Dr Oliver Mathematics
Further Mathematics
Numerical Solutions
Past Examination Questions

This booklet consists of 28 questions across a variety of examination topics.
The total number of marks available is 227.

There are three examples of this technique.

1 Interval bisection

We take, for example, $(a, \mathrm{f}(a)) < 0$ and $(b, \mathrm{f}(b)) > 0$. We then take

$$\left(\frac{a+b}{2}, \mathrm{f}\left(\frac{a+b}{2}\right)\right)$$

and it depends on the sign of

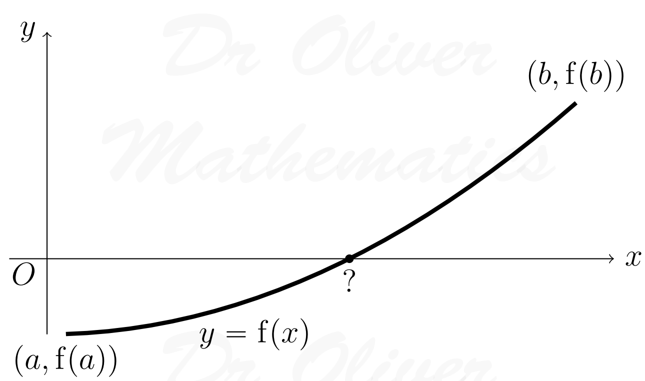
$$\mathrm{f}\left(\frac{a+b}{2}\right):$$

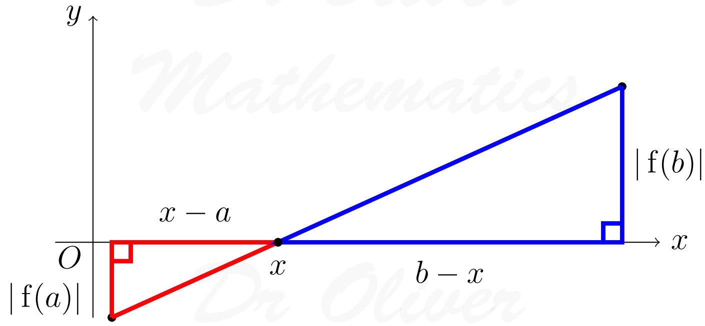
$$\left(\frac{a+b}{2}, \mathrm{f}\left(\frac{a+b}{2}\right)\right) < 0 \text{ then we take } \left(\frac{a+b}{2}, b\right)$$

and

$$\left(\frac{a+b}{2}, \mathrm{f}\left(\frac{a+b}{2}\right)\right) > 0 \text{ then we take } \left(a, \frac{a+b}{2}\right).$$

2 Linear interpolation





$$\begin{aligned}
&\frac{x-a}{b-x} = \frac{|\,\mathrm{f}(a)|}{|\,\mathrm{f}(b)|} \\
\Rightarrow\quad & |\,\mathrm{f}(b)|(x-a) = |\,\mathrm{f}(a)|(b-x) \\
\Rightarrow\quad & |\,\mathrm{f}(b)|x - |\,\mathrm{f}(b)|a = |\,\mathrm{f}(a)|b - |\,\mathrm{f}(a)|x \\
\Rightarrow\quad & |\,\mathrm{f}(a)|x + |\,\mathrm{f}(b)|x = a|\,\mathrm{f}(b)| + b|\,\mathrm{f}(a)| \\
\Rightarrow\quad & x\left(|\,\mathrm{f}(a)| + |\,\mathrm{f}(b)|\right) = a|\,\mathrm{f}(b)| + b|\,\mathrm{f}(a)| \\
\Rightarrow\quad & \boxed{x = \frac{a|\,\mathrm{f}(b)| + b|\,\mathrm{f}(a)|}{|\,\mathrm{f}(a)| + |\,\mathrm{f}(b)|}.}
\end{aligned}$$

3 Newton-Raphson

We take as a starting point

$$y - \mathrm{f}(a) = \mathrm{f}'(a)(x-a).$$

Set $y = 0$:

$$\begin{aligned}
-\,\mathrm{f}(a) = \mathrm{f}'(a)(x-a) &\Rightarrow x - a = -\frac{\mathrm{f}(a)}{\mathrm{f}'(a)} \\
&\Rightarrow x = a - \frac{\mathrm{f}(a)}{\mathrm{f}'(a)}.
\end{aligned}$$

That is, if a is an first approximation to a root of $\mathrm{f}(x) = 0$, a better approximation is, in general,

$$a - \frac{\mathrm{f}(a)}{\mathrm{f}'(a)};$$

i.e.,

$$\boxed{x_{n+1} = x_n - \frac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}.}$$

4 Questions

1. Figure 1 shows part of the graph of $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x \sin x + 2x - 3.$$

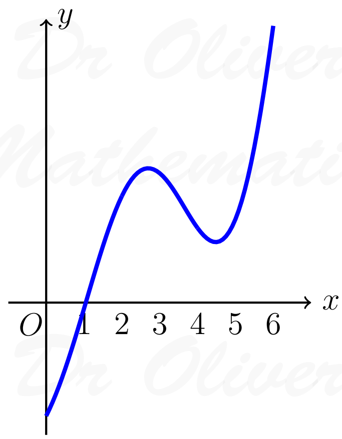


Figure 1: $\mathrm{f}(x) = x \sin x + 2x - 3$

The equation $\mathrm{f}(x) = 0$ has a single root α.

(a) Taking $x_1 = 1$ as a first approximation to α, apply the Newton-Raphson procedure once to $\mathrm{f}(x)$ to find a second approximation to α, to 3 significant figures. (5)

Solution

$$\begin{aligned} \mathrm{f}(x) = x \sin x + 2x - 3 &\Rightarrow \mathrm{f}(1) = -0.158\,529\,015\,2 \text{ (FCD)} \\ \mathrm{f}'(x) = \sin x + x \cos x + 2 &\Rightarrow \mathrm{f}'(1) = 3.381\,773\,291 \text{ (FCD)} \end{aligned}$$

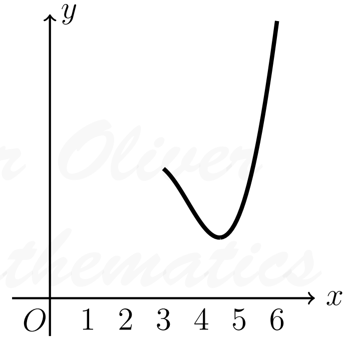
and

$$\begin{aligned} x_2 &= x_1 - \frac{\mathrm{f}(1)}{\mathrm{f}'(1)} \\ &= 1 - \frac{-0.158\,529\,015\,2}{3.381\,773\,291} \text{ (FCD)} \\ &= 1.046\,877\,482 \text{ (FCD)} \\ &= \underline{\underline{1.05 \text{ (3 sf)}}}. \end{aligned}$$

(b) Given $x_1 = 5$ as taken as a first approximation to α, apply the Newton-Raphson procedure,

(i) use Figure 1 to produce a rough sketch of $y = \mathrm{f}(x)$ for $3 \leqslant x \leqslant 6$ (1)

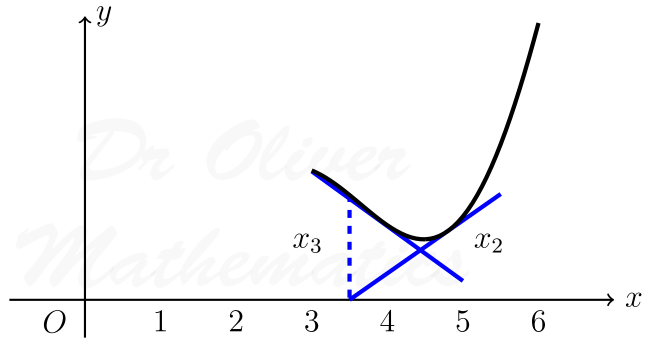
Solution



and by drawing suitable tangents, and without any further calculations,

(ii) show the approximate positions of x_2 and x_3, the second and third approximations to α. (1)

Solution



2.

$$\mathrm{f}(x) = 1 - \mathrm{e}^x + 3\sin 2x.$$

The equation $\mathrm{f}(x) = 0$ has a root α in $1.0 < x < 1.4$.

(a) Starting with the interval $(1.0, 1.4)$, use interval bisection three times to find of value of α to one decimal place. (3)

Solution

x	$\mathrm{f}(x)$
1	1.0096
1.2	-0.293
1.4	-2.05

and it is $(1.0, 1.2)$;

x	$\mathrm{f}(x)$
1	1.0096
1.1	0.4213
1.2	-0.293

and it is $(1.1, 1.2)$;

x	$\mathrm{f}(x)$
1.1	0.4213
1.15	0.078
1.2	-0.293

and $\underline{\underline{\alpha = 1.2 \text{ (1 dp)}}}$.

(b) Taking your answer to part (a) as a first approximation to α, apply the Newton-Raphson procedure once to $\mathrm{f}(x)$ to find a second approximation to α. (4)

Solution

$$\mathrm{f}(x) = 1 - \mathrm{e}^x + 3\sin 2x \Rightarrow \mathrm{f}(1.2) = -0.293\,727\,381\,1 \text{ (FCD)}$$
$$\mathrm{f}'(x) = -\mathrm{e}^x + 6\cos 2x \Rightarrow \mathrm{f}'(1.2) = -7.744\,479\,216 \text{ (FCD)}$$

and

$$\begin{aligned} x_2 &= x_1 - \frac{\mathrm{f}(1.2)}{\mathrm{f}'(1.2)} \\ &= 1.2 - \frac{-0.293\,727\,381\,1}{-7.744\,479\,216} \text{ (FCD)} \\ &= \underline{\underline{1.162\,072\,675 \text{ (FCD)}}}. \end{aligned}$$

(c) By considering the change in sign of $f(x)$ over an appropriate interval, show your your to part (b) is accurate to 2 decimal places. (2)

Solution

$f(1.155) = 0.042\ldots$
$f(1.165) = -0.027\ldots$
It is a continuous function and there is a change of sign.
Hence $1.155 < \alpha < 1.165$ and $\underline{\underline{\alpha = 1.16 \text{ (2 dp)}}}$.

3.

$$f(x) = 0.25x - 2 + 4\sin\sqrt{x}.$$

(a) Show that the equation $f(x) = 0$ has a root α between $x = 0.24$ and $x = 0.28$. (2)

Solution

$f(0.24) = -0.057\ldots$
$f(0.28) = 0.089\ldots$
It is a continuous function and there is a change of sign.
Hence $\underline{\underline{0.24 < \alpha < 0.28}}$.

(b) Starting with the interval $[0.24, 0.28]$, use interval bisection three times to find an interval of width 0.005 which contains α. (3)

Solution

x	$f(x)$
0.24	−0.057
0.26	0.0173
0.28	0.0891

and it is $[0.24, 0.26]$;

x	$f(x)$
0.24	−0.057
0.25	−0.019
0.26	0.0173

and it is $[0.25, 0.26]$;

x	$\mathrm{f}(x)$
0.25	-0.019
0.255	-0.001)
0.26	0.0173

and it is $\underline{\underline{[0.255, 0.26]}}$.

The equation $\mathrm{f}(x) = 0$ also has a root β between $x = 10.75$ and $x = 11.25$.

(c) Taking 11 as a first approximation to β, apply the Newton-Raphson procedure once to $\mathrm{f}(x)$ to find a second approximation to β. Give your answer to 2 decimal places. (6)

Solution

$$\mathrm{f}(x) = 0.25x - 2 + 4\sin\sqrt{x} \Rightarrow \mathrm{f}(11) = 0.053\,440\,866\,28 \text{ (FCD)}$$
$$\mathrm{f}'(x) = 0.25 + \frac{2}{\sqrt{x}}\cos\sqrt{x} \Rightarrow \mathrm{f}'(11) = -0.343\,809\,071\,2 \text{ (FCD)}$$

and

$$\begin{aligned} x_2 &= x_1 - \frac{\mathrm{f}(11)}{\mathrm{f}'(11)} \\ &= 11 - \frac{0.053\,440\,866\,28}{-0.343\,809\,071\,2} \text{ (FCD)} \\ &= 11.155\,437\,63 \text{ (FCD)} \\ &= \underline{\underline{11.16 \text{ (2 dp)}}}. \end{aligned}$$

4.

$$\mathrm{f}(x) = \ln x + x - 3, x > 0.$$

(a) Find $\mathrm{f}(2.0)$ and $\mathrm{f}(2.5)$, each to 4 decimal places, and show that the root α of the equation $\mathrm{f}(x) = 0$ satisfies $2.0 < \alpha < 2.5$. (3)

Solution

$\mathrm{f}(2.0) = -0.306\,852\,819\,4 \text{ (FCD)} = \underline{\underline{-0.3069 \text{ (4 dp)}}}$

$\mathrm{f}(2.5) = 0.416\,290\,731\,9 \text{ (FCD)} = \underline{\underline{0.4163 \text{ (4 dp)}}}$

It is a continuous function and there is a change of sign.
Hence $\underline{\underline{2.0 < \alpha < 2.5}}$.

(b) Use linear interpolation with your values of f(2.0) and f(2.5) to estimate α, giving your answer to 3 decimal places. (2)

Solution

$$\begin{aligned}\alpha &= \frac{2|f(2.5)| + 2.5|f(2)|}{|f(2)| + |f(2.5)|} \\ &= \frac{2|0.4163| + 2.5|-0.3069|}{|-0.3069| + |0.4163|} \\ &= 2.212\,181\,969 \text{ (FCD)} \\ &= \underline{\underline{2.212 \text{ (3 dp)}}}.\end{aligned}$$

(c) Taking 2.25 as a first approximation to α, apply the Newton-Raphson process once to $\mathrm{f}(x)$ to find a second approximation to α, giving your answer to 3 decimal places. (5)

Solution

$$\begin{aligned}\mathrm{f}(x) &= \ln x + x - 3 \Rightarrow \mathrm{f}(2.25) = 0.060\,930\,216\,22 \text{ (FCD)} \\ \mathrm{f}'(x) &= \frac{1}{x} + 1 \Rightarrow \mathrm{f}'(2.25) = \tfrac{13}{9}\end{aligned}$$

and

$$\begin{aligned}x_2 &= x_1 - \frac{\mathrm{f}(2.25)}{\mathrm{f}'(2.25)} \\ &= 2.25 - \frac{0.060\,930\,216\,22}{\frac{13}{9}} \text{ (FCD)} \\ &= 2.207\,817\,543 \text{ (FCD)} \\ &= \underline{\underline{2.208 \text{ (3 dp)}}}.\end{aligned}$$

(d) Show that your answer in part (d) gives α correct to 3 decimal places. (2)

Solution

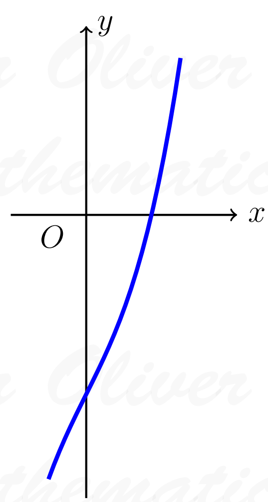
$\mathrm{f}(2.2075) = -0.00063\ldots$
$\mathrm{f}(2.2085) = 0.00081\ldots$
It is a continuous function and there is a change of sign.
Hence $2.2075 < \alpha < 2.2085$ and $\underline{\underline{\alpha = 2.208 \text{ (3 dp)}}}$.

5.

$$\mathrm{f}(x) = x^3 + 8x - 19.$$

(a) Show that the equation $\mathrm{f}(x) = 0$ has only one real root. (2)

Solution



$\mathrm{f}'(x) = 3x^2 + 8 > 0$ meaning it just $\underline{\underline{\text{one real root}}}$.

(b) Show that the equation $\mathrm{f}(x) = 0$ lies between 1 and 2. (2)

Solution

$\mathrm{f}(1) = -10$
$\mathrm{f}(2) = 5$
It is a continuous function and there is a change of sign.
Hence $\underline{\underline{1 < x < 2}}$.

(c) Obtain an approximation to the real root of $\mathrm{f}(x) = 0$ by performing two applications of Newton-Raphson procedure to$\mathrm{f}(x)$, using $x = 2$ as the first approximation. Give your answer to 3 decimal places. (4)

Solution

$$f(x) = x^3 + 8x - 19 \Rightarrow f(2) = 5$$
$$f'(x) = 3x^2 + 8 \Rightarrow f'(2) = 20$$

and

$$\begin{aligned} x_2 &= x_1 - \frac{f(2)}{f'(2)} \\ &= 2 - \frac{5}{20} \\ &= 1.75. \end{aligned}$$

Now,

$$f(1.75) = \tfrac{23}{64} \text{ and } f'(1.75) = \tfrac{275}{16}$$

and

$$\begin{aligned} x_3 &= x_2 - \frac{f(1.75)}{f'(1.75)} \\ &= 1.75 - \frac{\frac{23}{64}}{\frac{275}{16}} \\ &= 1.72\dot{9}\dot{0} \\ &= \underline{\underline{1.729 \text{ (3 dp)}}}. \end{aligned}$$

(d) By considering the change in sign of $f(x)$ over an appropriate interval, show that your answer to part (c) is accurate to 3 decimal places. (2)

Solution

$f(1.7285) = -0.0077\ldots$
$f(1.7295) = 0.0092\ldots$
It is a continuous function and there is a change of sign.
Hence $1.7285 < x < 1.7295$ and $\underline{\underline{x = 1.729 \text{ (3 dp)}}}$.

6.

$$f(x) = 3x^2 + x - \tan\left(\frac{x}{2}\right) - 2, -\pi < x < \pi.$$

The equation $f(x) = 0$ has a root α in the interval $[0.7, 0.8]$.

(a) Use linear interpolation, on the values at the end points of this interval, to obtain an approximation to α. Give your answer to 3 decimal places. (4)

Solution

$f(0.7) = -0.195\,028\,494\,8$ (FCD)
$f(0.8) = 0.297\,206\,781\,3$ (FCD)

$$\begin{aligned}\alpha &= \frac{0.7|\,f(0.8)| + 0.8|\,f(0.7)|}{|\,f(0.7)| + |\,f(0.8)|}\\ &= 0.739\,620\,991\,1 \text{ (FCD)}\\ &= \underline{\underline{0.740 \text{ (3 dp)}}}.\end{aligned}$$

(b) Taking 0.75 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α. Give your answer to 3 decimal places. (4)

Solution

$$\begin{aligned}f(x) &= 3x^2 + x - \tan\left(\frac{x}{2}\right) - 2 \Rightarrow f(0.75) = 0.043\,873\,424\,07\\ f'(x) &= 6x + 1 - \tfrac{1}{2}\sec^2\left(\frac{x}{2}\right) \Rightarrow f'(0.75) = 4.922\,529\,059\end{aligned}$$

and

$$\begin{aligned}x_2 &= 0.75 - \frac{f(0.75)}{f'(0.75)}\\ &= 0.741\,087\,218\,9 \text{ (FCD)}\\ &= \underline{\underline{0.741 \text{ (3 dp)}}}.\end{aligned}$$

7.

$$f(x) = 4\cos x + e^{-x}.$$

(a) Show that the equation $f(x) = 0$ has a root α between 1.6 and 1.7. (2)

Solution

$f(1.6) = 0.085\ldots$
$f(1.7) = -0.33\ldots$
It is a continuous function and there is a change of sign.
Hence $\underline{\underline{1.6 < \alpha < 1.7}}$.

(b) Taking 1.6 as a first approximation to α, apply the Newton-Raphson process once (4) to $\mathrm{f}(x)$ to find a second approximation to α. Give your answer to 3 significant figures.

Solution

$$\mathrm{f}(x) = 4\cos x + \mathrm{e}^{-x} \Rightarrow \mathrm{f}(1.6) = 0.085\,098\,428\,79$$
$$\mathrm{f}'(x) = -4\sin x - \mathrm{e}^{-x} \Rightarrow \mathrm{f}'(1.6) = -4.200\,190\,93$$

and

$$\begin{aligned} x_2 &= 1.6 - \frac{\mathrm{f}(1.6)}{\mathrm{f}'(1.6)} \\ &= 1.620\,260\,61 \text{ (FCD)} \\ &= \underline{\underline{1.62 \text{ (3 sf)}}}. \end{aligned}$$

8.

$$\mathrm{f}(x) = 3\sqrt{x} + \frac{18}{\sqrt{x}} - 20.$$

(a) Show that the equation $\mathrm{f}(x) = 0$ has a root α between $[1.1, 1.2]$. (2)

Solution

$\mathrm{f}(1.1) = 0.308\ldots$
$\mathrm{f}(1.2) = -0.281\ldots$
It is a continuous function and there is a change of sign.
Hence $\underline{\underline{1.1 < \alpha < 1.2}}$.

(b) Find $\mathrm{f}'(x)$. (3)

Solution

$$\begin{aligned} \mathrm{f}(x) = 3\sqrt{x} + \frac{18}{\sqrt{x}} - 20 &\Rightarrow \mathrm{f}(x) = 3x^{\frac{1}{2}} + 18x^{-\frac{1}{2}} - 20 \\ &\Rightarrow \underline{\underline{\mathrm{f}'(x) = \tfrac{3}{2}x^{-\frac{1}{2}} - 9x^{-\frac{3}{2}}}}. \end{aligned}$$

(c) Using $x_0 = 1.1$ as a first approximation to α, apply the Newton-Raphson process once to $\mathrm{f}(x)$ to find a second approximation to α, giving your answer to 3 significant figures. (4)

Solution

$$f(x) = 3x^{\frac{1}{2}} + 18x^{-\frac{1}{2}} - 20 \Rightarrow f(1.1) = 0.308\,753\,150\,9$$
$$f'(x) = \tfrac{3}{2}x^{-\frac{1}{2}} - 9x^{-\frac{3}{2}} \Rightarrow f'(1.1) = -6.370\,863\,665$$

and

$$\begin{aligned} x_2 &= 1.1 - \frac{f(1.1)}{f'(1.1)} \\ &= 1.148\,463\,312 \text{ (FCD)} \\ &= \underline{\underline{1.15 \text{ (3 sf)}}}. \end{aligned}$$

9. Figure 2 shows part of the curve with equation $y = f(x)$, where

$$f(x) = 1 - x - \sin(x^2).$$

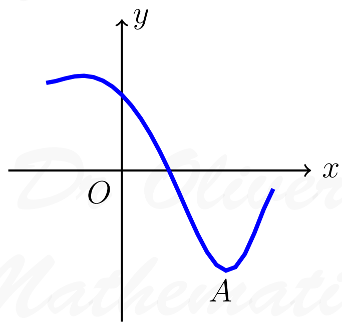


Figure 2: $f(x) = 1 - x - \sin(x^2)$

The point A, with x-coordinate p, is a stationary point on the curve. The equation $f(x) = 0$ has a root α in the interval $0.6 < \alpha < 0.7$.

(a) Explain why $x_0 = p$ is not suitable to use as a first approximation to α when applying the Newton-Raphson process to $f(x)$. (1)

Solution

Because it is stationary, i.e., the tangent is parallel to the x-axis

(b) Using $x_0 = 0.6$ as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places. (5)

Solution

$$f(x) = 1 - x - \sin(x^2) \Rightarrow f(0.6) = 0.047\,725\,766\,72$$
$$f'(x) = -1 - 2x\cos(x^2) \Rightarrow f'(0.6) = -2.123\,076\,188$$

and

$$\begin{aligned} x_2 &= 0.6 - \frac{f(0.6)}{f'(0.6)} \\ &= 0.622\,479\,535\,6 \text{ (FCD)} \\ &= \underline{\underline{0.622 \text{ (3 dp)}}}. \end{aligned}$$

(c) By considering the change in sign of $f(x)$ over an appropriate interval, show your your to part (b) is accurate to 3 decimal places. (2)

Solution

$f(0.6215) = 0.0017\ldots$
$f(0.6225) = -0.00038\ldots$
It is a continuous function and there is a change of sign.
Hence $0.6215 < \alpha < 0.6225$ and $\underline{\underline{\alpha = 0.622 \text{ (3 dp)}}}$.

10. Given that α is the only real root of the equation

$$\sin 2x - \ln 3x = 0,$$

(a) show that $0.8 < \alpha < 0.9$. (2)

Solution

$f(0.8) = 0.124\ldots$
$f(0.9) = -0.019\ldots$
It is a continuous function and there is a change of sign.
Hence $\underline{\underline{0.8 < \alpha < 0.9}}$.

(b) Taking 0.9 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places. (5)

Solution

$$f(x) = \sin 2x - \ln 3x \Rightarrow f(0.9) = -0.019\,404\,142\,13$$
$$f'(x) = 2\cos 2x - \frac{1}{x} \Rightarrow f'(0.9) = -1.565\,515\,3$$

and

$$\begin{aligned} x_2 &= 0.9 - \frac{f(0.9)}{f'(0.9)} \\ &= 0.887\,605\,268\,3 \text{ (FCD)} \\ &= \underline{\underline{0.888 \text{ (3 dp)}}}. \end{aligned}$$

(c) Use linear interpolation once on the interval $[0.8, 0.9]$ to find another approximation to α, giving your answer to 3 decimal places. (3)

Solution

$f(0.8) = 0.124\,104\,865\,7$ (FCD)
$f(0.9) = -0.019\,404\,142\,13$ (FCD)

$$\begin{aligned} \alpha &= \frac{0.8|f(0.9)| + 0.9|f(0.8)|}{|f(0.8)| + |f(0.9)|} \\ &= 0.886\,478\,798\,5 \text{ (FCD)} \\ &= \underline{\underline{0.886 \text{ (3 dp)}}}. \end{aligned}$$

11. Given that α is the only real root of the equation

$$x^3 - x^2 - 6 = 0,$$

(a) show that $2.2 < \alpha < 2.3$. (2)

Solution

$f(2.2) = -0.192$
$f(2.3) = 0.877$
It is a continuous function and there is a change of sign.
Hence $\underline{\underline{2.2 < \alpha < 2.3}}$.

(b) Taking 2.2 as a first approximation to α, apply the Newton-Raphson process once (5) to $\mathrm{f}(x)$ to find a second approximation to α, giving your answer to 3 decimal places.

Solution

$$\begin{aligned}\mathrm{f}(x) &= x^3 - x^2 - 6 \Rightarrow \mathrm{f}(2.2) = -0.192 \\ \mathrm{f}'(x) &= 3x^2 - 2x \Rightarrow \mathrm{f}'(2.2) = 10.12\end{aligned}$$

and

$$\begin{aligned}x_2 &= 2.2 - \frac{\mathrm{f}(2.2)}{\mathrm{f}'(2.2)} \\ &= 2.218\,972\,332 \text{ (FCD)} \\ &= \underline{\underline{2.219 \text{ (3 dp)}}}.\end{aligned}$$

(c) Use linear interpolation once on the interval $[2.2, 2.3]$ to find another approximation to α, giving your answer to 3 decimal places. (3)

Solution

$\mathrm{f}(2.2) = -0.192$
$\mathrm{f}(2.3) = 0.877$

$$\begin{aligned}\alpha &= \frac{2.2|\,\mathrm{f}(2.3)| + 2.3|\,\mathrm{f}(2.2)|}{|\,\mathrm{f}(2.2)| + |\,\mathrm{f}(2.3)|} \\ &= 2.171\,970\,803 \text{ (FCD)} \\ &= \underline{\underline{2.172 \text{ (3 dp)}}}.\end{aligned}$$

12.

$$\mathrm{f}(x) = x\cos x - 2x + 5.$$

(a) Show that $\mathrm{f}(x) = 0$ has a root α in the interval $[2, 2.1]$. (2)

Solution

$\mathrm{f}(2) = 0.167\ldots$
$\mathrm{f}(2.1) = -0.260\ldots$
It is a continuous function and there is a change of sign.
Hence $\underline{\underline{2 < \alpha < 2.1}}$.

(b) Taking 2 as a first approximation to α, apply the Newton-Raphson process once to $\mathrm{f}(x)$ to find a second approximation to α, giving your answer to 2 decimal places. (5)

Solution

$$\mathrm{f}(x) = x\cos x - 2x + 5 \Rightarrow \mathrm{f}(2) = 0.167\,706\,326\,9 \text{ (FCD)}$$
$$\mathrm{f}'(x) = \cos x - x\sin x - 2 \Rightarrow \mathrm{f}'(2) = -4.234\,741\,69 \text{ (FCD)}$$

and

$$\begin{aligned} x_2 &= 2 - \frac{\mathrm{f}(2)}{\mathrm{f}'(2)} \\ &= 2.036\,602\,493 \text{ (FCD)} \\ &= \underline{\underline{2.04 \text{ (2 dp)}}}. \end{aligned}$$

(c) Show that your answer to part (b) gives α to 2 decimal places. (2)

Solution

$\mathrm{f}(2.035) = 0.018\ldots$
$\mathrm{f}(2.045) = -0.023\ldots$
It is a continuous function and there is a change of sign.
Hence $2.035 < \alpha < 2.045$ and $\underline{\underline{\alpha = 2.04 \text{ (2 dp)}}}$.

13.

$$\mathrm{f}(x) = 3x^2 - \frac{11}{x^2}.$$

(a) Write down, to 3 decimal places, the value of $\mathrm{f}(1.3)$ and the value of $\mathrm{f}(1.4)$. (1)

Solution

$\mathrm{f}(1.3) = -1.438\,875\,74 \text{ (FCD)} = \underline{\underline{-1.439 \text{ (3 dp)}}}$

$\mathrm{f}(1.4) = 0.267\,755\,102 \text{ (FCD)} = \underline{\underline{0.268 \text{ (3 dp)}}}$

The equation $\mathrm{f}(x) = 0$ has a root α between 1.3 and 1.4.

(b) Starting with the interval $[1.3, 1.4]$, use interval bisection three times to find an interval of width 0.025 which contains α. (3)

Solution

x	$\mathrm{f}(x)$
1.3	-1.438
1.35	-0.568
1.4	0.267

and it is $[1.35, 1.4]$;

x	$\mathrm{f}(x)$
1.35	-0.568
1.375	-0.146
1.4	0.267

and it is $\underline{\underline{[1.375, 1.4]}}$.

(c) Taking 1.4 as a first approximation to α, apply the Newton-Raphson process once to $\mathrm{f}(x)$ to find a second approximation to α, giving your answer to 3 decimal places. (5)

Solution

$$\mathrm{f}(x) = 3x^2 - \frac{11}{x^2} \Rightarrow \mathrm{f}(x) = 3x^2 - 11x^{-2}$$
$$\Rightarrow \mathrm{f}'(x) = 6x + 22x^{-3}$$

and we have

$$\mathrm{f}'(1.4) = 16.417\,492\,71 \text{ (FCD)}.$$

Now,

$$\begin{aligned} x_2 &= 1.4 - \frac{\mathrm{f}(1.4)}{\mathrm{f}'(1.4)} \\ &= 1.383\,675948\,493 \text{ (FCD)} \\ &= \underline{\underline{1.384 \text{ (2 dp)}}}. \end{aligned}$$

14.

$$f(x) = x^3 - \frac{7}{x} + 2, x > 0.$$

(a) Show that the equation $f(x) = 0$ has a root α between 1.4 and 1.5. (2)

Solution

$f(1.4) = -\frac{32}{125}$
$f(1.5) = \frac{17}{24}$
It is a continuous function and there is a change of sign.
Hence $\underline{\underline{1.4 < \alpha < 1.5}}$.

(b) Starting with the interval $[1.4, 1.5]$, use interval bisection three times to find an interval of width 0.025 which contains α. (3)

Solution

x	$f(x)$
1.4	−0.256
1.45	0.221
1.5	0.708

and it is $[1.4, 1.45]$;

x	$f(x)$
1.4	−0.256
1.425	−0.018
1.45	0.221

and it is $\underline{\underline{[1.425, 1.45]}}$.

(c) Taking 1.45 as a first approximation to α, apply the Newton-Raphson process once to (5)

$$f(x) = x^3 - \frac{7}{x} + 2$$

to find a second approximation to α, giving your answer to 3 decimal places.

Solution

$$\mathrm{f}(x) = x^3 - \frac{7}{x} + 2 \Rightarrow \mathrm{f}(x) = x^3 - 7x^{-1} + 2$$
$$\Rightarrow \mathrm{f}'(x) = 3x^2 + 7x^{-2}.$$

Now,

$$\mathrm{f}(1.45) = 0.221\,038\,793\,1 \text{ (FCD) and } \mathrm{f}'(1.45) = 9.636\,869\,798 \text{ (FCD)}.$$

and

$$\begin{aligned} x_2 &= 1.45 - \frac{\mathrm{f}(1.45)}{\mathrm{f}'(1.45)} \\ &= 1.427\,063\,217 \text{ (FCD)} \\ &= \underline{\underline{1.427 \text{ (2 dp)}}}. \end{aligned}$$

15.

$$\mathrm{f}(x) = 5x^2 - 4x^{\frac{3}{2}} - 6, x \geqslant 0.$$

The root α the equation $\mathrm{f}(x) = 0$ lies in the interval $[1.6, 1.8]$.

(a) Use linear interpolation once on the interval $[1.6, 1.8]$ to find to approximation to α. Give your answer to 3 decimal places. (4)

Solution

$\mathrm{f}(1.6) = -1.295\,430\,81$ (FCD)
$\mathrm{f}(1.8) = 0.540\,186\,337\,2$ (FCD)

$$\begin{aligned} \alpha &= \frac{1.6|\mathrm{f}(1.8)| + 1.8|\mathrm{f}(1.6)|}{|\mathrm{f}(1.6)| + |\mathrm{f}(1.8)|} \\ &= 1.741\,143\,899 \text{ (FCD)} \\ &= \underline{\underline{1.741 \text{ (3 dp)}}}. \end{aligned}$$

(b) Differentiate $\mathrm{f}(x)$ to find $\mathrm{f}'(x)$. (2)

Solution

$$\mathrm{f}(x) = 5x^2 - 4x^{\frac{3}{2}} - 6 \Rightarrow \underline{\underline{\mathrm{f}'(x) = 10x - 6x^{\frac{1}{2}}}}.$$

(c) Taking 1.7 as a first approximation to α, apply the Newton-Raphson process once (4) to $f(x)$ to find a second approximation to α. Give your answer to 3 decimal places.

Solution

$$f(1.7) = -0.416\,115\,271\,1 \text{ (FCD) and } f'(1.7) = 9.176\,957\,114 \text{ (FCD)}.$$

and

$$\begin{aligned} x_2 &= 1.7 - \frac{f(1.7)}{f'(1.7)} \\ &= 1.745\,343\,491 \text{ (FCD)} \\ &= \underline{\underline{1.745 \text{ (3 dp)}}}. \end{aligned}$$

16.

$$f(x) = 3^x + 3x - 7.$$

(a) Show that $f(x) = 0$ has a root α between $x = 1$ and $x = 2$. (2)

Solution

$f(1) = -1$
$f(2) = 8$
It is a continuous function and there is a change of sign.
Hence $\underline{\underline{1 < \alpha < 2}}$.

(b) Starting with the interval $[1, 2]$, use interval bisection three times to find an interval of width 0.25 which contains α. (3)

Solution

x	$f(x)$
1	-1
1.5	2.6961
2	8

and it is $[1, 1.5]$;

x	$\mathrm{f}(x)$
1	-1
1.25	0.6982
1.5	2.6961

and it is $\underline{\underline{[1, 1.25]}}$.

17.

$$\mathrm{f}(x) = x^2 + \frac{5}{2x} - 3x - 1, x \neq 0.$$

(a) Differentiate $\mathrm{f}(x)$ to find $\mathrm{f}'(x)$. (2)

Solution

$$\mathrm{f}(x) = x^2 + \frac{5}{2x} - 3x - 1 \Rightarrow \mathrm{f}(x) = x^2 + \tfrac{5}{2}x^{-1} - 3x - 1$$
$$\Rightarrow \underline{\underline{\mathrm{f}'(x) = 2x - \tfrac{5}{2}x^{-2} - 3}}.$$

The root α the equation $\mathrm{f}(x) = 0$ lies in the interval $[0.7, 0.9]$.

(b) Taking 0.8 as a first approximation to α, apply the Newton-Raphson process once to $\mathrm{f}(x)$ to find a second approximation to α. Give your answer to 3 decimal places. (4)

Solution

$$\mathrm{f}(0.8) = \tfrac{73}{200} \text{ and } \mathrm{f}'(0.8) = -\tfrac{849}{160}.$$

Now,

$$\begin{aligned} x_2 &= 0.8 - \frac{\mathrm{f}(0.8)}{\mathrm{f}'(0.8)} \\ &= 0.868\,786\,808 \text{ (FCD)} \\ &= \underline{\underline{0.869 \text{ (3 dp)}}}. \end{aligned}$$

18. (a) Show that $\mathrm{f}(x) = x^4 + x - 1$ has a real root α in the interval $[0.5, 1.0]$. (2)

Solution

$\mathrm{f}(0.5) = -\frac{7}{16}$
$\mathrm{f}(2) = 1$
It is a continuous function and there is a change of sign.
Hence $\underline{\underline{0.5 < \alpha < 1}}$.

(b) Starting with the interval $[0.5, 1.0]$, use interval bisection three times to find an interval of width 0.125 which contains α. (3)

Solution

x	$\mathrm{f}(x)$
0.5	−0.437
0.75	0.4611
1	1

and it is $[0.5, 0.75]$;

x	$\mathrm{f}(x)$
0.5	−0.437
0.625	−0.222
0.75	0.4611

and it is $\underline{\underline{[0.625, 0.75]}}$.

(c) Taking 0.75 as a first approximation to α, apply the Newton-Raphson process once to $\mathrm{f}(x)$ to find a second approximation to α. Give your answer to 3 decimal places. (5)

Solution

$$\mathrm{f}(x) = x^4 + x - 1 \Rightarrow \mathrm{f}(0.75) = \tfrac{17}{256}$$
$$\mathrm{f}'(x) = 4x^3 + 1 \Rightarrow \mathrm{f}'(0.75) = \tfrac{43}{16}$$

and

$$\begin{aligned} x_2 &= 0.75 - \frac{\mathrm{f}(0.75)}{\mathrm{f}'(0.75)} \\ &= 0.725\,290\,697\,7 \text{ (FCD)}. \end{aligned}$$

Now,

$$f(x_2) = 2.015\,719\,042 \times 10^{-3} \text{ and } f'(x_2) = 2.526\,146\,811$$

and

$$\begin{aligned} x_3 &= 0.725\,290\,697\,7 - \frac{2.015\,719\,042 \times 10^{-3}}{2.526\,146\,811} \text{ (FCD)} \\ &= 0.724\,492\,755\,5 \text{ (FCD)} \\ &= \underline{\underline{0.724 \text{ (3 dp)}}}. \end{aligned}$$

19. (6)

$$f(x) = x^2 + \frac{3}{4\sqrt{x}} - 3x - 7, x > 0.$$

The root α the equation $f(x) = 0$ lies in the interval $[3, 5]$.
Taking 4 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α. Give your answer to 2 decimal places.

Solution

$$\begin{aligned} f(x) = x^2 + \frac{3}{4\sqrt{x}} - 3x - 7 &\Rightarrow f(x) = x^2 + \tfrac{3}{4}x^{-\frac{1}{2}} - 3x - 7 \\ &\Rightarrow f'(x) = 2x - \tfrac{3}{8}x^{-\frac{3}{2}} - 3. \end{aligned}$$

Now,

$$f(4) = -\tfrac{21}{8} \text{ and } f'(4) = \tfrac{317}{64}$$

and

$$\begin{aligned} x_2 &= 4 - \frac{f(4)}{f'(4)} \\ &= 4.529\,968\,454 \text{ (FCD)} \\ &= \underline{\underline{4.53 \text{ (2 dp)}}}. \end{aligned}$$

20.

$$f(x) = \tan\left(\frac{x}{2}\right) + 3x - 6, -\pi < x < \pi.$$

(a) Show that $f(x) = 0$ has a root α in the interval $[1, 2]$. (2)

Solution

$f(1) = -2.453\,697\,51$ (FCD)
$f(2) = 1.557\,407\,725$ (FCD)
It is a continuous function and there is a change of sign.
Hence $\underline{\underline{1 < \alpha < 2}}$.

(b) Use linear interpolation once on the interval $[2.2, 2.3]$ to find another approximation to α, giving your answer to 3 decimal places. (3)

Solution

$$\begin{aligned}\alpha &= \frac{1|\,f(2)| + 2|\,f(1)|}{|\,f(1)| + |\,f(2)|} \\ &= 1.611\,726\,037 \text{ (FCD)} \\ &= \underline{\underline{1.61 \text{ (2 dp)}}}.\end{aligned}$$

21.

$$f(x) = 2x^{\frac{1}{2}} + x^{-\frac{1}{2}} - 5, x > 0.$$

(a) Find $f'(x)$. (2)

Solution

$$f(x) = 2x^{\frac{1}{2}} + x^{-\frac{1}{2}} - 5 \Rightarrow \underline{\underline{f'(x) = x^{-\frac{1}{2}} - \tfrac{1}{2}x^{-\frac{3}{2}}}}.$$

The equation $f(x) = 0$ has a root α in the interval $[4.5, 5.5]$.

(b) Using $x_0 = 5$ as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α, giving your answer to 3 significant figures. (4)

Solution

$$f(5) = -0.080\,650\,449\,5 \text{ (FCD) and } f'(5) = 0.402\,492\,235\,9 \text{ (FCD)}$$

and

$$\begin{aligned} x_1 &= 5 - \frac{\mathrm{f}(5)}{\mathrm{f}'(5)} \\ &= 5.200\,377\,653 \text{ (FCD)} \\ &= \underline{\underline{5.20 \text{ (2 dp)}}}. \end{aligned}$$

22.

$$\mathrm{f}(x) = \cos(x^2) - x + 3, 0 < x < \pi.$$

(a) Show that the equation $\mathrm{f}(x) = 0$ has a root α in the interval $[2.5, 3]$. (2)

Solution

$\mathrm{f}(2.5) = 1.499\,449\,418$ (FCD)
$\mathrm{f}(3) = -0.911\,130\,261\,1$ (FCD)
It is a continuous function and there is a change of sign.
Hence $\underline{\underline{2.5 < \alpha < 3}}$.

(b) Use linear interpolation once on the interval $[2.5, 3]$ to find to approximation to α, giving your answer to 2 decimal places. (3)

Solution

$$\begin{aligned} \alpha &= \frac{2.5|\mathrm{f}(3)| + 3|\mathrm{f}(2.5)|}{|\mathrm{f}(2.5)| + |\mathrm{f}(3)|} \\ &= 2.811\,014\,282 \text{ (FCD)} \\ &= \underline{\underline{2.81 \text{ (2 dp)}}}. \end{aligned}$$

23.

$$\mathrm{f}(x) = \tfrac{1}{2}x^4 - x^3 + x - 3.$$

(a) Show that $\mathrm{f}(x) = 0$ has a root α in the interval $x = 2$ and $x = 2.5$. (2)

Solution

$\mathrm{f}(2) = -1$
$\mathrm{f}(2.5) = 3.406\,25$

It is a continuous function and there is a change of sign.
Hence $\underline{\underline{2 < \alpha < 2.5}}$.

(b) Starting with the interval $[2, 2.5]$, use interval bisection twice to find an interval of width 0.125 which contains α. (3)

Solution

x	$f(x)$
2	-1
2.25	0.673
2.5	3.406

and it is $[2, 2.25]$;

x	$f(x)$
2	-1
2.125	-0.275
2.25	0.673

and it is $\underline{\underline{[2.125, 2.25]}}$.

The equation $f(x) = 0$ has a root β in the interval $[-2, -1]$.

(c) Taking -1.5 as a first approximation to β, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to β. Give your answer to 2 decimal places. (5)

Solution

$$f(x) = \tfrac{1}{2}x^4 - x^3 + x - 3 \Rightarrow f(-1.5) = \tfrac{45}{32}$$
$$f'(x) = 2x^3 - 3x^2 + 1 \Rightarrow f'(-1.5) = -\tfrac{25}{2}$$

and

$$\begin{aligned} x_2 &= -1.5 - \frac{f(-1.5)}{f'(-1.5)} \\ &= -1.3875 \text{ (FCD)} \\ &= \underline{\underline{-1.39 \text{ (2 dp)}}}. \end{aligned}$$

24.

$$f(x) = x^3 - \frac{5}{2x^{\frac{3}{2}}} + 2x - 3, x > 0.$$

(a) Show that $f(x) = 0$ has a root α in the interval $[1.1, 1.5]$. (2)

Solution

$f(1.1) = -1.635\,960\,43$ (FCD)
$f(1.5) = 2.014\,172\,3$ (FCD)
It is a continuous function and there is a change of sign.
Hence $\underline{\underline{1.1 < \alpha < 1.5}}$.

(b) Find $f'(x)$. (2)

Solution

$$f(x) = x^3 - \frac{5}{2x^{\frac{3}{2}}} + 2x - 3 \Rightarrow f(x) = x^3 - \tfrac{5}{2}x^{-\frac{3}{2}} + 2x - 3$$
$$\Rightarrow \underline{\underline{f'(x) = 3x^2 + \tfrac{15}{4}x^{-\frac{5}{2}} + 2}}.$$

(c) Using $x_0 = 1.1$ as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places. (3)

Solution

$$f'(1.1) = 8.584\,946\,041 \text{ (FCD)}$$

and

$$\begin{aligned} x_1 &= 1.1 - \frac{f(1.1)}{f'(1.1)} \\ &= 1.290\,561\,527 \text{ (FCD)} \\ &= \underline{\underline{1.291 \text{ (3 dp)}}}. \end{aligned}$$

25.

$$f(x) = 3\cos 2x + x - 2, -\pi \leqslant x \leqslant \pi.$$

(a) Show that $f(x) = 0$ has a root α in the interval $[2, 3]$. (2)

Solution

$f(2) = -1.960\,930\,863$ (FCD)
$f(3) = 3.880\,510\,86$ (FCD)
It is a continuous function and there is a change of sign.
Hence $\underline{\underline{2 < \alpha < 3}}$.

(b) Use linear interpolation once on the interval $[2, 3]$ to find to approximation to α. Give your answer to 3 decimal places. (3)

Solution

$$\begin{aligned}\alpha &= \frac{2|\,f(3)| + 3|\,f(2)|}{|\,f(2)| + |\,f(3)|} \\ &= 2.335\,692\,96 \text{ (FCD)} \\ &= \underline{\underline{2.336 \text{ (3 dp)}}}.\end{aligned}$$

(c) The equation $f(x) = 0$ has a root β in the interval $[-1, -0]$. Starting with the interval $[-1, 0]$, use interval bisection to find an interval of width 0.25 which contains β. (4)

Solution

x	$f(x)$
-1	-4.248
-0.5	-0.879
0	1

and it is $[-0.5, 0]$;

x	$f(x)$
-0.5	-0.879
-0.25	0.3827
0	1

and it is $\underline{\underline{[-0.5, -0.25]}}$.

26. In the interval $13 < x < 14$, the equation

$$3 + x \sin\left(\frac{x}{4}\right) = 0,$$

where x is measured in radians, has exactly one root.

(a) Starting with the interval $[13, 14]$, use interval bisection twice to find an interval of width 0.25 which contains α. (3)

Solution

x	$\mathrm{f}(x)$
13	1.5934
13.5	−0.122
14	−1.009

and it is $[13, 13.5]$;

x	$\mathrm{f}(x)$
13	1.5934
13.25	0.7464
13.5	−0.122

and it is $\underline{\underline{[13.25, 13.5]}}$.

(b) Use linear interpolation once on the interval $[2, 3]$ to find to approximation to α. Give your answer to 3 decimal places. (4)

Solution

$\mathrm{f}(13) = 1.593\,463\,251$ (FCD)
$\mathrm{f}(14) = -1.910\,965\,188$ (FCD)

$$\begin{aligned}\alpha &= \frac{13|\,\mathrm{f}(14)| + 14|\,\mathrm{f}(13)|}{|\,\mathrm{f}(13)| + |\,\mathrm{f}(14)|} \\ &= 13.454\,699\,9 \text{ (FCD)} \\ &= \underline{\underline{13.455 \text{ (3 dp)}}}.\end{aligned}$$

27.

$$f(x) = 3x^{\frac{3}{2}} - 25x^{-\frac{1}{2}} - 125, x > 0.$$

(a) Find $f'(x)$. (2)

Solution

$$f(x) = 3x^{\frac{3}{2}} - 25x^{-\frac{1}{2}} - 125 \Rightarrow \underline{\underline{f'(x) = \tfrac{9}{2}x^{\frac{1}{2}} + \tfrac{25}{2}x^{-\frac{3}{2}}}}.$$

The equation $f(x) = 0$ has a root α in the interval $[12, 13]$.

(b) Using $x_0 = 12.5$ as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places. (4)

Solution

$$f(12.5) = 0.511\,453\,660\,6 \text{ (FCD) and } f'(12.5) = 16.192\,745\,29 \text{ (FCD)}$$

and

$$\begin{aligned} x_1 &= 12.5 - \frac{f(12.5)}{f'(12.5)} \\ &= 12.468\,414\,64 \text{ (FCD)} \\ &= \underline{\underline{12.468 \text{ (3 dp)}}}. \end{aligned}$$

28.

$$f(x) = \tfrac{1}{3}x^2 + \frac{4}{x^2} - 2x - 1, \, x > 0.$$

(a) Show that the equation $f(x) = 0$ has a root α in the interval $[6, 7]$. (2)

Solution

$f(6) = -\frac{8}{9}$
$f(7) = \frac{208}{147}$
It is a continuous function and there is a change of sign.
Hence $\underline{\underline{6 < \alpha < 7}}$.

(b) Taking 6 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 2 decimal places. (5)

Solution

$$\begin{aligned} \mathrm{f}(x) = \tfrac{1}{3}x^2 + \frac{4}{x^2} - 2x - 1 &\Rightarrow \mathrm{f}(x) = \tfrac{1}{3}x^2 + 4x^{-2} - 2x - 1 \\ &\Rightarrow \mathrm{f}'(x) = \tfrac{2}{3}x - 8x^{-3} - 2 \end{aligned}$$

and

$$\begin{aligned} x_1 &= 6 - \frac{\mathrm{f}(6)}{\mathrm{f}'(6)} \\ &= 6.452\,830\,189 \text{ (FCD)} \\ &= \underline{\underline{6.45 \text{ (2 dp)}}}. \end{aligned}$$