Dr Oliver Mathematics
Mathematics: Advanced Higher
2009 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. (a) Given (3)

$$f(x) = (x+1)(x-2)^3,$$

obtain the values of x for which $f'(x) = 0$

Solution

We use the product rule:

$$u = x + 1 \Rightarrow \frac{du}{dx} = 1$$
$$v = (x-2)^3 \Rightarrow \frac{dv}{dx} = 3(x-2)^2$$

Now,

$$\begin{aligned} f'(x) &= (x+1) \cdot 3(x-2)^2 + 1 \cdot (x-2)^3 \\ &= (x-2)^2 \left[3(x+1) + (x-2)\right] \\ &= (4x+1)(x-2)^2. \end{aligned}$$

Finally,

$$\begin{aligned} f'(x) = 0 &\Rightarrow (4x+1)(x-2)^2 = 0 \\ &\Rightarrow \underline{\underline{x = -\tfrac{1}{4} \text{ or } x = 2}}. \end{aligned}$$

(b) Calculate the gradient of the curve defined by (4)

$$\frac{x^2}{y} + x = y - 5$$

at the point $(3, -1)$.

Solution

We use the product rule and implicit differentation:

$$\begin{aligned} x^2y^{-1}+x=y-5 &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(x^2y^{-1}+x)=\frac{\mathrm{d}}{\mathrm{d}x}(y-5) \\ &\Rightarrow 2x\cdot y^{-1}+x^2\cdot\left(-y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x}\right)+1=\frac{\mathrm{d}y}{\mathrm{d}x} \\ &\Rightarrow 2xy^{-1}+1-x^2y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}x} \\ &\Rightarrow 2xy^{-1}+1=\frac{\mathrm{d}y}{\mathrm{d}x}+x^2y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x} \\ &\Rightarrow 2xy^{-1}+1=\frac{\mathrm{d}y}{\mathrm{d}x}(1+x^2y^{-2}) \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2xy^{-1}+1}{1+x^2y^{-2}}. \end{aligned}$$

Finally, at $(3,-1)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\cdot3\cdot-1}{1+3^2\cdot(-1)^{-2}}=\underline{\underline{-\tfrac{1}{2}}}.$$

2. Given the matrix

$$\mathbf{A}=\begin{pmatrix} t+4 & 3t \\ 3 & 5 \end{pmatrix},$$

(a) find $\mathbf{A}^{-1}$ in terms of t when $\mathbf{A}$ is non-singular. (3)

Solution

$$\begin{aligned} \det\mathbf{A} &= 5(t+4)-3(3t) \\ &= (5t+20)-9t \\ &= 20-4t. \end{aligned}$$

Finally,

$$\underline{\underline{\mathbf{A}^{-1}=\frac{1}{20-4t}\begin{pmatrix} 5 & -3t \\ -3 & t+4 \end{pmatrix}}}.$$

(b) Write down the value of t such that $\mathbf{A}$ is singular. (1)

Solution

$$20-4t=0\Rightarrow \underline{\underline{t=5}}.$$

(c) Given that the transpose of $\mathbf{A}$ is (1)

$$\begin{pmatrix} 6 & 3 \\ 6 & 5 \end{pmatrix},$$

find t.

Solution

$$t + 4 = 6 \Rightarrow \underline{\underline{t = 2}}.$$

3. Given that (4)

$$x^2 \mathrm{e}^y \frac{\mathrm{d}y}{\mathrm{d}x} = 1,$$

and $y = 0$ when $x = 1$, find y in terms of x.

Solution

$$\begin{aligned} x^2 \mathrm{e}^y \frac{\mathrm{d}y}{\mathrm{d}x} = 1 &\Rightarrow \mathrm{e}^y \, \mathrm{d}y = x^{-2} \, \mathrm{d}x \\ &\Rightarrow \int \mathrm{e}^y \, \mathrm{d}y = \int x^{-2} \, \mathrm{d}x \\ &\Rightarrow \mathrm{e}^y = -x^{-1} + c. \end{aligned}$$

Now, $y = 0$ when $x = 1$:

$$1 = -1 + c \Rightarrow c = 2$$

and so

$$\begin{aligned} \mathrm{e}^y = -x^{-1} + 2 &\Rightarrow \mathrm{e}^y = -\frac{1}{x} + 2 \\ &\Rightarrow \mathrm{e}^y = \frac{2x - 1}{x} \\ &\Rightarrow \underline{\underline{y = \ln\left(\frac{2x-1}{x}\right)}}. \end{aligned}$$

4. Prove by induction that, for all positive integers n, (5)

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}.$$

Solution

$\underline{n=1}$: LHS $= \frac{1}{1\times 2} = \frac{1}{2}$ and RHS $= 1 - \frac{1}{1+1} = \frac{1}{2}$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$\sum_{r=1}^{k} \frac{1}{r(r+1)} = 1 - \frac{1}{k+1}.$$

Then,

$$\begin{aligned}\sum_{r=1}^{k+1} \frac{1}{r(r+1)} &= \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)} \\ &= 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= 1 + \frac{1-(k+2)}{(k+1)(k+2)} \\ &= 1 + \frac{-(k+1)}{(k+1)(k+2)} \\ &= 1 - \frac{1}{k+2}\end{aligned}$$

and so the result is true for $n = k+1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{Z}^+$, as required.

5. Show that (4)

$$\int_{\ln \frac{3}{2}}^{\ln 2} \frac{e^x + e^{-x}}{e^x - e^{-x}} \, dx = \ln\left(\frac{9}{5}\right).$$

Solution

Notice how the bottom line is the derivative of the top line:

$$\begin{aligned}\int_{\ln \frac{3}{2}}^{\ln 2} \frac{e^x + e^{-x}}{e^x - e^{-x}} \, dx &= \left[\ln(e^x - e^{-x})\right]_{x=\ln \frac{3}{2}}^{\ln 2} \\ &= \ln(2 - \tfrac{1}{2}) - \ln(\tfrac{3}{2} - \tfrac{2}{3}) \\ &= \ln(\tfrac{3}{2}) - \ln(\tfrac{5}{6}) \\ &= \underline{\underline{\ln\left(\frac{9}{5}\right)}},\end{aligned}$$

as required

6. (a) Express (3)

$$z = \frac{(1+2\mathrm{i})^2}{7-\mathrm{i}}$$

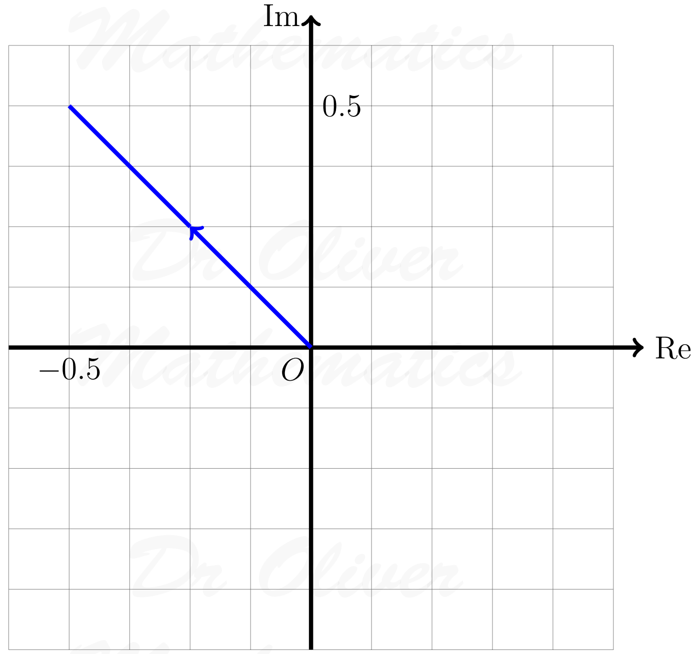
in the form $a + \mathrm{i}b$ where a and b are real numbers.

Solution

$$\begin{aligned} z &= \frac{(1+2\mathrm{i})^2}{7-\mathrm{i}} \\ &= \frac{1+4\mathrm{i}+4\mathrm{i}^2}{7-\mathrm{i}} \\ &= \frac{-3+4\mathrm{i}}{7-\mathrm{i}} \\ &= \frac{-3+4\mathrm{i}}{7-\mathrm{i}} \times \frac{7+\mathrm{i}}{7+\mathrm{i}} \\ &= \frac{-21-3\mathrm{i}+28\mathrm{i}+4\mathrm{i}^2}{49+1} \\ &= \frac{-25+25\mathrm{i}}{50} \\ &= \underline{\underline{-\tfrac{1}{2}+\tfrac{1}{2}\mathrm{i}}}. \end{aligned}$$

(b) Show z on an Argand diagram and evaluate $|z|$ and $\arg(z)$. (3)

Solution



$$
\begin{aligned}
|z| &= \sqrt{(\tfrac{1}{2})^2 + (\tfrac{1}{2})^2} \\
&= \sqrt{\tfrac{1}{2}} \\
&= \underline{\underline{\tfrac{\sqrt{2}}{2}}}
\end{aligned}
$$

and

$$
\begin{aligned}
\arg(z) &= \tan^{-1} \frac{\frac{1}{2}}{-\frac{1}{2}} \\
&= \tan^{-1}(-1) \\
&= -\tfrac{1}{4}\pi \text{ (no)}, \underline{\underline{\tfrac{3}{4}\pi \text{ (yes)}}}.
\end{aligned}
$$

7. Use the substitution $x = 2\sin\theta$ to obtain the exact value of (6)

$$\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4 - x^2}} \, \mathrm{d}x.$$

Solution

Would it be better if we had

$$x = 2\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{2}\right)?$$

(Just saying...) Now,

$$\begin{aligned} x = 2\sin\theta &\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\cos\theta \\ &\Rightarrow \mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta \end{aligned}$$

and

$$\begin{aligned} x = 0 &\Rightarrow \theta = 0, \\ x = \sqrt{2} &\Rightarrow \theta = \tfrac{1}{4}\pi. \end{aligned}$$

Finally,

$$\begin{aligned} \int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,\mathrm{d}x &\Rightarrow \int_0^{\frac{1}{4}\pi} \frac{(2\sin\theta)^2}{\sqrt{4-(2\sin\theta)^2}}\,2\cos\theta\,\mathrm{d}\theta \\ &\Rightarrow \int_0^{\frac{1}{4}\pi} \frac{8\sin^2\theta\cos\theta}{\sqrt{4-4\sin^2\theta}}\,\mathrm{d}\theta \\ &\Rightarrow \int_0^{\frac{1}{4}\pi} \frac{8\sin^2\theta\cos\theta}{\sqrt{4\cos^2\theta}}\,\mathrm{d}\theta \\ &\Rightarrow \int_0^{\frac{1}{4}\pi} \frac{8\sin^2\theta\cos\theta}{2\cos\theta}\,\mathrm{d}\theta \\ &\Rightarrow \int_0^{\frac{1}{4}\pi} 4\sin^2\theta\,\mathrm{d}\theta \\ &\Rightarrow \int_0^{\frac{1}{4}\pi} 2\cdot 2\sin^2\theta\,\mathrm{d}\theta \\ &\Rightarrow 2\int_0^{\frac{1}{4}\pi} (1-\cos 2\theta)\,\mathrm{d}\theta \\ &\Rightarrow 2\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_{x=0}^{\frac{1}{4}\pi} \\ &\Rightarrow 2\left\{\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\right) - (0-0)\right\} \\ &\Rightarrow \underline{\underline{\tfrac{1}{2}\pi - 1}}. \end{aligned}$$

8. (a) Write down the binomial expansion of $(1+x)^5$. (1)

Solution

$$\begin{aligned}(1+x)^5 &= 1 + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + x^5 \\ &= \underline{\underline{1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5}}.\end{aligned}$$

(b) Hence show that $(0.9)^5$ is 0.590 49. (2)

Solution

$$\begin{aligned}(0.9)^5 &= [1 + (-0.1)]^5 \\ &= 1 + 5(-0.1) + 10(-0.1)^2 + 10(-0.1)^3 + 5(-0.1)^4 + (-0.1)^5 \\ &= 1 - 0.5 + 0.1 - 0.01 + 0.0005 - 0.00001 \\ &= \underline{\underline{0.59049}},\end{aligned}$$

as required.

9. Use integration by parts to obtain the exact value of (5)

$$\int_0^1 x \tan^{-1} x^2 \,\mathrm{d}x.$$

Solution

$$u = \tan^{-1} x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2x}{1+(x^2)^2}$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = x \Rightarrow v = \tfrac{1}{2}x^2.$$

Finally,

$$\begin{aligned}\int_0^1 x\tan^{-1}x^2\,\mathrm{d}x &= \left[\tfrac{1}{2}x^2\tan^{-1}x^2\right]_{x=0}^{1} - \int_0^1 \frac{2x^3}{2(1+x^4)}\,\mathrm{d}x \\ &= \left(\tfrac{1}{2}\tan^{-1}1 - 0\right) - \tfrac{1}{4}\int_0^1 \frac{4x^3}{1+x^4}\,\mathrm{d}x \\ &= \tfrac{1}{8}\pi - \tfrac{1}{4}\left[\ln\left|1+x^4\right|\right]_{x=0}^{1} \\ &= \tfrac{1}{8}\pi - \tfrac{1}{4}(\ln 2 - 0) \\ &= \underline{\underline{\tfrac{1}{8}\pi - \tfrac{1}{4}\ln 2}}.\end{aligned}$$

10. Use the Euclidean algorithm to obtain the greatest common divisor of 1 326 and 14 654, expressing it in the form (4)

$$1\,326a + 14\,654b,$$

where a and b are integers.

Solution

$$\begin{aligned}14\,654 &= 1\,326 \times 11 + 68 \\ 1\,326 &= 68 \times 19 + 34 \\ 68 &= 34 \times 2 + 0\end{aligned}$$

and

$$\begin{aligned}34 &= 1\,326 - 68 \times 19 \\ &= 1\,326 - 19(14\,654 - 1\,326 \times 11) \\ &= 1\,326 - 14\,654 \times 19 + 1\,326 \times 209 \\ &= \underline{\underline{1\,326 \times 210 - 14\,654 \times 19}}.\end{aligned}$$

11. The curve (5)

$$y = x^{2x^2+1}$$

is defined for $x > 0$.

Obtain the values of y and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at the point where $x = 1$.

Solution

$$x = 1 \Rightarrow \underline{\underline{y = 1}}.$$

Now,

$$\begin{aligned} y = x^{2x^2+1} &\Rightarrow \ln y = \ln x^{2x^2+1} \\ &\Rightarrow \ln y = (2x^2+1)\ln x \\ &\Rightarrow \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x^2+1}{x} + 4x\ln x \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = (x^{2x^2+1})\left(\frac{2x^2+1}{x} + 4x\ln x\right), \end{aligned}$$

and

$$x = 1 \Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 3}}.$$

12. (a) The first two terms of a geometric sequence are $a_1 = p$ and $a_2 = p^2$. Obtain expressions for S_n and S_{2n} in terms of p, where (2)

$$S_k = \sum_{j=1}^{k} a_j.$$

Solution

$a = p$ and $r = p$:

$$\begin{aligned} S_n &= \sum_{j=1}^{n} a_j \\ &= p + p^2 + \ldots p^n \\ &= \underline{\underline{\frac{p - p^{n+1}}{1-p}}} \end{aligned}$$

and

$$S_{2n} = \underline{\underline{\frac{p - p^{2n+1}}{1-p}}}.$$

(b) Given that (2)

$$S_{2n} = 65 S_n$$

show that $p^n = 64$.

Solution

$$\begin{aligned} S_{2n} = 65S_n \Rightarrow \frac{p - p^{2n+1}}{1-p} &= \frac{65(p - p^{n+1})}{1-p} \\ \Rightarrow p - p^{2n+1} &= 65(p - p^{n+1}) \\ \Rightarrow p(1 - p^{2n}) &= 65p(1 - p^n) \\ \Rightarrow (1-p^n)(1+p^n) &= 65(1-p^n) \text{ (as } p > 0) \\ \Rightarrow 1 + p^n &= 65 \\ \Rightarrow \underline{\underline{p^n = 64}}, \end{aligned}$$

as required.

(c) Given also that $a_3 = 2p$ and that $p > 0$, obtain the exact value of p and hence the value of n. (2)

Solution

$$\begin{aligned} \frac{p^2}{p} = \frac{2p}{p^2} &\Rightarrow p^4 = 2p^2 \\ &\Rightarrow p^4 - 2p^2 = 0 \\ &\Rightarrow p^2(p^2 - 2) = 0 \end{aligned}$$

and, as $p > 0$, $\underline{\underline{p = \sqrt{2}}}$. Finally,

$$\begin{aligned} \sqrt{2}^n = 64 &\Rightarrow 2^{\frac{1}{2}n} = 2^6 \\ &\Rightarrow \tfrac{1}{2}n = 6 \\ &\Rightarrow \underline{\underline{n = 12}}. \end{aligned}$$

13. The function f(x) is defined by

$$f(x) = \frac{x^2 + 2x}{x^2 - 1}, \; x \neq \pm 1.$$

(a) Obtain equations for the asymptotes of the graph of f(x). (3)

Solution

Horizontally:

$$
\begin{aligned}
\mathrm{f}(x) &= \frac{x^2 + 2x}{x^2 - 1} \\
&= \frac{1 + \frac{2}{x}}{1 - \frac{1}{x^2}} \\
&\to \frac{1 + 0}{1 - 0} \\
&= 1 \text{ as } x \to \infty
\end{aligned}
$$

and the asymptote is at $\underline{\underline{y = 1}}$.

Vertically: the denominator is zero:

$$x^2 - 1 = 0 \Rightarrow \underline{\underline{x = \pm 1}}.$$

(b) Show that $\mathrm{f}(x)$ is a strictly decreasing function. (3)

Solution

We use the quotient rule:

$$
\begin{aligned}
\mathrm{f}'(x) &= \frac{(x^2 - 1) \cdot (2x + 2) - (x^2 + 2x) \cdot (2x)}{(x^2 - 1)^2} \\
&= \frac{(2x^3 + 2x^2 - 2x - 2) - (2x^3 + 4x^2)}{(x^2 - 1)^2} \\
&= \frac{-2x^2 - 2x - 2}{(x^2 - 1)^2} \\
&= -\frac{2(x^2 + x + 1)}{(x^2 - 1)^2} \\
&= -\frac{2[(x + \frac{1}{2})^2 + \frac{3}{4}]}{(x^2 - 1)^2} \\
&< 0,
\end{aligned}
$$

and, hence, $\mathrm{f}(x)$ is a $\underline{\underline{\text{strictly decreasing function}}}$.

(c) Find the coordinates of the points where the graph of $\mathrm{f}(x)$ crosses

(i) the x-axis, and (1)

Solution

$$\begin{aligned} y = 0 &\Rightarrow x^2 + 2x = 0 \\ &\Rightarrow x(x+2) = 0 \\ &\Rightarrow x = -2 \text{ or } x = 0, \end{aligned}$$

and hence the points are $\underline{\underline{(-2, 0)}}$ and $\underline{\underline{(0, 0)}}$.

(ii) the horizontal asymptote. (1)

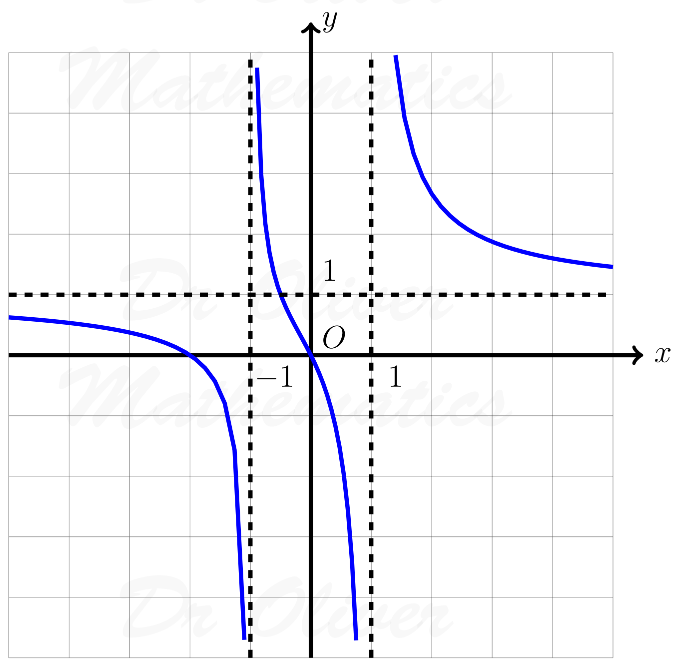
Solution

$$\begin{aligned} \frac{x^2 + 2x}{x^2 - 1} = 1 &\Rightarrow x^2 + 2x = x^2 - 1 \\ &\Rightarrow 2x = -1 \\ &\Rightarrow x = -\tfrac{1}{2}, \end{aligned}$$

and hence the point is $\underline{\underline{(-\tfrac{1}{2}, 1)}}$.

(d) Sketch the graph of $\mathrm{f}(x)$, showing clearly all relevant features. (2)

Solution



14. (a) Express (4)

$$\frac{x^2+6x-4}{(x+2)^2(x-4)}$$

in partial fractions.

Solution

$$\frac{x^2+6x-4}{(x+2)^2(x-4)} \equiv \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-4}$$
$$\equiv \frac{A(x+2)(x-4) + B(x-4) + C(x+2)^2}{(x+2)^2(x-4)}$$

and so

$$x^2+6x-4 \equiv A(x+2)(x-4) + B(x-4) + C(x+2)^2.$$

$\underline{x=-2}$: $-12 = -6B \Rightarrow B = 2.$
$\underline{x=4}$: $36 = 36C \Rightarrow C = 1.$
$\underline{x=0}$: $-4 = -8A - 4B + 4C \Rightarrow -8A = 0 \Rightarrow A = 0.$
Hence,

$$\frac{x^2+6x-4}{(x+2)^2(x-4)} \equiv \underline{\underline{\frac{2}{(x+2)^2} + \frac{1}{x-4}}}.$$

(b) Hence, or otherwise, obtain the first three non-zero terms in the Maclaurin expansion of (5)

$$\frac{x^2+6x-4}{(x+2)^2(x-4)}.$$

Solution

Let

$$\mathrm{f}(x) = 2(x+2)^{-2} + (x-4)^{-1}$$
$$\mathrm{f}(0) = \tfrac{1}{4}$$

$$\mathrm{f}'(x) = -4(x+2)^{-3} - (x-4)^{-2}$$
$$\mathrm{f}'(0) = -\tfrac{9}{16}$$

$$\mathrm{f}''(x) = 12(x+2)^{-4} + 2(x-4)^{-3}$$
$$\mathrm{f}''(0) = \tfrac{23}{32}.$$

Hence, the first three non-zero terms in the Maclaurin expansion are

$$\underline{\underline{\tfrac{1}{4} - \tfrac{9}{16}x + \tfrac{23}{64}x^2 + \dots}}$$

15. (a) Solve the differential equation (6)

$$(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = (x+1)^4,$$

given that $y = 16$ when $x = 1$, expressing the answer in the form $y = \mathrm{f}(x)$.

Solution

$$\begin{aligned} &(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = (x+1)^4 \\ \Rightarrow\quad &\frac{\mathrm{d}y}{\mathrm{d}x} + \left(-\frac{3}{x+1}\right)y = (x+1)^3. \end{aligned}$$

Now,

$$\begin{aligned} \mathrm{IF} &= \mathrm{e}^{\int \left(-\frac{3}{x+1}\right)\,\mathrm{d}x} \\ &= \mathrm{e}^{-3\ln(x+1)} \\ &= \mathrm{e}^{\ln(x+1)^{-3}} \\ &= (x+1)^{-3} \end{aligned}$$

and so the the differential equation is

$$\begin{aligned} &\frac{\mathrm{d}y}{\mathrm{d}x} + \left(-\frac{3}{x+1}\right)y = (x+1)^3 \\ \Rightarrow\quad &(x+1)^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} - 3(x+1)^{-4} = 1 \\ \Rightarrow\quad &\frac{\mathrm{d}}{\mathrm{d}x}\left[y(x+1)^{-3}\right] = 1 \\ \Rightarrow\quad &y(x+1)^{-3} = x + c \\ \Rightarrow\quad &y = (x+c)(x+1)^3. \end{aligned}$$

Next, $y = 16$ when $x = 1$ so

$$16 = (1+c)\times 2^3 \Rightarrow c = 1$$

and so the solution is

$$\underline{\underline{y = (x+1)^4}}.$$

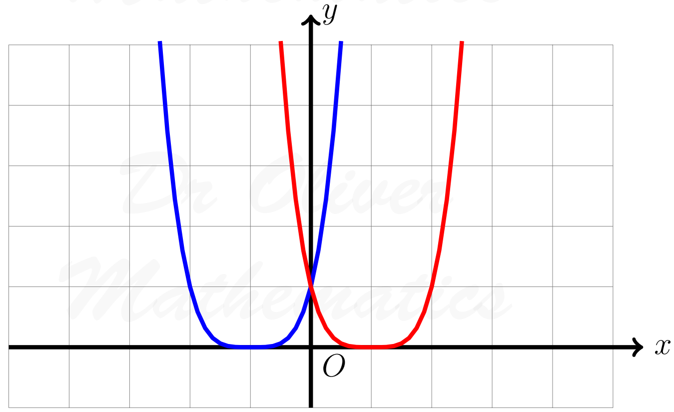
(b) Hence, find the area enclosed by the graphs of $y = \mathrm{f}(x)$, $y = (1-x)^4$, and the x-axis. (4)

Solution

Where do the two graphs cross?

$$\begin{aligned}(x+1)^4 = (1-x)^4 &\Rightarrow x+1 = 1-x \text{ or } x+1 = -(1-x)\\ &\Rightarrow x = 0 \text{ or } 0 = 2 \text{ (no solutions!)}.\end{aligned}$$

Next, a picture:



Finally,

$$\begin{aligned}\text{area} &= \int_{-1}^{0} (x+1)^4 \, \mathrm{d}x + \int_{0}^{1} (1-x)^4 \, \mathrm{d}x\\ &= \left[\tfrac{1}{5}(x+1)^5\right]_{x=-1}^{0} + \left[-\tfrac{1}{5}(1-x)^5\right]_{x=0}^{1}\\ &= \left(\tfrac{1}{5} - 0\right) + \left(0 + \tfrac{1}{5}\right)\\ &= \underline{\underline{\tfrac{2}{5}}}.\end{aligned}$$

16. (a) Use Gaussian elimination to solve the following system of equations: (5)

$$\begin{aligned}x + y - z &= 6\\ 2x - 3y + 2z &= 2\\ -5x + 2y - 4z &= 1.\end{aligned}$$

Solution

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & -3 & 2 & 2 \\ -5 & 2 & -4 & 1 \end{array}\right)$$

Do $R_2 - 2R_1$ and $R_3 + 5R_1$:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & -5 & 4 & -10 \\ 0 & 7 & -9 & 31 \end{array}\right)$$

Do $R_3 + \frac{7}{5}R_2$:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & -5 & 4 & -10 \\ 0 & 0 & -\frac{17}{5} & 17 \end{array}\right)$$

Finally,

$$\begin{aligned} -\tfrac{17}{5}z = 17 &\Rightarrow z = -5 \\ -5y - 20 = -10 &\Rightarrow y = -2 \\ x - 2 + 5 = 6 &\Rightarrow x = 3; \end{aligned}$$

hence,

$$\underline{\underline{x = 3, y = -2, z = -5}}.$$

(b) Show that the line of intersection, L, of the planes $x+y-z = 6$ and $2x-3y+2z = 2$ has parametric equations (2)

$$\begin{aligned} x &= \lambda \\ y &= 4\lambda - 14 \\ z &= 5\lambda - 20. \end{aligned}$$

Solution

$\underline{x + y - z = 6}$:

$$\lambda + (4\lambda - 14) - (5\lambda - 20) = 6 \quad \checkmark$$

$\underline{2x - 3y + 2z = 2}$:

$$2\lambda - 3(4\lambda - 14) + 2(5\lambda - 20) = 2 \quad \checkmark$$

So, the line does have the parametric equations.

(c) Find the acute angle between line L and the plane $-5x + 2y - 4z = 1$. (4)

Solution

Now, re-cast the parametric equations into a straight line:

$$\frac{x}{1} = \frac{y + 14}{4} = \frac{z + 20}{5}$$

and the direction of L is $\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$. Next,

$$\begin{aligned} & \sin\theta = \frac{|(\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) \cdot (-5\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})|}{\sqrt{1^2 + 4^2 + 5^2} \cdot \sqrt{5^2 + 2^2 + 4^2}} \\ \Rightarrow \quad & \sin\theta = \frac{|-5 + 8 - 20|}{\sqrt{42} \cdot \sqrt{45}} \\ \Rightarrow \quad & \sin\theta = \frac{17}{\sqrt{42} \cdot \sqrt{45}} \\ \Rightarrow \quad & \theta = 23.019\,049\,32 \text{ (FCD)} \\ \Rightarrow \quad & \underline{\underline{\theta = 23.0^\circ \text{ (1 dp)}}}. \end{aligned}$$