \text{P}(GG) \\
&= (\tfrac{x}{3x} \times \tfrac{x-1}{3x-1}) + (\tfrac{5}{3x} \times \tfrac{4}{3x-1}) + (\tfrac{3x-x-5}{3x} \times \tfrac{3x-x-6}{3x-1}) \\
&= \frac{x(x-1) + 20 + (2x-5)(2x-6)}{3x(3x-1)}
\end{aligned}$$

$\times$	$2x$	-5
$2x$	$4x^2$	$-10x$
-6	$-12x$	$+30$

$$\begin{aligned}
&= \frac{x^2 - x + 20 + 4x^2 - 22x + 30}{9x^2 - 3x} \\
&= \underline{\underline{\frac{5x^2 - 23x + 50}{9x^2 - 3x}}}
\end{aligned}$$

hence, $\underline{\underline{a = 5}}$, $\underline{\underline{b = -23}}$, $\underline{\underline{c = 50}}$, $\underline{\underline{d = 9}}$, and $\underline{\underline{e = -3}}$.