Dr Oliver Mathematics
Mathematics
Trigonometry Part 2
Past Examination Questions

This booklet consists of 49 questions across a variety of examination topics.
The total number of marks available is 473.

1. (a) Given that $\sin^2\theta + \cos^2\theta \equiv 1$, show that $1 + \tan^2\theta \equiv \sec^2\theta$. (2)

Solution

$$\begin{aligned}\sin^2\theta + \cos^2\theta \equiv 1 &\Rightarrow \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta} \\ &\Rightarrow \underline{\underline{\tan^2\theta + 1 \equiv \sec^2\theta}}.\end{aligned}$$

(b) Solve, for $0° \leqslant \theta < 360°$, the equation (6)

$$2\tan^2\theta + \sec\theta = 1,$$

giving your answer to 1 decimal place.

Solution

$$\begin{aligned}2\tan^2\theta + \sec\theta = 1 &\Rightarrow 2(\sec^2\theta - 1) + \sec\theta = 1 \\ &\Rightarrow 2\sec^2\theta - 2 + \sec\theta = 1 \\ &\Rightarrow 2\sec^2\theta + \sec\theta - 3 = 0 \\ &\Rightarrow (2\sec\theta + 3)(\sec\theta - 1) = 0 \\ &\Rightarrow \sec\theta = -\tfrac{3}{2} \text{ or } \sec\theta = 1 \\ &\Rightarrow \cos\theta = -\tfrac{2}{3} \text{ or } \cos\theta = 1.\end{aligned}$$

$\underline{\cos\theta = -\tfrac{2}{3}}$:

$$\begin{aligned}\cos\theta = -\tfrac{2}{3} &\Rightarrow \theta = 131.810\,314\,9,\ 228.189\,685\,1 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\theta = 131.8,\ 228.2 \text{ (1 dp)}}}.\end{aligned}$$

$\underline{\cos\theta = 1}$:

$$\cos\theta = 1 \Rightarrow \underline{\underline{\theta = 0}}.$$

2. (a) Using the identity $\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$, prove that (2)

$$\cos 2A \equiv 1 - 2\sin^2 A.$$

Solution

$$\begin{aligned}\cos 2A &\equiv \cos(A+A) \\ &\equiv \cos A \cos A - \sin A \sin A \\ &\equiv \cos^2 A - \sin^2 A \\ &\equiv (1-\sin^2 A) - \sin^2 A \\ &\equiv \underline{\underline{1 - 2\sin^2 A}}.\end{aligned}$$

(b) Show that (4)

$$2\sin 2\theta - 3\cos 2\theta - 3\sin\theta + 3 \equiv \sin\theta(4\cos\theta + 6\sin\theta - 3).$$

Solution

$$\begin{aligned} & 2\sin 2\theta - 3\cos 2\theta - 3\sin\theta + 3 \\ \Rightarrow\ & 4\sin\theta\cos\theta - 3(1 - 2\sin^2\theta) - 3\sin\theta + 3 \\ \Rightarrow\ & 4\sin\theta\cos\theta - 3 + 6\sin^2\theta - 3\sin\theta + 3 \\ \Rightarrow\ & 4\sin\theta\cos\theta + 6\sin^2\theta - 3\sin\theta \\ \Rightarrow\ & \underline{\underline{\sin\theta(4\cos\theta + 6\sin\theta - 3)}}.\end{aligned}$$

(c) Express $4\cos\theta + 6\sin\theta - 3$ in the form $R\sin(\theta+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. (4)

Solution

$$R\sin(\theta+\alpha) \equiv R\sin\theta\cos\alpha + R\sin\alpha\cos\theta$$

and we have

$$R\cos\alpha = 6 \text{ and } R\sin\alpha = 4.$$

Now,

$$R = \sqrt{6^2 + 4^2} = \underline{\underline{2\sqrt{13}}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{2}{3} \Rightarrow \underline{\underline{\alpha = 0.588\,002\,603\,5 \text{ (FCD)}}}.$$

(d) Hence, for $0 \leqslant x < \pi$, solve (5)

$$2\sin 2\theta = 3(\cos 2\theta + \sin\theta - 1),$$

giving your answer in radians to 3 significant figures, where appropriate.

Solution

$$\begin{aligned} & 2\sin 2\theta = 3(\cos 2\theta + \sin\theta - 1) \\ \Rightarrow\ & 2\sin 2\theta - 3\cos 2\theta - 3\sin\theta + 3 = 0 \\ \Rightarrow\ & \sin\theta(4\cos\theta + 6\sin\theta - 3) = 0 \\ \Rightarrow\ & \sin\theta = 0 \text{ or } 4\cos\theta + 6\sin\theta = 3 \\ \Rightarrow\ & \sin\theta = 0 \text{ or } 2\sqrt{13}\sin(\theta + 0.588\ldots) = 3. \end{aligned}$$

$\underline{\sin\theta = 0}$:

$$\sin\theta = 0 \Rightarrow \underline{\underline{\theta = 0}}.$$

$\underline{2\sqrt{13}\sin(\theta + 0.588\ldots) = 3}$:

$$\begin{aligned} & 2\sqrt{13}\sin(\theta + 0.588\ldots) = 3 \\ \Rightarrow\ & \sin(\theta + 0.588\ldots) = \tfrac{3\sqrt{13}}{26} \\ \Rightarrow\ & \theta + 0.588\ldots = 0.429\ldots \text{ (not a solution)}, 2.712\ldots \\ \Rightarrow\ & \theta = 2.124\,520\,201 \text{ (FCD)} \\ \Rightarrow\ & \underline{\underline{\theta = 2.12 \text{ (3 sf)}}}. \end{aligned}$$

3.

$$\mathrm{f}(x) = 12\cos x - 4\sin x.$$

Given that $\mathrm{f}(x) = R\cos(x + \alpha)$ where $R > 0$ and $0° \leqslant \alpha \leqslant 90°$,

(a) find the value of R and find the value of α. (4)

Solution

$$R\cos(x + \alpha) \equiv R\cos x\cos\alpha - R\sin x\sin\alpha$$

and we have

$$R\cos\alpha = 12 \text{ and } R\sin\alpha = 4.$$

Now,

$$R = \sqrt{12^2 + 4^2} = \underline{\underline{4\sqrt{10}}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{1}{3} \Rightarrow \alpha = 18.434\,948\,82 \text{ (FCD)} = \underline{\underline{18.4 \text{ (1 dp)}}}.$$

(b) Hence solve the equation (5)

$$12\cos x - 4\sin x = 7,$$

for $0° \leqslant x \leqslant 360°$, giving your answers to 1 decimal place.

Solution

$$\begin{aligned} 12\cos x - 4\sin x = 7 &\Rightarrow 4\sqrt{10}\cos(x+\alpha) = 7 \\ &\Rightarrow \cos(x+\alpha) = \tfrac{7\sqrt{10}}{40} \\ &\Rightarrow x + 18.434\ldots = 56.399\ldots \text{ or } 303.600\ldots \\ &\Rightarrow x = 37.964\ldots \text{ or } 285.165\ldots \\ &\Rightarrow \underline{\underline{x = 38.8 \text{ or } 285.2 \text{ (1 dp)}}}. \end{aligned}$$

(c) Write down the minimum value of $12\cos x - 4\sin x$. (1)

Solution

$\underline{\underline{-4\sqrt{10}}}$.

(d) Find, to 2 decimal places, the smallest positive value of x for which this minimum value occurs. (2)

Solution

$$\begin{aligned} \cos(x+\alpha) = -1 &\Rightarrow x + 18.434\ldots = 180 \\ &\Rightarrow x = 161.565\,051\,2 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 161.57 \text{ (2 dp)}}}. \end{aligned}$$

4. (a) Show that

(i) $\dfrac{\cos 2x}{\cos x + \sin x} \equiv \cos x - \sin x,\ x \neq (n - \frac{1}{4})\pi,\ n \in \mathbb{Z}.$ (2)

Solution

$$\begin{aligned}\frac{\cos 2x}{\cos x + \sin x} &\equiv \frac{\cos^2 x - \sin^2 x}{\cos x + \sin x} \\ &\equiv \frac{(\cos x - \sin x)(\cos x + \sin x)}{\cos x + \sin x} \\ &\equiv \underline{\underline{\cos x - \sin x}}.\end{aligned}$$

(ii) $\frac{1}{2}(\cos 2x - \sin 2x) \equiv \cos^2 x - \cos x \sin x - \frac{1}{2}$. (3)

Solution

$$\begin{aligned}\tfrac{1}{2}(\cos 2x - \sin 2x) &\equiv \tfrac{1}{2}(2\cos^2 x - 1 - 2\sin x \cos x) \\ &\equiv \underline{\underline{\cos^2 x - \cos x \sin x - \tfrac{1}{2}}}.\end{aligned}$$

(b) Hence, or otherwise, show that the equation (3)

$$\cos\theta\left(\frac{\cos 2\theta}{\cos\theta + \sin\theta}\right) = \frac{1}{2}$$

can be written as

$$\sin 2\theta = \cos 2\theta.$$

Solution

$$\begin{aligned}\cos\theta\left(\frac{\cos 2\theta}{\cos\theta + \sin\theta}\right) = \tfrac{1}{2} &\Rightarrow \cos\theta(\cos\theta - \sin\theta) = \tfrac{1}{2} \\ &\Rightarrow \cos^2\theta - \sin\theta\cos\theta = \tfrac{1}{2} \\ &\Rightarrow \tfrac{1}{2}(1 + \cos 2\theta) - \tfrac{1}{2}\sin 2\theta = \tfrac{1}{2} \\ &\Rightarrow \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta - \tfrac{1}{2}\sin 2\theta = \tfrac{1}{2} \\ &\Rightarrow \tfrac{1}{2}\cos 2\theta - \tfrac{1}{2}\sin 2\theta = 0 \\ &\Rightarrow \underline{\underline{\cos 2\theta = \sin 2\theta}}.\end{aligned}$$

(c) Solve, for $0 \leqslant x < 2\pi$, (4)

$$\sin 2\theta = \cos 2\theta,$$

giving your answers in terms of π.

Solution

$$\begin{aligned} \sin 2\theta = \cos 2\theta &\Rightarrow \tan 2\theta = 1 \\ &\Rightarrow 2\theta = \tfrac{\pi}{4}, \tfrac{5\pi}{4}, \tfrac{9\pi}{4}, \tfrac{13\pi}{4} \\ &\Rightarrow \underline{\underline{\theta = \tfrac{\pi}{8}, \tfrac{5\pi}{8}, \tfrac{9\pi}{8}, \tfrac{13\pi}{8}}}. \end{aligned}$$

5. (a) Using $\sin^2\theta + \cos^2\theta \equiv 1$, show that $\operatorname{cosec}^2\theta - \cot^2\theta \equiv 1$. (2)

Solution

$$\begin{aligned} \sin^2\theta + \cos^2\theta \equiv 1 &\Rightarrow \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} \equiv \frac{1}{\sin^2\theta} \\ &\Rightarrow 1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta \\ &\Rightarrow \underline{\underline{\operatorname{cosec}^2\theta - \cot^2\theta \equiv 1}}. \end{aligned}$$

(b) Hence, or otherwise, prove that (2)

$$\operatorname{cosec}^4\theta - \cot^4\theta \equiv \operatorname{cosec}^2\theta + \cot^2\theta.$$

Solution

$$\begin{aligned} \operatorname{cosec}^4\theta - \cot^4\theta &\equiv (\operatorname{cosec}^2\theta - \cot^2\theta)(\operatorname{cosec}^2\theta + \cot^2\theta) \\ &\equiv \underline{\underline{\operatorname{cosec}^2\theta + \cot^2\theta}}. \end{aligned}$$

(c) Solve, for $90^\circ < x < 180^\circ$, (6)

$$\operatorname{cosec}^4\theta - \cot^4\theta = 2 - \cot\theta.$$

Solution

$$\begin{aligned} & \operatorname{cosec}^4\theta - \cot^4\theta = 2 - \cot\theta \\ \Rightarrow\ & \operatorname{cosec}^2\theta + \cot^2\theta = 2 - \cot\theta \\ \Rightarrow\ & (1 + \cot^2\theta) + \cot^2\theta = 2 - \cot\theta \\ \Rightarrow\ & 2\cot^2\theta + \cot\theta - 1 = 0 \\ \Rightarrow\ & (2\cot\theta - 1)(\cot\theta + 1) = 0 \\ \Rightarrow\ & \cot\theta = -1 \text{ (as } \cot\theta = \tfrac{1}{2} \text{ does not have a solution)} \\ \Rightarrow\ & \underline{\underline{\theta = 135}}. \end{aligned}$$

6. (a) Given that $\cos A = \frac{3}{4}$, $270^\circ \leqslant x < 360^\circ$, find the exact value of $\sin 2A$. (5)

Solution

$$\begin{aligned} \sin^2 A + \cos^2 A \equiv 1 &\Rightarrow \sin^2 A + \tfrac{9}{16} \equiv 1 \\ &\Rightarrow \sin^2 A = \tfrac{7}{16} \\ &\Rightarrow \sin A = -\tfrac{\sqrt{7}}{4}, \end{aligned}$$

as $\sin A < 0$. Hence

$$\sin 2A = 2\sin A\cos A = 2 \times (-\tfrac{\sqrt{7}}{4}) \times \tfrac{3}{4} = \underline{\underline{-\tfrac{3\sqrt{7}}{8}}}$$

(b) (i) Show that $\cos(2x + \frac{\pi}{3}) + \cos(2x - \frac{\pi}{3}) \equiv \cos 2x$. (3)

Solution

$$\begin{aligned} & \cos(2x + \tfrac{\pi}{3}) + \cos(2x - \tfrac{\pi}{3}) \\ \equiv\ & (\cos 2x \cos\tfrac{\pi}{3} - \sin 2x \sin\tfrac{\pi}{3}) + (\cos 2x\cos\tfrac{\pi}{3} + \sin 2x \sin\tfrac{\pi}{3}) \\ \equiv\ & 2\cos 2x\cos\tfrac{\pi}{3} \\ \equiv\ & \underline{\underline{\cos 2x}}. \end{aligned}$$

Given that

$$y = 3\sin^2 x + \cos(2x + \tfrac{\pi}{3}) + \cos(2x - \tfrac{\pi}{3}),$$

(ii) show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sin 2x$. (4)

Solution

$$\begin{aligned} & y = 3\sin^2 x + \cos(2x + \tfrac{\pi}{3}) + \cos(2x - \tfrac{\pi}{3}) \\ \Rightarrow\ & y = 3\sin^2 x + \cos 2x \\ \Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = 6\sin x \cos x - 2\sin 2x \\ \Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = 3\sin 2x - 2\sin 2x \\ \Rightarrow\ & \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \sin 2x}}. \end{aligned}$$

7. (a) By writing $\sin 3\theta$ as $\sin(2\theta + \theta)$, show that (5)

$$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta.$$

Solution

$$\begin{aligned} \sin 3\theta &\equiv \sin 2\theta \cos\theta + \sin\theta \cos 2\theta \\ &\equiv (2\sin\theta\cos\theta)\cos\theta + \sin\theta(1 - 2\sin^2\theta) \\ &\equiv 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta \\ &\equiv 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta \\ &\equiv 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta \\ &\equiv \underline{\underline{3\sin\theta - 4\sin^3\theta}}. \end{aligned}$$

(b) Given that $\sin\theta = \frac{\sqrt{3}}{4}$, find the exact value of $\sin 3\theta$. (2)

Solution

$$\sin 3\theta = 3 \times \tfrac{\sqrt{3}}{4} - 4(\tfrac{\sqrt{3}}{4})^3 = \tfrac{3\sqrt{3}}{4} - \tfrac{3\sqrt{3}}{16} = \underline{\underline{\tfrac{9\sqrt{3}}{16}}}.$$

8.

$$y = \sqrt{3}\cos x + \sin x.$$

(a) Express the equation of the curve in the form $y = R\sin(x + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. (4)

Solution

$$R\sin(x+\alpha) \equiv R\sin x\cos\alpha + R\sin\alpha\cos x$$

and we have

$$R\sin\alpha = \sqrt{3} \text{ and } R\cos\alpha = 1.$$

Now,

$$R = \sqrt{(\sqrt{3})^2 + 1^2} = \underline{\underline{2}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \sqrt{3} \Rightarrow \underline{\underline{\alpha = \tfrac{\pi}{3}}}.$$

(b) Find the values of x, $0 \leqslant x \leqslant 2\pi$, for which $y = 1$. (4)

Solution

$$\begin{aligned} 2\sin(x + \tfrac{\pi}{3}) = 1 &\Rightarrow \sin(x + \tfrac{\pi}{3}) = \tfrac{1}{2} \\ &\Rightarrow x + \tfrac{\pi}{3} = \tfrac{5\pi}{6},\ \tfrac{13\pi}{6} \\ &\Rightarrow \underline{\underline{x = \tfrac{\pi}{2},\ \tfrac{11\pi}{6}}}. \end{aligned}$$

9. (a) Prove that $\sec^2 x - \operatorname{cosec}^2 x \equiv \tan^2 x - \cot^2 x$. (3)

Solution

$$\begin{aligned} \sec^2 x - \operatorname{cosec}^2 x &\equiv (\tan^2 x - 1) - (\cot^2 x - 1) \\ &\equiv \underline{\underline{\tan^2 x - \cot^2 x}}. \end{aligned}$$

Given that

$$y = \arccos x,\ -1 \leqslant x \leqslant 1,\ \text{and } 0 \leqslant y \leqslant \pi,$$

(i) express $\arcsin x$ in terms of y. (2)

Solution

$$\begin{aligned} y = \arccos x &\Rightarrow x = \cos y \\ &\Rightarrow x = \sin(\tfrac{\pi}{2} - y) \\ &\Rightarrow \underline{\underline{\arcsin x = \tfrac{\pi}{2} - y}}. \end{aligned}$$

(ii) Hence evaluate $\arccos x + \arcsin x$. Give your answer in terms of x. (1)

Solution

$$\arccos x + \arcsin x = y + (\tfrac{\pi}{2} - y) = \underline{\underline{\tfrac{\pi}{2}}}.$$

10. (a) Express $3\sin x + 2\cos x$ in the form $R\sin(x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. (4)

Solution

$$R\sin(x+\alpha) \equiv R\sin x\cos\alpha + R\sin\alpha\cos x$$

and we have

$$R\sin\alpha = 2 \text{ and } R\cos\alpha = 3.$$

Now,

$$R = \sqrt{3^2 + 2^2} = \underline{\underline{\sqrt{13}}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \tfrac{2}{3} \Rightarrow \underline{\underline{\alpha = 0.588\,002\,603\,5 \text{ (FCD)}}}.$$

(b) Hence find the greatest value of $(3\sin x + 2\cos x)^4$. (2)

Solution

$(\sqrt{13})^4 = \underline{\underline{169}}$.

(c) Solve, for $0 < x < 2\pi$, the equation (5)

$$3\sin x + 2\cos x = 1,$$

giving your answers to 3 decimal places.

Solution

$$\begin{aligned}
3\sin x + 2\cos x = 1 &\Rightarrow \sqrt{13}\sin(x + 0.588\,002\ldots) = 1 \\
&\Rightarrow \sin(x + 0.588\,002\ldots) = \tfrac{\sqrt{13}}{13} \\
&\Rightarrow x + 0.588\,002\ldots = 2.860\,557\ldots,\ 6.564\,220\ldots \\
&\Rightarrow x = 2.272\,555\ldots,\ 5.976\,217\ldots \\
&\Rightarrow \underline{\underline{x = 2.273,\ 5.976 \text{ (3 dp)}}}.
\end{aligned}$$

11. (a) Prove that (4)

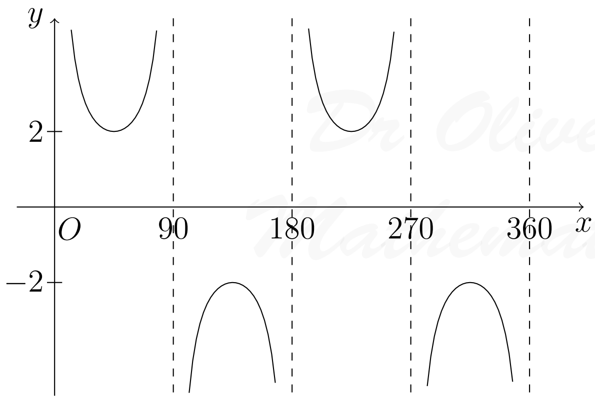
$$\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = 2\operatorname{cosec} 2\theta,\ \theta \neq 90n^\circ.$$

Solution

$$\begin{aligned}\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} &\equiv \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} \\ &\equiv \frac{1}{\sin\theta\cos\theta} \\ &\equiv \frac{2}{2\sin\theta\cos\theta} \\ &\equiv \frac{2}{\sin 2\theta} \\ &\equiv \underline{\underline{2\operatorname{cosec} 2\theta}}.\end{aligned}$$

(b) Sketch the graph of $y = 2\operatorname{cosec} 2\theta$ for $0^\circ < \theta < 360^\circ$. (2)

Solution



(c) Solve, for $0^\circ < \theta < 360^\circ$, the equation (6)

$$\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = 3,$$

giving your answers to 1 decimal place.

Solution

$$\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = 3 \Rightarrow 2\operatorname{cosec} 2\theta = 3$$
$$\Rightarrow \operatorname{cosec} 2\theta = \tfrac{3}{2}$$
$$\Rightarrow \sin 2\theta = \tfrac{2}{3}$$
$$\Rightarrow 2\theta = 41.810\ldots,\ 138.189\ldots,\ 401.810\ldots,\ 498.189\ldots$$
$$\Rightarrow \theta = 20.905\ldots,\ 69.094\ldots,\ 200.905\ldots,\ 249.094\ldots$$
$$\Rightarrow \underline{\underline{\theta = 20.9,\ 69.1,\ 200.9,\ 249.1\ (1\text{ dp})}}.$$

12. (a) Use the double angle formulae and the identity (4)

$$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$$

to obtain an expression for $\cos 3x$ in terms of powers of $\cos x$ only.

Solution

$$\begin{aligned}
\cos 3x &\equiv \cos(2x+x)\\
&\equiv \cos 2x\cos x - \sin 2x\sin x\\
&\equiv (2\cos^2 x - 1)\cos x - (2\sin x\cos x)\sin x\\
&\equiv 2\cos^3 x - \cos x - 2\sin^2 x\cos x\\
&\equiv 2\cos^3 x - \cos x - 2(1-\cos^2 x)\cos x\\
&\equiv 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x\\
&\equiv \underline{\underline{4\cos^3 x - 3\cos x}}.
\end{aligned}$$

(b) (i) Prove that (4)

$$\frac{\cos x}{1+\sin x} + \frac{1+\sin x}{\cos x} \equiv 2\sec x,\ x \neq (2n+1)\frac{\pi}{2}.$$

Solution

$$\begin{aligned}\frac{\cos x}{1+\sin x}+\frac{1+\sin x}{\cos x} &\equiv \frac{\cos^2 x+(1+\sin x)^2}{\cos x(1+\sin x)}\\ &\equiv \frac{\cos^2 x+1+2\sin x+\sin^2 x}{\cos x(1+\sin x)}\\ &\equiv \frac{2+2\sin x}{\cos x(1+\sin x)}\\ &\equiv \frac{2(1+\sin x)}{\cos x(1+\sin x)}\\ &\equiv \frac{2}{\cos x}\\ &\equiv \underline{\underline{2\sec x}}.\end{aligned}$$

(ii) Hence find, for $0 < x < 2\pi$, all the solutions of (3)

$$\frac{\cos x}{1+\sin x}+\frac{1+\sin x}{\cos x}=4.$$

Solution

$$\begin{aligned}\frac{\cos x}{1+\sin x}+\frac{1+\sin x}{\cos x}=4 &\Rightarrow 2\sec x=4\\ &\Rightarrow \cos x=\tfrac{1}{2}\\ &\Rightarrow \underline{\underline{x=\tfrac{\pi}{3},\ \tfrac{5\pi}{3}}}.\end{aligned}$$

13. A curve C has equation

$$y=3\sin 2x+4\cos 2x,\ -\pi \leqslant x \leqslant \pi.$$

The point $P(0,4)$ lies on C.

(a) Find an equation of the normal to the curve C at A. (5)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x}=6\cos 2x-8\sin 2x \Rightarrow \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0}=6$$

and so

$$m_T=-\tfrac{1}{6}.$$

$$y - 4 = -\tfrac{1}{6}(x - 0) \Rightarrow 6y - 24 = -x$$
$$\Rightarrow \underline{\underline{x + 6y - 24 = 0}}.$$

(b) Express y in the form $R\sin(2x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the value of α to 3 significant figures. (4)

Solution

$$R\sin(2x + \alpha) \equiv R\sin 2x\cos\alpha + R\sin\alpha\cos 2x$$

and we have

$$R\sin\alpha = 4 \text{ and } R\cos\alpha = 3.$$

Now,

$$R = \sqrt{3^2 + 4^2} = \underline{\underline{5}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \tfrac{4}{3} \Rightarrow \alpha = 0.927\,295\,218 \text{ (FCD)} = \underline{\underline{0.927 \text{ (3 sf)}}}.$$

(c) Find the coordinates of the points of intersection of the curve C with the x-axis. Give your answers to 2 decimal places. (4)

Solution

$$\begin{aligned} 3\sin 2x + 4\cos 2x = 0 &\Rightarrow 5\sin(2x + \alpha) = 0 \\ &\Rightarrow 2x + \alpha = -\pi,\ 0,\ \pi,\ 2\pi, \\ &\Rightarrow 2x = -\pi - \alpha,\ -\alpha,\ \pi - \alpha,\ 2\pi - \alpha \\ &\Rightarrow x = \tfrac{1}{2}(-\pi - \alpha),\ -\tfrac{1}{2}\alpha,\ \tfrac{1}{2}(\pi - \alpha),\ \tfrac{1}{2}(2\pi - \alpha) \\ &\Rightarrow \underline{\underline{x = -2.03,\ -0.46,\ 1.11,\ 2.68 \text{ (2 dp)}}}. \end{aligned}$$

14.

$$f(x) = 5\cos x + 12\sin x.$$

Given that $f(x) = R\cos(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$,

(a) find the value of R and find the value of α to 3 decimal places. (4)

Solution

$$R\cos(x - \alpha) \equiv R\cos x\cos\alpha + R\sin x\sin\alpha$$

and we have

$$R\cos\alpha = 5 \text{ and } R\sin\alpha = 12.$$

Now,

$$R = \sqrt{5^2 + 12^2} = \underline{\underline{13}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \tfrac{12}{5} \Rightarrow \alpha = 1.176\,005\,207 \text{ (FCD)} = \underline{\underline{1.176 \text{ (3 sf)}}}.$$

(b) Hence solve the equation (5)

$$5\cos x + 12\sin x = 6$$

for $0 \leqslant x < 2\pi$.

Solution

$$\begin{aligned} 5\cos x + 12\sin x = 6 &\Rightarrow 13\cos(x-\alpha) = 6 \\ &\Rightarrow \cos(x-\alpha) = \tfrac{6}{13} \\ &\Rightarrow x - \alpha = -1.091\,067\,689,\ 1.091\,067\,689 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 0.084\,937\,517\,98,\ 2.267\,072\,896 \text{ (FCD)}}}. \end{aligned}$$

(c) (i) Write down the maximum value of $5\cos x + 12\sin x$. (1)

Solution

$\underline{\underline{13}}$.

(ii) Find the smallest positive value of x for which this maximum value occurs. (2)

Solution

$$\cos(x-\alpha) = 1 \Rightarrow x - \alpha = 0 \Rightarrow \underline{\underline{x = 1.176\,005\,207 \text{ (FCD)}}}.$$

15. (a) Given that $\sin^2\theta + \cos^2\theta \equiv 1$, show that $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$. (2)

Solution

$$\begin{aligned} \sin^2\theta + \cos^2\theta \equiv 1 &\Rightarrow \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} \equiv \frac{1}{\sin^2\theta} \\ &\Rightarrow \underline{\underline{1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta}}. \end{aligned}$$

(b) Solve, for $0^\circ \leqslant \theta < 180^\circ$, the equation (6)

$$2\cot^2\theta - 9\operatorname{cosec}\theta = 3,$$

giving your answers to 1 decimal place.

Solution

$$\begin{aligned}
& 2\cot^2\theta - 9\operatorname{cosec}\theta = 3 \\
\Rightarrow\quad & 2(\operatorname{cosec}^2\theta - 1) - 9\operatorname{cosec}\theta = 3 \\
\Rightarrow\quad & 2\operatorname{cosec}^2\theta - 3 - 9\operatorname{cosec}\theta = 3 \\
\Rightarrow\quad & 2\operatorname{cosec}^2\theta - 9\operatorname{cosec}\theta - 5 = 0 \\
\Rightarrow\quad & (2\operatorname{cosec}\theta + 1)(\operatorname{cosec}\theta - 5) = 0 \\
\Rightarrow\quad & \operatorname{cosec}\theta = 5 \text{ (as } \operatorname{cosec}\theta = -\tfrac{1}{2} \text{ does not have a solution)} \\
\Rightarrow\quad & \sin\theta = \tfrac{1}{5} \\
\Rightarrow\quad & \theta = 11.536\,959\,03,\ 168.463\,041 \text{ (FCD)} \\
\Rightarrow\quad & \underline{\underline{\theta = 11.5,\ 168.5 \text{ (1 dp)}}}
\end{aligned}$$

16. (a) (i) By writing $3\theta = 2\theta + \theta$, show that (4)

$$\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta.$$

Solution

$$\begin{aligned}
\sin 3\theta &\equiv \sin(2\theta + \theta) \\
&\equiv \sin 2\theta\cos\theta + \sin\theta\cos 2\theta \\
&\equiv (2\sin\theta\cos\theta)\cos\theta + \sin\theta(1 - 2\sin^2\theta) \\
&\equiv 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta \\
&\equiv 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta \\
&\equiv 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta \\
&\equiv \underline{\underline{3\sin\theta - 4\sin^3\theta}}.
\end{aligned}$$

(ii) Hence, or otherwise, for $0 < \theta < \frac{\pi}{3}$, solve (5)

$$8\sin^3\theta - 6\sin\theta + 1 = 0.$$

Give your answers in terms of π.

Solution

$$\begin{aligned} 8\sin^3\theta - 6\sin\theta + 1 = 0 &\Rightarrow 6\sin\theta - 8\sin^3\theta = 1 \\ &\Rightarrow 3\sin\theta - 4\sin^3\theta = \tfrac{1}{2} \\ &\Rightarrow \sin 3\theta = \tfrac{1}{2} \\ &\Rightarrow 3\theta = \tfrac{\pi}{6}, \tfrac{5\pi}{6} \\ &\Rightarrow \underline{\underline{\theta = \tfrac{\pi}{18}, \tfrac{5\pi}{18}}}. \end{aligned}$$

(b) Using $\sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha$, or otherwise, show that (4)

$$\sin 15^\circ = \tfrac{1}{4}(\sqrt{6} - \sqrt{2}).$$

Solution

$$\begin{aligned} \sin 15^\circ &= \sin(45 - 30)^\circ \\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \\ &= \tfrac{\sqrt{2}}{2} \times \tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{2}}{2} \times \tfrac{1}{2} \\ &= \tfrac{\sqrt{6}}{4} \times \tfrac{\sqrt{2}}{4} \\ &= \underline{\underline{\tfrac{1}{4}(\sqrt{6} - \sqrt{2})}}. \end{aligned}$$

17. (a) Express $3\cos\theta + 4\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$. (4)

Solution

$$R\cos(\theta - \alpha) \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$$

and we have

$$R\cos\alpha = 3 \text{ and } R\sin\alpha = 4.$$

Now,

$$R = \sqrt{3^2 + 4^2} = \underline{\underline{5}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \tfrac{4}{3} \Rightarrow \underline{\underline{\alpha = 53.130\,102\,35 \text{ (FCD)}}}.$$

(b) Hence find the maximum value of $3\cos\theta + 4\sin\theta$ and the smallest positive value of θ for which this maximum occurs. (3)

Solution

The maximum value is $\underline{\underline{5}}$ and this is achieved when

$$\cos(\theta - \alpha) = 1 \Rightarrow \theta = \alpha = \underline{\underline{53.130\,102\,35 \text{ (FCD)}}}.$$

The temperature, $f(t)$, of a warehouse is modelled using the equation

$$f(t) = 10 + 3\cos(15t)^\circ + 4\sin(15t)^\circ,$$

where t is the time in hours from midday and $0 \leqslant t \leqslant 24$.

(c) Calculate the minimum temperature of the warehouse as given by this model. (2)

Solution

The minimum temperature is $10 - 5 = \underline{\underline{5^\circ}}$.

(d) Find the value of t when this minimum temperature occurs. (3)

Solution

$$\begin{aligned} \cos(15t - \alpha) = -1 &\Rightarrow 15t - \alpha = 180 \\ &\Rightarrow 15t = 233.130\,102\,4 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{t = 15.542\,006\,82 \text{ (FCD)}}}. \end{aligned}$$

18. (a) Use the identity $\sin^2\theta + \cos^2\theta \equiv 1$ to prove that $\tan^2\theta \equiv \sec^2\theta - 1$. (2)

Solution

$$\begin{aligned} \sin^2\theta + \cos^2\theta \equiv 1 &\Rightarrow \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta} \\ &\Rightarrow \underline{\underline{\tan^2\theta \equiv \sec^2\theta - 1}}. \end{aligned}$$

(b) Solve, for $0^\circ \leqslant \theta < 360^\circ$, the equation (6)

$$2\tan^2\theta + 4\sec\theta + \sec^2\theta = 2.$$

Solution

$$\begin{aligned} 2\tan^2\theta + 4\sec\theta + \sec^2\theta = 2 &\Rightarrow 2(\sec^2\theta - 1) + 4\sec\theta + \sec^2\theta = 2 \\ &\Rightarrow 2\sec^2\theta - 2 + 4\sec\theta + \sec^2\theta - 2 = 0 \\ &\Rightarrow 3\sec^2\theta + 4\sec\theta - 4 = 0 \\ &\Rightarrow (3\sec\theta - 2)(\sec\theta + 2) = 0 \\ &\Rightarrow \sec\theta = -2 \text{ (only)} \\ &\Rightarrow \cos\theta = -\tfrac{1}{2} \\ &\Rightarrow \underline{\underline{\theta = 120,\ 240}}. \end{aligned}$$

19. (a) Use the identity $\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$ to show that (2)

$$\cos 2A \equiv 1 - 2\sin^2 A.$$

Solution

$$\begin{aligned} \cos 2A &\equiv \cos(A+A) \\ &\equiv \cos^2 A - \sin^2 A \\ &\equiv (1 - \sin^2 A) - \sin^2 A \\ &\equiv \underline{\underline{1 - 2\sin^2 A}}. \end{aligned}$$

The curves C_1 and C_2 have equations

$$C_1 : \ y = 3\sin 2x \text{ and } C_2 : \ y = 4\sin^2 x - 2\cos 2x.$$

(b) Show that the x-coordinates of the points where C_1 and C_2 intersect satisfy the equation (3)

$$4\cos 2x + 3\sin 2x = 2.$$

Solution

$$\begin{aligned} 3\sin 2x = 4\sin^2 x - 2\cos 2x &\Rightarrow 3\sin 2x = 2(2\sin^2 x) - 2\cos 2x \\ &\Rightarrow 2\cos 2x + 3\sin 2x = 2(1-\cos 2x) \\ &\Rightarrow 2\cos 2x + 3\sin 2x = 2 \\ &\Rightarrow \underline{\underline{4\cos 2x + 3\sin 2x = 2.}} \end{aligned}$$

(c) Express $4\cos 2x + 3\sin 2x = 2$ in the form $R\cos(2x - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$, giving the value of α to 2 decimal places. (3)

Solution

$$R\cos(2x-\alpha) \equiv R\cos 2x\cos\alpha + R\sin 2x\sin\alpha$$

and we have

$$R\cos\alpha = 4 \text{ and } R\sin\alpha = 3.$$

Now,

$$R = \sqrt{4^2+3^2} = \underline{\underline{5}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \tfrac{3}{4} \Rightarrow \alpha = 36.869\,897\,65 \text{ (FCD)} = \underline{\underline{36.87 \text{ (2 dp)}}}.$$

(d) Hence find, for $0^\circ \leqslant x < 180^\circ$, all the solutions of (4)

$$4\cos 2x + 3\sin 2x = 2,$$

giving your answers to 1 decimal place.

Solution

$$\begin{aligned} 4\cos 2x + 3\sin 2x = 2 &\Rightarrow 5\cos(2x-\alpha) = 2 \\ &\Rightarrow \cos(2x-\alpha) = \tfrac{2}{5} \\ &\Rightarrow 2x - \alpha = 66.421\ldots,\ 293.578\ldots \\ &\Rightarrow 2x = 103.291\ldots,\ 330.448\ldots \\ &\Rightarrow x = 51.645\ldots,\ 165.224\ldots \\ &\Rightarrow \underline{\underline{x = 51.6,\ 165.2 \text{ (1 dp)}}}. \end{aligned}$$

20. (a) Write down $\sin 2x$ in terms of $\sin x$ and $\cos x$. (1)

Solution

$$\sin 2x = \underline{\underline{2\sin x \cos x}}.$$

(b) Find, for $0 < x < \pi$, all the solutions of the equation (5)

$$\operatorname{cosec} x - 8\cos x = 0,$$

giving your answers to 2 decimal places.

Solution

$$\begin{aligned}\operatorname{cosec} x - 8\cos x = 0 &\Rightarrow \frac{1}{\sin x} = 8\cos x \\ &\Rightarrow 8\sin x\cos x = 1 \\ &\Rightarrow 4\sin 2x = 1 \\ &\Rightarrow \sin 2x = \tfrac{1}{4} \\ &\Rightarrow 2x = 0.252\,680\,255\,1,\ 2.888\,912\,398\ \text{(FCD)} \\ &\Rightarrow x = 0.126\,340\,127\,6,\ 1.444\,456\,199\ \text{(FCD)} \\ &\Rightarrow \underline{\underline{x = 0.13,\ 1.44\ \text{(2 dp)}}}.\end{aligned}$$

21. (a) Express $5\cos x - 3\sin x$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. (4)

Solution

$$R\cos(x + \alpha) \equiv R\cos x\cos\alpha - R\sin x\sin\alpha$$

and we have

$$R\cos\alpha = 5 \text{ and } R\sin\alpha = 3.$$

Now,

$$R = \sqrt{5^2 + 3^2} = \underline{\underline{\sqrt{34}}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \tfrac{3}{5} \Rightarrow \underline{\underline{\alpha = 0.540\,419\,350\,3\ \text{(FCD)}}}.$$

(b) Hence, or otherwise, solve the equation (5)

$$5\cos x - 3\sin x = 4$$

for $0 \leqslant x < 2\pi$, giving your answers to 2 decimal places.

Solution

$$\begin{aligned}5\cos x - 3\sin x = 4 &\Rightarrow \sqrt{34}\cos(x+\alpha) = 4\\ &\Rightarrow \cos(x+\alpha) = \tfrac{2\sqrt{34}}{17}\\ &\Rightarrow x+\alpha = 0.814\,826\,916\,4,\ 5.468\,358\,91 \text{ (FCD)}\\ &\Rightarrow x = 0.274\,407\,416\,1,\ 4.927\,938\,891 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{x = 0.27,\ 4.93 \text{ (2 dp)}}}.\end{aligned}$$

22. Solve

$$\operatorname{cosec}^2 2x - \cot 2x = 1$$

for $0^\circ \leqslant x \leqslant 180^\circ$.

Solution

$$\begin{aligned}\operatorname{cosec}^2 2x - \cot 2x = 1 &\Rightarrow (\cot^2 2x + 1) - \cot 2x = 1\\ &\Rightarrow \cot^2 2x - \cot 2x = 0\\ &\Rightarrow \cot 2x(\cot 2x - 1) = 0\\ &\Rightarrow \cot 2x = 0 \text{ or } \cot 2x = 1.\end{aligned}$$

$\underline{\cot 2x = 0}$:

$$\begin{aligned}\cot 2x = 0 &\Rightarrow \tan 2x = \infty\\ &\Rightarrow 2x = 90,\ 270\\ &\Rightarrow \underline{\underline{x = 45,\ 135}}.\end{aligned}$$

$\underline{\cot 2x = 1}$:

$$\begin{aligned}\cot 2x = 1 &\Rightarrow \tan 2x = 1\\ &\Rightarrow 2x = 45,\ 225\\ &\Rightarrow \underline{\underline{x = 22.5,\ 112.5}}.\end{aligned}$$

23. (a) Show that (2)

$$\frac{\sin 2\theta}{1+\cos 2\theta} \equiv \tan\theta.$$

Solution

$$\begin{aligned}\frac{\sin 2\theta}{1+\cos 2\theta} &\equiv \frac{2\sin\theta\cos\theta}{1+(2\cos^2\theta-1)}\\ &\equiv \frac{2\sin\theta\cos\theta}{2\cos^2\theta}\\ &\equiv \frac{\sin\theta}{\cos\theta}\\ &\equiv \underline{\underline{\tan\theta}}.\end{aligned}$$

(b) Hence find, for $-180^\circ \leqslant \theta < 180^\circ$, all the solutions of (3)

$$\frac{2\sin 2\theta}{1+\cos 2\theta} = 1.$$

Give your answers to 1 decimal place.

Solution

$$\begin{aligned}\frac{2\sin 2\theta}{1+\cos 2\theta} = 1 &\Rightarrow 2\tan\theta = 1\\ &\Rightarrow \tan\theta = \tfrac{1}{2}\\ &\Rightarrow \theta = -153.434\,948\,8,\ 26.565\,051\,118 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{\theta = -153.4,\ 26.6 \text{ (1 dp)}}}.\end{aligned}$$

24. (a) Express $2\sin\theta - 1.5\cos\theta$ in the form $R\sin(\theta-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the value of α to 4 decimal places. (3)

Solution

$$R\sin(\theta-\alpha) \equiv R\sin\theta\cos\alpha - R\sin\alpha\cos\theta$$

and we have

$$R\sin\alpha = 1.5 \text{ and } R\cos\alpha = 2.$$

Now,

$$R = \sqrt{1.5^2+2^2} = \underline{\underline{2.5}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \tfrac{3}{4} \Rightarrow \alpha = 0.643\,501\,108\,8 \text{ (FCD)} = \underline{\underline{0.6435 \text{ (4 dp)}}}.$$

(b) Find the maximum value of $2\sin\theta - 1.5\cos\theta$ and find the value of θ, for $0 \leqslant x < \pi$, at which this maximum occurs. (3)

Solution

The maximum is $\underline{\underline{2.5}}$ and this is at

$$\begin{aligned}\sin(\theta - \alpha) = 1 &\Rightarrow \theta - \alpha = \tfrac{\pi}{2} \\ &\Rightarrow \underline{\underline{\theta = 2.214\,297\,436 \text{ (FCD)}}}.\end{aligned}$$

Tom models the height of sea water, H metres, on a particular day by the equation

$$H = 6 + 2\sin\left(\frac{4\pi t}{25}\right) - 1.5\cos\left(\frac{4\pi t}{25}\right),\ 0 \leqslant t < 12,$$

where t hours is the number of hours after midday.

(c) Calculate the maximum value of H predicted by this model and the value of t, to 2 decimal places, when this maximum occurs. (3)

Solution

The maximum is $6 + 2.5 = \underline{\underline{8.5 \text{ metres}}}$ and this is at

$$\begin{aligned}\sin\left(\frac{4\pi t}{25} - \alpha\right) = 1 &\Rightarrow \frac{4\pi t}{25} - \alpha = \tfrac{\pi}{2} \\ &\Rightarrow \frac{4\pi t}{25} = 2.214\,297\,436 \text{ (FCD)} \\ &\Rightarrow t = 4.405\,204\,779 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{t = 4.41 \text{ (2 dp)}}}.\end{aligned}$$

(d) Calculate, to the nearest minute, the times when the height of sea water is predicted, by this model, to be 7 metres. (6)

Solution

$$
\begin{aligned}
& 6 + 2\sin\left(\frac{4\pi t}{25}\right) - 1.5\cos\left(\frac{4\pi t}{25}\right) = 7 \\
\Rightarrow\ & 6 + 2.5\sin\left(\frac{4\pi t}{25} - \alpha\right) = 7 \\
\Rightarrow\ & 2.5\sin\left(\frac{4\pi t}{25} - \alpha\right) = 1 \\
\Rightarrow\ & \sin\left(\frac{4\pi t}{25} - \alpha\right) = \tfrac{2}{5} \\
\Rightarrow\ & \frac{4\pi t}{25} - \alpha = 0.411\,516\,846\,1,\ 2.730\,075\,808 \text{ (FCD)} \\
\Rightarrow\ & \frac{4\pi t}{25} = 1.055\,017\,955,\ 3.373\,576\,916 \text{ (FCD)} \\
\Rightarrow\ & t = 2.098\,891\,532,\ 6.711\,518\,027 \text{ (FCD)} \\
\Rightarrow\ & \underline{\underline{\text{time} = 14:06,\ 18:43}}.
\end{aligned}
$$

25. (a) Express $7\cos x - 24\sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the value of α to 3 decimal places. (3)

Solution

$$R\cos(x + \alpha) \equiv R\cos x\cos\alpha - R\sin x\sin\alpha$$

and we have

$$R\cos\alpha = 7 \text{ and } R\sin\alpha = 24.$$

Now,

$$R = \sqrt{7^2 + 24^2} = \underline{\underline{25}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \tfrac{24}{7} \Rightarrow \alpha = 1.287\,002\,218 \text{ (FCD)} = \underline{\underline{1.287 \text{ (3 dp)}}}.$$

(b) Hence write down the minimum value of $7\cos x - 24\sin x$. (1)

Solution

$\underline{\underline{-25}}$.

(c) Solve, for $0 \leqslant x < 2\pi$, the equation (5)

$$7\cos x - 24\sin x = 10,$$

giving your answers to 2 decimal place

Solution

$$\begin{aligned}7\cos x - 24\sin x = 10 &\Rightarrow 25\cos(x+\alpha) = 10\\ &\Rightarrow \cos(x+\alpha) = \tfrac{2}{5}\\ &\Rightarrow x+\alpha = 5.123\,905\,826,\ 7.442\,464\,788\ \text{(FCD)}\\ &\Rightarrow x = 3.836\,903\,609,\ 6.155\,2462\,57\ \text{(FCD)}\\ &\Rightarrow \underline{\underline{x = 3.84,\ 6.16\ (2\ \text{dp})}}.\end{aligned}$$

26. Find all the solutions of

$$2\cos 2\theta = 1 - 2\sin\theta$$

in the interval $0^\circ \leqslant \theta < 360^\circ$.

Solution

$$\begin{aligned}2\cos 2\theta = 1 - 2\sin\theta &\Rightarrow 2(1 - 2\sin^2\theta) = 1 - 2\sin\theta\\ &\Rightarrow 2 - 4\sin^2\theta = 1 - 2\sin\theta\\ &\Rightarrow 4\sin^2\theta - 2\sin\theta - 1 = 0\\ &\Rightarrow \sin\theta = \frac{2\pm\sqrt{20}}{8} = \frac{1\pm\sqrt{5}}{4}\\ &\Rightarrow \underline{\underline{\theta = 54,\ 126,\ 198,\ 342}}.\end{aligned}$$

27. (a) Prove that (4)

$$\frac{1}{\sin 2\theta} - \frac{\cos 2\theta}{\sin 2\theta} \equiv \tan\theta,\ \theta \neq 90n^\circ,\ n \in \mathbb{Z}.$$

Solution

$$\frac{1}{\sin 2\theta} - \frac{\cos 2\theta}{\sin 2\theta} \equiv \frac{1-\cos 2\theta}{\sin 2\theta}$$
$$\equiv \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$$
$$\equiv \frac{2\sin^2\theta}{2\sin\theta\cos\theta}$$
$$\equiv \frac{\sin\theta}{\cos\theta}$$
$$\equiv \underline{\underline{\tan\theta}}.$$

(b) Hence, or otherwise,

(i) show that $\tan 15^\circ = 2-\sqrt{3}$, (3)

Solution

$$\tan 15^\circ = \frac{1-\cos 30^\circ}{\sin 30^\circ} = \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \underline{\underline{2-\sqrt{3}}}.$$

(ii) solve, for $0^\circ < x < 360^\circ$, (5)

$$\operatorname{cosec} 4x - \cot 4x = 1.$$

Solution

$$\operatorname{cosec} 4x - \cot 4x = 1 \Rightarrow \frac{1}{\sin 4x} - \frac{\cos 4x}{\sin 4x} = 1$$
$$\Rightarrow \tan 2x = 1$$
$$\Rightarrow 2x = 45,\ 225,\ 405,\ 585$$
$$\Rightarrow \underline{\underline{x = 22.5,\ 112.5,\ 202.5,\ 292.5}}.$$

28. (a) Express $2\cos 3x - 3\sin 3x$ in the form $R\cos(3x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. (4)
Give your answers to to 3 significant figures.

Solution

$$R\cos(3x+\alpha) \equiv R\cos 3x\cos\alpha - R\sin 3x\sin\alpha$$

and we have

$$R\cos\alpha = 2 \text{ and } R\sin\alpha = 3.$$

Now,

$$R = \sqrt{2^2 + 3^2} = \sqrt{13} = \underline{\underline{3.61 \text{ (3 sf)}}}$$

and

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \tfrac{3}{2} \Rightarrow \alpha = 0.982\,793\,723\,2 \text{ (FCD)} = \underline{\underline{0.983 \text{ (3 sf)}}}.$$

$$\mathrm{f}(x) = \mathrm{e}^{2x} \cos 3x.$$

(b) Show that $\mathrm{f}'(x)$ can be written in the form (5)

$$\mathrm{f}'(x) = R\mathrm{e}^{2x} \cos(3x + \alpha),$$

where R and α are the constants found in part (a).

Solution

$$\begin{aligned}\mathrm{f}(x) = \mathrm{e}^{2x} \cos 3x &\Rightarrow \mathrm{f}'(x) = 2e^{2x} \cos 3x - 3e^{2x} \sin 3x \\ &\Rightarrow \mathrm{f}'(x) = e^{2x}(2 \cos 3x - 3 \sin 3x) \\ &\Rightarrow \underline{\underline{\mathrm{f}'(x) = R\mathrm{e}^{2x} \cos(3x + \alpha)}}.\end{aligned}$$

(c) Hence, or otherwise, find the smallest positive value of x which the curve with equation $y = \mathrm{f}(x)$ has a turning point. (3)

Solution

$$\begin{aligned}R\mathrm{e}^{2x} \cos(3x + \alpha) = 0 &\Rightarrow \cos(3x + \alpha) = 0 \\ &\Rightarrow 3x + \alpha = \tfrac{\pi}{2} \\ &\Rightarrow 3x = 0.588\,002\,603\,5 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 0.196\,000\,867\,8 \text{ (FCD)}}}.\end{aligned}$$

29. Solve, for $0^\circ \leqslant \theta \leqslant 180^\circ$,

$$2 \cot^2 3\theta = 7 \operatorname{cosec} 3\theta - 5.$$

Give your answers in degrees to 1 decimal place.

Solution

$$\begin{aligned}
2\cot^2 3\theta = 7\operatorname{cosec} 3\theta - 5 &\Rightarrow 2(\operatorname{cosec}^2 3\theta - 1) = 7\operatorname{cosec} 3\theta - 5 \\
&\Rightarrow 2\operatorname{cosec}^2 3\theta - 2 = 7\operatorname{cosec} 3\theta - 5 \\
&\Rightarrow 2\operatorname{cosec}^2 3\theta - 2 - 7\operatorname{cosec} 3\theta + 3 = 0 \\
&\Rightarrow (2\operatorname{cosec} 3\theta - 1)(\operatorname{cosec} 3\theta - 3) = 0 \\
&\Rightarrow \operatorname{cosec} 3\theta = 3 \text{ (only)} \\
&\Rightarrow \sin 3\theta = \tfrac{1}{3} \\
&\Rightarrow 3\theta = 19.471\ldots,\ 160.528\ldots,\ 379.471\ldots,\ 520.528\ldots \\
&\Rightarrow \theta = 6.490\ldots,\ 53.509\ldots\ 126.490\ldots,\ 173.509\ldots \\
&\Rightarrow \underline{\underline{\theta = 6.5,\ 53.5,\ 126.5,\ 173.5}}.
\end{aligned}$$

30. (a) Starting from the formulae for $\sin(A+B)$ and $\cos(A+B)$, prove that (4)

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Solution

$$\begin{aligned}
\tan(A+B) &\equiv \frac{\sin(A+B)}{\cos(A+B)} \\
&\equiv \frac{\sin A \cos B + \sin B \cos A}{\cos A \cos B - \sin A \sin B} \\
&\equiv \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\sin B \cos A}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}} \\
&\equiv \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A \sin B}{\cos A \cos B}} \\
&\equiv \underline{\underline{\frac{\tan A + \tan B}{1 - \tan A \tan B}}}.
\end{aligned}$$

(b) Deduce that (3)

$$\tan\left(\theta + \tfrac{\pi}{6}\right) \equiv \frac{1 + \sqrt{3}\tan\theta}{\sqrt{3} - \tan\theta}.$$

Solution

$$\begin{aligned}\tan\left(\theta+\tfrac{\pi}{6}\right) &= \frac{\tan\theta+\tan\frac{\pi}{6}}{1-\tan\theta\tan\frac{\pi}{6}}\\ &= \frac{\tan\theta+\frac{\sqrt{3}}{3}}{1-\frac{\sqrt{3}}{3}\tan\theta}\\ &= \underline{\underline{\frac{1+\sqrt{3}\tan\theta}{\sqrt{3}-\tan\theta}}}.\end{aligned}$$

(c) Hence, or otherwise, solve, for $0 \leqslant x \leqslant \pi$, (6)

$$1+\sqrt{3}\tan\theta = (\sqrt{3}-\tan\theta)\tan(\pi-\theta).$$

Give your answers as multiples of π.

Solution

$$\begin{aligned} & 1+\sqrt{3}\tan\theta = (\sqrt{3}-\tan\theta)\tan(\pi-\theta)\\ \Rightarrow\ & \frac{1+\sqrt{3}\tan\theta}{\sqrt{3}-\tan\theta} = \tan(\pi-\theta)\\ \Rightarrow\ & \tan\left(\theta+\tfrac{\pi}{6}\right) = \tan(\pi-\theta).\end{aligned}$$

$\underline{\tan\left(\theta+\tfrac{\pi}{6}\right) = \tan(\pi-\theta)}$:

$$\begin{aligned}\tan\left(\theta+\tfrac{\pi}{6}\right) = \tan(\pi-\theta) &\Rightarrow \theta+\tfrac{\pi}{6} = \pi-\theta\\ &\Rightarrow 2\theta = \tfrac{5\pi}{6}\\ &\Rightarrow \underline{\underline{\theta = \tfrac{5\pi}{12}}}.\end{aligned}$$

$\underline{\tan\left(\theta+\tfrac{\pi}{6}\right) = \tan(2\pi-\theta)}$:

$$\begin{aligned}\tan\left(\theta+\tfrac{\pi}{6}\right) = \tan(2\pi-\theta) &\Rightarrow \theta+\tfrac{\pi}{6} = 2\pi-\theta\\ &\Rightarrow 2\theta = \tfrac{11\pi}{6}\\ &\Rightarrow \underline{\underline{\theta = \tfrac{11\pi}{12}}}.\end{aligned}$$

31. (a) Express $4\operatorname{cosec}^2 2\theta - \operatorname{cosec}^2\theta$ in terms of $\sin\theta$ and $\cos\theta$. (2)

Solution

$$\begin{aligned} 4\operatorname{cosec}^2 2\theta - \operatorname{cosec}^2\theta &\equiv \frac{4}{\sin^2 2\theta} - \frac{1}{\sin^2\theta} \\ &\equiv \frac{4}{(2\sin\theta\cos\theta)^2} - \frac{1}{\sin^2\theta} \\ &\equiv \underline{\underline{\frac{4}{4\sin^2\theta\cos^2\theta} - \frac{1}{\sin^2\theta}}}. \end{aligned}$$

(b) Hence show that (4)

$$4\operatorname{cosec}^2 2\theta - \operatorname{cosec}^2\theta = \sec^2\theta.$$

Solution

$$\begin{aligned} 4\operatorname{cosec}^2 2\theta - \operatorname{cosec}^2\theta &\equiv \frac{4}{4\sin^2\theta\cos^2\theta} - \frac{1}{\sin^2\theta} \\ &\equiv \frac{1}{\sin^2\theta\cos^2\theta} - \frac{1}{\sin^2\theta} \\ &\equiv \frac{1-\cos^2\theta}{\sin^2\theta\cos^2\theta} \\ &\equiv \frac{\sin^2\theta}{\sin^2\theta\cos^2\theta} \\ &\equiv \frac{1}{\cos^2\theta} \\ &\equiv \underline{\underline{\sec^2\theta}}. \end{aligned}$$

(c) Hence or otherwise solve, for $0 < \theta < \pi$, (3)

$$4\operatorname{cosec}^2 2\theta - \operatorname{cosec}^2\theta = 4,$$

giving your answers in terms of π.

Solution

$$\begin{aligned} 4\operatorname{cosec}^2 2\theta - \operatorname{cosec}^2\theta = 4 &\Rightarrow \sec^2\theta = 4 \\ &\Rightarrow \cos^2\theta = \tfrac{1}{4} \\ &\Rightarrow \cos\theta = \tfrac{1}{2} \text{ or } \cos\theta = -\tfrac{1}{2}. \end{aligned}$$

$\underline{\cos\theta = \frac{1}{2}}$:

$$\cos\theta = \tfrac{1}{2} \Rightarrow \underline{\underline{\theta = \tfrac{\pi}{3}}}.$$

$\underline{\cos\theta = -\frac{1}{2}}$:

$$\cos\theta = -\tfrac{1}{2} \Rightarrow \underline{\underline{\theta = \tfrac{2\pi}{3}}}.$$

32.

$$\mathrm{f}(x) = 7\cos 2x - 24\sin 2x.$$

Given that $\mathrm{f}(x) = R\cos(2x + \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$,

(a) find the value of R and find the value of α. (3)

Solution

$$R\cos(2x + \alpha) \equiv R\cos 2x\cos\alpha - R\sin 2x\sin\alpha$$

and we have

$$R\cos\alpha = 7 \text{ and } R\sin\alpha = 24.$$

Now,

$$R = \sqrt{7^2 + 24^2} = \underline{\underline{25}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \tfrac{24}{7} \Rightarrow \underline{\underline{\alpha = 73.739\,795\,29 \text{ (FCD)}}}.$$

(b) Hence solve the equation (5)

$$7\cos 2x - 24\sin 2x = 12.5,$$

for $0^\circ \leqslant x \leqslant 180^\circ$, giving your answers to 1 decimal place.

Solution

$$\begin{aligned} 7\cos 2x - 24\sin 2x = 12.5 &\Rightarrow 25\cos(2x + \alpha) = 12.5 \\ &\Rightarrow \cos(2x + \alpha) = \tfrac{1}{2} \\ &\Rightarrow 2x + \alpha = 300,\ 420 \\ &\Rightarrow 2x = 226.260\,204\,7,\ 346.260\,204\,7 \text{ (FCD)} \\ &\Rightarrow x = 113.130\,2102\,4,\ 173.130\,104\,4 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 113.1,\ 173.1 \text{ (1 dp)}}}. \end{aligned}$$

(c) Express $14\cos^2 x - 48\sin x\cos x$ in the form $a\cos 2x + b\sin 2x + c$, where a, b, and c are constants to be found. (2)

Solution

$$\begin{aligned}14\cos^2 x - 48\sin x\cos x &\equiv 7(2\cos^2 x) - 24\sin 2x\\ &\equiv 7(\cos 2x + 1) - 24\sin 2x\\ &\equiv \underline{\underline{7\cos 2x - 24\sin 2x + 7}}.\end{aligned}$$

(d) Hence, using your answers to parts (a) and (c), deduce the maximum value of (2)

$$14\cos^2 x - 48\sin x\cos x.$$

Solution

$$14\cos^2 x - 48\sin x\cos x \equiv 7\cos 2x - 24\sin 2x + 7 \equiv 25\cos(2x+\alpha) + 7;$$

hence, $25 + 7 = \underline{\underline{32}}$.

33. (a) Express $6\cos\theta + 8\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the value of α to 3 decimal places. (4)

Solution

$$R\cos(\theta-\alpha) \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$$

and we have

$$R\cos\alpha = 6 \text{ and } R\sin\alpha = 8.$$

Now,

$$R = \sqrt{6^2 + 8^2} = \underline{\underline{10}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \tfrac{4}{3} \Rightarrow \alpha = 0.927\,295\,218 \text{ (FCD)} = \underline{\underline{0.927 \text{ (3 dp)}}}.$$

(b)

$$\mathrm{p}(\theta) = \frac{4}{12 + 6\cos\theta + 8\sin\theta},\ 0 \leqslant \theta \leqslant 2\pi.$$

Calculate

(i) the maximum value of $\mathrm{p}(\theta)$, (2)

Solution

$$\mathrm{p}(\theta) = \frac{4}{12 + 10\cos(\theta - \alpha)}$$

has a maximum of $\frac{4}{12-10} = \underline{\underline{2}}$.

(ii) the value of θ at which the maximum occurs. (2)

Solution

$$\theta - \alpha = \pi \Rightarrow \underline{\underline{\theta = 4.068\,887\,872 \text{ (FCD)}}}.$$

34. (a) Without using a calculator, find the exact value of (5)

$$(\sin 22.5^\circ + \cos 22.5^\circ)^2.$$

Solution

$$\begin{aligned}(\sin 22.5^\circ + \cos 22.5^\circ)^2 &= \sin^2 22.5^\circ + 2\sin 22.5^\circ \cos 22.5^\circ + \cos^2 22.5^\circ \\ &= 1 + \sin 45^\circ \\ &= \underline{\underline{\tfrac{2+\sqrt{2}}{2}}}.\end{aligned}$$

(b) (i) Show that $\cos 2\theta + \sin\theta = 1$ may be written in the form (2)

$$k\sin^2\theta - \sin\theta = 0,$$

stating the value of k.

Solution

$$\begin{aligned}\cos 2\theta + \sin\theta = 1 &\Rightarrow (1 - 2\sin^2\theta) + \sin\theta = 1 \\ &\Rightarrow \underline{\underline{2\sin^2\theta - \sin\theta = 0}}.\end{aligned}$$

(ii) Hence solve, for $0^\circ \leqslant \theta < 360^\circ$, the equation (4)

$$\cos 2\theta + \sin\theta = 1.$$

Solution

$$\begin{aligned}\cos 2\theta + \sin\theta = 1 &\Rightarrow 2\sin^2\theta - \sin\theta = 0\\ &\Rightarrow \sin\theta(2\sin\theta - 1) = 0\\ &\Rightarrow \sin\theta = 0 \text{ or } \sin\theta = \tfrac{1}{2}.\end{aligned}$$

$\underline{\sin\theta = 0}$:

$$\sin\theta = 0 \Rightarrow \underline{\underline{\theta = 0,\ 180}}.$$

$\underline{\sin\theta = \frac{1}{2}}$:

$$\sin\theta = \tfrac{1}{2} \Rightarrow \underline{\underline{\theta = 30,\ 150}}.$$

35. Given that

$$2\cos(x+50)^\circ = \sin(x+40)^\circ,$$

(a) show, without a calculator, that (4)

$$\tan x^\circ = \tfrac{1}{3}\tan 40^\circ.$$

Solution

$$\begin{aligned}& 2\cos(x+50)^\circ = \sin(x+40)^\circ\\ \Rightarrow\ & 2(\cos x^\circ \cos 50^\circ - \sin x^\circ \sin 50^\circ) = \sin x^\circ \cos 40^\circ + \sin 40^\circ \cos x^\circ\\ \Rightarrow\ & 2\cos x^\circ \sin 40^\circ - 2\sin x^\circ \cos 40^\circ = \sin x^\circ \cos 40^\circ + \sin 40^\circ \cos x^\circ\\ \Rightarrow\ & \cos x^\circ \sin 40^\circ = 3\sin x^\circ \cos 40^\circ\\ \Rightarrow\ & \frac{\sin x^\circ}{\cos x^\circ} = \frac{\sin 40^\circ}{3\cos 40^\circ}\\ \Rightarrow\ & \underline{\underline{\tan x^\circ = \tfrac{1}{3}\tan 40^\circ}}.\end{aligned}$$

(b) Hence solve, for $0 \leqslant \theta < 360$, (4)

$$2\cos(2\theta+50)^\circ = \sin(2\theta+40)^\circ,$$

giving your answers to 1 decimal place.

Solution

$$
\begin{aligned}
& 2\cos(2\theta+50)^\circ = \sin(2\theta+40)^\circ \\
\Rightarrow\ & \tan 2\theta^\circ = \tfrac{1}{3}\tan 40^\circ \\
\Rightarrow\ & 2\theta = 15.626\ldots,\ 195.626\ldots,\ 375.626\ldots,\ 555.626\ldots \\
\Rightarrow\ & \underline{\underline{\theta = 7.8,\ 97.8,\ 197.8,\ 277.8\ (1\text{ dp})}}.
\end{aligned}
$$

36. Kate crosses a road, of constant width 7 m, in order to take a photograph of a marathon runner, John, approaching at 3 ms^{-1}. Kate is 24 m ahead of John when she starts to cross the road from the fixed point A. John passes her as she reaches the other side of the road at a variable point B, as shown in Figure 1.

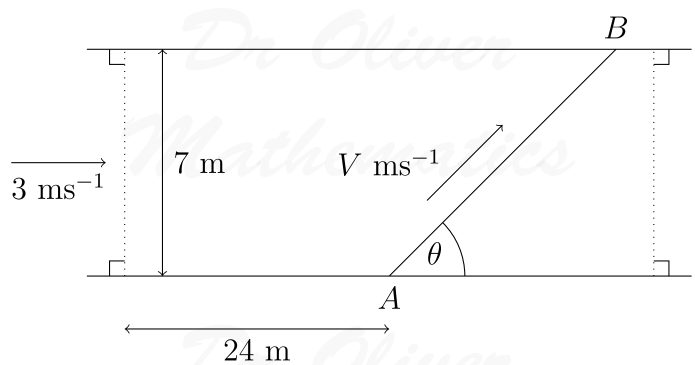


Figure 1: the road

Kate's speed is V ms^{-1} and she moves in a straight line, which makes an angle θ, $0^\circ < \theta < 150^\circ$, with the edge of the road. You may assume that V is given by the formula

$$V = \frac{21}{24\sin\theta + 7\cos\theta},\ 0^\circ < \theta < 150^\circ.$$

(a) Express $24\sin\theta + 7\cos\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$, giving the value of α to 2 decimal places. (3)

Solution

$$R\cos(\theta-\alpha) \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$$

and we have

$$R\cos\alpha = 7 \text{ and } R\sin\alpha = 24.$$

Now,

$$R = \sqrt{7^2+24^2} = \underline{\underline{25}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \tfrac{24}{7} \Rightarrow \alpha = 73.739\,795\,29 \text{ (FCD)} = \underline{\underline{73.74 \text{ (2 dp)}}}.$$

Given that θ varies,

(b) find the minimum value of V. (2)

Solution

The maximum value of the denominator is 25 which gives $\underline{\underline{V = \tfrac{21}{25} \text{ ms}^{-1}}}$.

Given that Kate's speed has the value found in part (b),

(c) find the distance AB. (3)

Solution

$$\begin{aligned} AB &= \frac{7}{\sin\alpha} \\ &= \underline{\underline{7\tfrac{7}{24} \text{ or } 7.291\dot{6} \text{ m (FCD)}}}. \end{aligned}$$

Given instead that Kate's speed is 1.68 ms^{-1},

(d) find the two possible values of the angle θ, given that $0^\circ < \theta < 150^\circ$. (6)

Solution

$$\begin{aligned} 25\cos(\theta - \alpha) = \frac{21}{1.68} &\Rightarrow 25\cos(\theta - \alpha) = 12.5 \\ &\Rightarrow \cos(\theta - \alpha) = \tfrac{1}{2} \\ &\Rightarrow \theta - \alpha = -60,\ 60 \\ &\Rightarrow \underline{\underline{\theta = 13.739\,795\,29,\ 133.739\,795\,29 \text{ (FCD)}}}. \end{aligned}$$

37.

$$f(x) = 7\cos x + \sin x.$$

Given that $f(x) = R\cos(x - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$,

(a) find the value of R and find the value of α to one decimal place. (3)

Solution

$$R\cos(x-\alpha) \equiv R\cos x\cos\alpha + R\sin x\sin\alpha$$

and we have

$$R\cos\alpha = 7 \text{ and } R\sin\alpha = 1.$$

Now,

$$R = \sqrt{7^2+1^2} = \underline{\underline{5\sqrt{2}}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \tfrac{1}{7} \Rightarrow \alpha = 8.130\,102\,354 \text{ (FCD)} = \underline{\underline{8.1 \text{ (1 dp)}}}.$$

(b) Hence solve the equation (5)

$$7\cos x + \sin x = 5$$

for $0^\circ < x < 360^\circ$, giving your answers to one decimal place.

Solution

$$\begin{aligned} 7\cos x + \sin x = 5 &\Rightarrow 5\sqrt{2}\cos(x-\alpha) = 5 \\ &\Rightarrow \cos(x-\alpha) = \frac{1}{\sqrt{2}} \\ &\Rightarrow x-\alpha = 45,\ 315 \\ &\Rightarrow x = 53.130\,102\,35,\ 323.130\,102\,4 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 53.1,\ 323.1 \text{ (1 dp)}}}. \end{aligned}$$

(c) State the values of k for which the equation (2)

$$7\cos x + \sin x = k$$

has only one solution in the interval $0^\circ < x < 360^\circ$.

Solution

There is one solution only if

$$\frac{k}{5\sqrt{2}} = \pm 1 \Rightarrow \underline{\underline{k = \pm 5\sqrt{2}}}.$$

38. (a) Use an appropriate double angle formula to show that (3)

$$\operatorname{cosec} 2x \equiv \lambda \operatorname{cosec} x \sec x,$$

and state the value of the constant λ.

Solution

$$\begin{aligned} \operatorname{cosec} 2x &\equiv \frac{1}{\sin 2x} \\ &\equiv \frac{1}{2\sin x \cos x} \\ &\equiv \underline{\underline{\tfrac{1}{2} \operatorname{cosec} x \sec x}}. \end{aligned}$$

(b) Solve, for $0 \leqslant \theta < 2\pi$, the equation (6)

$$3\sec^2\theta + 3\sec\theta = 2\tan^2\theta.$$

Give your answers in terms of π.

Solution

$$\begin{aligned} 3\sec^2\theta + 3\sec\theta = 2\tan^2\theta &\Rightarrow 3\sec^2\theta + 3\sec\theta = 2(\sec^2\theta - 1) \\ &\Rightarrow 3\sec^2\theta + 3\sec\theta = 2\sec^2\theta - 2 \\ &\Rightarrow \sec^2\theta + 3\sec\theta + 2 = 0 \\ &\Rightarrow (\sec\theta + 2)(\sec\theta + 1) = 0 \\ &\Rightarrow \sec\theta = -2 \text{ or } \sec\theta = -1 \\ &\Rightarrow \cos\theta = -\tfrac{1}{2} \text{ or } \cos\theta = -1. \end{aligned}$$

$\underline{\cos\theta = -\tfrac{1}{2}}$:

$$\cos\theta = -\tfrac{1}{2} \Rightarrow \underline{\underline{\theta = \tfrac{2\pi}{3}, \tfrac{4\pi}{3}}}.$$

$\underline{\cos\theta = -1}$:

$$\cos\theta = -1 \Rightarrow \underline{\underline{\theta = \pi}}.$$

39. (a) Show that (5)

$$\operatorname{cosec} 2x + \cot 2x \equiv \cot x,\ x \neq 90^\circ,\ n \in \mathbb{Z}.$$

Solution

$$\begin{aligned}
\operatorname{cosec} 2x + \cot 2x &\equiv \frac{1}{\sin 2x} + \frac{\cos 2x}{\sin 2x} \\
&\equiv \frac{1 - \cos 2x}{\sin 2x} \\
&\equiv \frac{1 - (2\cos^2 x - 1)}{2 \sin x \cos x} \\
&\equiv \frac{2\cos^2 x}{2 \sin x \cos x} \\
&\equiv \frac{2\cos^2 x}{2 \sin x \cos x} \\
&\equiv \frac{\cos x}{\sin x} \\
&\equiv \underline{\underline{\cot x}}.
\end{aligned}$$

(b) Hence, or otherwise, solve, for $0^\circ \leqslant \theta < 180^\circ$, (5)

$$\operatorname{cosec}(4\theta + 10)^\circ + \cot(4\theta + 10)^\circ = \sqrt{3}.$$

You must show your working.

Solution

$$\begin{aligned}
& \operatorname{cosec}(4\theta + 10)^\circ + \cot(4\theta + 10)^\circ = \sqrt{3} \\
\Rightarrow \quad & \cot(2\theta + 5)^\circ = \sqrt{3} \\
\Rightarrow \quad & \tan(2\theta + 5)^\circ = \tfrac{1}{\sqrt{3}} \\
\Rightarrow \quad & 2\theta + 5 = 30,\ 210 \\
\Rightarrow \quad & 2\theta = 25,\ 205 \\
\Rightarrow \quad & \underline{\underline{\theta = 12.5,\ 102.5}}.
\end{aligned}$$

40. (a) Express $2 \sin \theta - 4 \cos \theta$ in the form $R \sin(\theta - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the value of α to 3 decimal places. (3)

Solution

$$R \sin(\theta - \alpha) \equiv R \sin \theta \cos \alpha - R \sin \alpha \cos \theta$$

and we have

$$R \sin \alpha = 4 \text{ and } R \cos \alpha = 2.$$

Now,

$$R = \sqrt{4^2 + 2^2} = \underline{\underline{2\sqrt{5}}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = 2 \Rightarrow \alpha = 1.107\,148\,718 \text{ (FCD)} = \underline{\underline{1.107 \text{ (3 dp)}}}.$$

$$\mathrm{H}(\theta) = 4 + 5(2\sin 3\theta - 4\cos 3\theta)^2.$$

Find

(b) (i) the maximum value of $\mathrm{H}(\theta)$, (1)

Solution

$$\text{Maximum value} = 4 + 5 \times (2\sqrt{5})^2 = \underline{\underline{104}}.$$

(ii) the smallest value of θ, for $0 \leqslant \theta < \pi$, at which this maximum value occurs. (2)

Solution

$$\theta - \alpha = \tfrac{\pi}{2} \Rightarrow 3\theta = 2.677\,945\,045 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\theta = 0.892\,648\,348\,2 \text{ (FCD)}}}.$$

(c) (i) the minimum value of $\mathrm{H}(\theta)$, (1)

Solution

$$\text{Minimum value} = 4 - 5 \times 0 = \underline{\underline{4}}.$$

(ii) the largest value of θ, for $0 \leqslant \theta < \pi$, at which this minimum value occurs. (2)

Solution

We cannot have $\theta - \alpha = 0$ (why?) and so we need $\theta - \alpha = 2\pi$:

$$\theta - \alpha = 2\pi \Rightarrow 3\theta = 7.390\,334\,025 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\theta = 2.463\,444\,675 \text{ (FCD)}}}.$$

41. (a) (i) Show that (4)

$$2\tan x - \cot x = 5\operatorname{cosec} x$$

may be rewritten in the form

$$a\cos^2 x + b\cos x + c = 0,$$

stating the values of the constants a, b, and c.

Solution

$$\begin{aligned} 2\tan x - \cot x = 5\operatorname{cosec} x &\Rightarrow \frac{2\sin x}{\cos x} - \frac{\cos x}{\sin x} = \frac{5}{\sin x} \\ &\Rightarrow 2\sin^2 x - \cos^2 x = 5\cos x \\ &\Rightarrow 2(1-\cos^2 x) - \cos^2 x = 5\cos x \\ &\Rightarrow 2 - 2\cos^2 x - \cos^2 x = 5\cos x \\ &\Rightarrow \underline{\underline{3\cos^2 x + 5\cos x - 2 = 0}}. \end{aligned}$$

(ii) Hence solve, for $0 \leqslant x < 2\pi$, the equation (4)

$$2\tan x - \cot x = 5\operatorname{cosec} x,$$

giving your answers to 3 significant figures.

Solution

$$\begin{aligned} 2\tan x - \cot x = 5\operatorname{cosec} x &\Rightarrow 3\cos^2 x + 5\cos x - 2 = 0 \\ &\Rightarrow (3\cos x - 1)(\cos x + 2) = 0 \\ &\Rightarrow \cos x = \tfrac{1}{3} \text{ (only)} \\ &\Rightarrow x = 1.230\,959\,417,\ 5.052\,225\,89 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 1.23,\ 5.05 \text{ (3 sf)}}}. \end{aligned}$$

(b) Show that (4)

$$\tan\theta + \cot\theta \equiv \lambda\operatorname{cosec} 2\theta,\ \theta \neq \tfrac{n\pi}{2},\ n \in \mathbb{Z},$$

stating the value of the constant λ.

Solution

$$
\begin{aligned}
\tan\theta + \cot\theta &\equiv \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \\
&\equiv \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} \\
&\equiv \frac{1}{\sin\theta\cos\theta} \\
&\equiv \frac{2}{2\sin\theta\cos\theta} \\
&\equiv \frac{2}{\sin 2\theta} \\
&\equiv \underline{\underline{2\operatorname{cosec} 2\theta}}.
\end{aligned}
$$

42. Figure 2 shows the curve C with equation $y = 6\cos x + 2.5\sin x$ for $0 \leqslant x \leqslant 2\pi$.

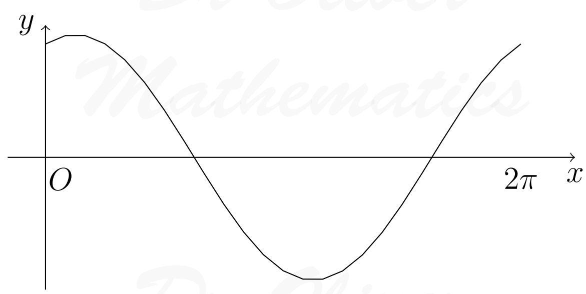


Figure 2: $y = 6\cos x + 2.5\sin x$

(a) Express $6\cos x + 2.5\sin x$ in the form $R\cos(x - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give the value of α to 3 decimal places. (3)

Solution

$$R\cos(x + \alpha) \equiv R\cos x\cos\alpha - R\sin x\sin\alpha$$

and we have

$$R\sin\alpha = 2.5 \text{ and } R\cos\alpha = 6.$$

Now,

$$R = \sqrt{6^2 + 2.5^2} = \underline{\underline{6.5}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \tfrac{5}{12} \Rightarrow \alpha = 0.394\,791\,119\,7 \text{ (FCD)} = \underline{\underline{0.395 \text{ (3 dp)}}}.$$

(b) Find the coordinates of the points on the graph where the curve C crosses the coordinate axes. (3)

Solution

$\underline{\underline{(0, 6)}}$, $\underline{\underline{(1.965\,587\,446, 0)\ \text{(FCD)}}}$, and $\underline{\underline{(5.107\,180\,1, 0)\ \text{(FCD)}}}$.

A student records the number of hours of daylight each Sunday throughout the year. She starts on the lasts Sunday in May, with a recording of 18 hours, and continues until her final recording 52 weeks later. She models her results with the continuous function given by

$$H = 12 + 6\cos\left(\frac{2\pi t}{52}\right) + 2.5\sin\left(\frac{2\pi t}{52}\right),\ 0 \leqslant t \leqslant 52,$$

where H is the number of hours of daylight and t is the number of weeks since her first recording. Use this function to find

(c) the maximum and minimum values of H predicted by the model, (3)

Solution

$H_{\max} = 12 + 6.5 = \underline{\underline{18.5}}$ and $H_{\min} = 12 - 6.5 = \underline{\underline{5.5}}$

(d) the values for t when $H = 16$, giving your answers to the nearest whole number. (6)

Solution

$$\begin{aligned}
& 12 + 6\cos\left(\frac{2\pi t}{52} - \alpha\right) + 2.5\sin\left(\frac{2\pi t}{52} - \alpha\right) = 16 \\
\Rightarrow\ & 12 + 6.5\cos\left(\frac{2\pi t}{52} - \alpha\right) = 16 \\
\Rightarrow\ & 6.5\cos\left(\frac{2\pi t}{52} - \alpha\right) = 4 \\
\Rightarrow\ & \cos\left(\frac{2\pi t}{52} - \alpha\right) = \tfrac{8}{13} \\
\Rightarrow\ & \frac{2\pi t}{52} - \alpha = 0.907\ldots,\ 5.375\ldots \\
\Rightarrow\ & \frac{2\pi t}{52} = 1.302\ldots,\ 5.770\ldots \\
\Rightarrow\ & t = 10.781\ldots,\ 47.753\ldots \\
\Rightarrow\ & \underline{\underline{t = 11,\ 48\ \text{(nearest whole number)}}}.
\end{aligned}$$

43. Given that

$$\tan\theta^\circ = p, \quad \text{where } p \text{ is a constant, } p \neq \pm 1,$$

use standard trigonometric identities, to find in terms of p,

(a) $\tan 2\theta^\circ$, (2)

Solution

$$\tan 2\theta^\circ = \frac{2\tan\theta}{1-\tan^2\theta} = \underline{\underline{\frac{2p}{1-p^2}}}.$$

(b) $\cos\theta^\circ$, (2)

Solution

$$\cos\theta^\circ = \frac{1}{\sec\theta} = \pm\frac{1}{\sqrt{1+\tan^2\theta}} = \underline{\underline{\pm\frac{1}{\sqrt{1+p^2}}}}.$$

(c) $\cot(\theta - 45)^\circ$. (2)

Solution

$$\cot(\theta-45)^\circ = \frac{1}{\tan(\theta-45)^\circ} = \frac{1+\tan\theta^\circ\tan 45^\circ}{\tan\theta^\circ - \tan 45^\circ} = \underline{\underline{\frac{1+p}{p-1}}}.$$

44.

$$\mathrm{g}(\theta) = 4\cos 2\theta + 2\sin 2\theta.$$

Given that $\mathrm{g}(\theta) = R\cos(2\theta - \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$,

(a) find the exact value of R and find the exact value of α to 2 decimal places. (3)

Solution

$$R\cos(2\theta-\alpha) \equiv R\cos 2\theta\cos\alpha + R\sin 2\theta\sin\alpha$$

and we have

$$R\sin\alpha = 2 \text{ and } R\cos\alpha = 4.$$

Now,

$$R = \sqrt{4^2+2^2} = \underline{\underline{2\sqrt{5}}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \tfrac{5}{12} \Rightarrow \alpha = 26.565\,051\,18 \text{ (FCD)} = \underline{\underline{26.57 \text{ (2 dp)}}}.$$

(b) Hence solve, for $-90° < \theta < 90°$, (5)

$$4\cos 2\theta + 2\sin 2\theta = 1,$$

giving your answers to one decimal place.

Solution

$$\begin{aligned} 4\cos 2\theta + 2\sin 2\theta = 1 &\Rightarrow 2\sqrt{5}\cos(2\theta - \alpha) = 1 \\ &\Rightarrow \cos(2\theta - \alpha) = \tfrac{\sqrt{5}}{10} \\ &\Rightarrow 2\theta - \alpha = -77.079\ldots,\ 77.079\ldots \\ &\Rightarrow 2\theta = -50.513\ldots,\ 103.644\ldots \\ &\Rightarrow \theta = -25.256\ldots,\ 51.822\ldots \\ &\Rightarrow \underline{\underline{\theta = -25.3,\ 51.8\ (1\text{ dp})}}. \end{aligned}$$

Given that k is a constant and the equation $g(\theta) = k$ has no solutions,

(c) state the possible values of k. (2)

Solution

$$\underline{\underline{|k| > 2\sqrt{5}}}.$$

45. (a) Prove that (5)

$$\sec 2A + \tan 2A \equiv \frac{\cos A + \sin A}{\cos A - \sin A},\ A \neq \tfrac{(2n+1)\pi}{4},\ n \in \mathbb{Z}.$$

Solution

$$\begin{aligned} \sec 2A + \tan 2A &\equiv \frac{1}{\cos 2A} + \frac{\sin 2A}{\cos 2A} \\ &\equiv \frac{1 + \sin 2A}{\cos 2A} \\ &\equiv \frac{\sin^2 A + \cos^2 A + 2\sin A\cos A}{\cos^2 A - \sin^2 A} \\ &\equiv \frac{(\sin A + \cos A)^2}{(\cos A - \sin A)(\cos A + \sin A)} \\ &\equiv \underline{\underline{\frac{\cos A + \sin A}{\cos A - \sin A}}}. \end{aligned}$$

(b) Hence solve, for $0 \leqslant A < 2\pi$, (4)

$$\sec 2A + \tan 2A = \tfrac{1}{2}.$$

Give your answers to 3 decimal places.

Solution

$$\begin{aligned}
\sec 2A + \tan 2A = \tfrac{1}{2} &\Rightarrow \frac{\cos A + \sin A}{\cos A - \sin A} = \tfrac{1}{2} \\
&\Rightarrow \cos A + \sin A = \tfrac{1}{2}(\cos A - \sin A) \\
&\Rightarrow \cos A + \sin A = \tfrac{1}{2}\cos A - \tfrac{1}{2}\sin A \\
&\Rightarrow \tfrac{3}{2}\sin A = -\tfrac{1}{2}\cos A \\
&\Rightarrow \tan A = -\tfrac{1}{3} \\
&\Rightarrow A = 2.819\,842\,099,\ 5.961\,434\,753\ \text{(FCD)} \\
&\Rightarrow \underline{\underline{A = 2.820,\ 5.961\ \text{(3 dp)}}}.
\end{aligned}$$

46. (a) Express $2\cos\theta - \sin\theta$ in the form $R\cos(\theta+\alpha)$, where R and α are constants, $R > 0$ and $0^\circ < \alpha < 90^\circ$. Give the exact value for R and give the exact value for α to 2 decimal places. (3)

Solution

$$R\cos(\theta+\alpha) \equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

and we have

$$R\sin\alpha = 1 \text{ and } R\cos\alpha = 2.$$

Now,

$$R = \sqrt{2^2 + 1^2} = \underline{\underline{\sqrt{5}}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \tfrac{1}{2} \Rightarrow \alpha = 26.565\,051\,18\ \text{(FCD)} = \underline{\underline{26.57\ \text{(2 dp)}}}.$$

(b) Hence solve, for $0^\circ < \theta < 360^\circ$, (5)

$$\frac{2}{2\cos\theta - \sin\theta - 1} = 15.$$

Give your answers to one decimal place.

Solution

$$\begin{aligned}\frac{2}{2\cos\theta-\sin\theta-1}=15 &\Rightarrow \frac{2\cos\theta-\sin\theta-1}{2}=\tfrac{1}{15}\\ &\Rightarrow 2\cos\theta-\sin\theta-1=\tfrac{2}{15}\\ &\Rightarrow 2\cos\theta-\sin\theta=\tfrac{17}{15}\\ &\Rightarrow \sqrt{5}\cos(\theta+\alpha)=\tfrac{17}{15}\\ &\Rightarrow \cos(\theta+\alpha)=\tfrac{17\sqrt{5}}{75}\\ &\Rightarrow \theta+\alpha=59.546\,290\,08,\ 300.453\,709\,9\ \text{(FCD)}\\ &\Rightarrow \theta=32.981\,238\,9,\ 273.888\,658\,7\ \text{(FCD)}\\ &\Rightarrow \underline{\underline{\theta=33.0,\ 273.9\ (1\text{ dp})}}.\end{aligned}$$

(c) Use your solutions to parts (a) and (b) to deduce the smallest positive value of θ for which (2)

$$\frac{2}{2\cos\theta+\sin\theta-1}=15.$$

Give your answer to one decimal place.

Solution

This is

$$\frac{2}{\sqrt{5}\cos(\theta-\alpha)-1}=15$$

so do '$\theta-\alpha$' instead of '$\theta+\alpha$':

$$\begin{aligned}\theta-\alpha=59.546\,290\,08\ \text{(FCD)} &\Rightarrow \theta=86.111\,341\,26\ \text{(FCD)}\\ &\Rightarrow \underline{\underline{\theta=86.1\ (1\text{ dp})}}.\end{aligned}$$

47. (a) Prove that (4)

$$2\cot 2x+\tan x\equiv\cot x,\ x\neq\tfrac{n\pi}{2},\ n\in\mathbb{Z}.$$

Solution

$$
\begin{aligned}
2\cot 2x + \tan x &\equiv \frac{2}{\tan 2x} + \tan x \\
&\equiv \frac{1-\tan^2 x}{\tan x} + \frac{\tan^2 x}{\tan x} \\
&\equiv \frac{1}{\tan x} \\
&\equiv \underline{\underline{\cot x}}.
\end{aligned}
$$

(b) Hence, or otherwise, solve, for $-\pi \leqslant x < \pi$, (6)

$$6\cot 2x + 3\tan x = \operatorname{cosec}^2 x - 2.$$

Give your answers to 3 decimal places.

Solution

$$
\begin{aligned}
6\cot 2x + 3\tan x = \operatorname{cosec}^2 x - 2 &\Rightarrow 3\cot x = (1+\cot^2 x) - 2 \\
&\Rightarrow \cot^2 x - 3\cot x - 1 = 0 \\
&\Rightarrow \cot x = \frac{3 \pm \sqrt{13}}{2} \\
&\Rightarrow \tan x = \frac{-3 \pm \sqrt{13}}{2}.
\end{aligned}
$$

$\underline{\tan x = \frac{-3+\sqrt{13}}{2}}$:

$$
\begin{aligned}
\tan x = \frac{-3+\sqrt{13}}{2} &\Rightarrow x = -2.847\,591\,352,\ 0.294\,001\,301\,8 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{x = -2.848,\ 0.294 \text{ (3 dp)}}}.
\end{aligned}
$$

$$
\begin{aligned}
\tan x = \frac{-3-\sqrt{13}}{2} &\Rightarrow x = 1.276\,795\,025,\ 1.864\,797\,629 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{x = -1.277,\ 1.865 \text{ (3 dp)}}}.
\end{aligned}
$$

48. (a) Write $5\cos\theta - 2\sin\theta$ in the form $R\cos(\theta + \alpha)$, where R and α are constants, $R > 0$ and $0 \leqslant \alpha < \frac{\pi}{2}$. Give the exact value of R and give the value of α in radians to 3 decimal places. (3)

Solution

$$R\cos(\theta+\alpha) \equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

and we have

$$R\sin\alpha = 2 \text{ and } R\cos\alpha = 5.$$

Now,

$$R = \sqrt{5^2+2^2} = \underline{\underline{\sqrt{29}}}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \tfrac{2}{5} \Rightarrow \alpha = 0.380\,506\,377\,1 \text{ (FCD)} = \underline{\underline{0.381 \text{ (3 dp)}}}.$$

(b) Show that the equation (2)

$$5\cot 2x - 3\operatorname{cosec} 2x = 2$$

can be rewritten in the form

$$5\cos 2x - 2\sin 2x = c$$

where c is a positive constant to be determined.

Solution

$$\begin{aligned} 5\cot 2x - 3\operatorname{cosec} 2x = 2 &\Rightarrow \frac{5\cos 2x}{\sin 2x} - \frac{3}{\sin 2x} = 2 \\ &\Rightarrow 5\cos 2x - 3 = 2\sin 2x \\ &\Rightarrow \underline{\underline{5\cos 2x - 2\sin 2x = 3}}. \end{aligned}$$

(c) Hence or otherwise, solve, for $0 \leqslant x < \pi$, (4)

$$5\cot 2x - 3\operatorname{cosec} 2x = 2,$$

giving your answers to 2 decimal places.

Solution

$$
\begin{aligned}
5\cot 2x - 3\operatorname{cosec} 2x = 2 &\Rightarrow 5\cos 2x - 2\sin 2x = 3 \\
&\Rightarrow \sqrt{29}\cos(2x+\alpha) = 3 \\
&\Rightarrow \cos(2x+\alpha) = \tfrac{3\sqrt{29}}{29} \\
&\Rightarrow 2x+\alpha = 0.979\,923\,576\,6,\ 5.303\,261\,731 \text{ (FCD)} \\
&\Rightarrow x = 0.299\,708\,599\,8,\ 2.461\,377\,677 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{x = 0.30,\ 2.46 \text{ (2 dp)}}}.
\end{aligned}
$$

49. (a) Prove that (4)

$$\sin 2x - \tan x \equiv \tan x \cos 2x,\ x \neq (2n+1)90^\circ, n \in \mathbb{Z}.$$

Solution

$$
\begin{aligned}
\sin 2x - \tan x &\equiv 2\sin x\cos x - \frac{\sin x}{\cos x} \\
&\equiv \frac{2\sin x\cos^2 x - \sin x}{\cos x} \\
&\equiv \frac{\sin x(2\cos^2 x - 1)}{\cos x} \\
&\equiv \underline{\underline{\tan x\cos 2x}}.
\end{aligned}
$$

(b) Given that $x \neq 90^\circ$ and $x \neq 270^\circ$, solve, for $0^\circ \leqslant x < 360^\circ$, (5)

$$\sin 2x - \tan x = 3\tan x\sin x.$$

Give your answers in degrees to one decimal place where appropriate.

Solution

$$
\begin{aligned}
\sin 2x - \tan x = 3\tan x\sin x &\Rightarrow \tan x\cos 2x = 3\tan x\sin x \\
&\Rightarrow \tan x\cos 2x - 3\tan x\sin x = 0 \\
&\Rightarrow \tan x(\cos 2x - 3\sin x) = 0 \\
&\Rightarrow \tan x(1 - 2\sin^2 x - 3\sin x) = 0 \\
&\Rightarrow \tan x = 0 \text{ or } 2\sin^2 x + 3\sin x - 1 = 0 \\
&\Rightarrow \tan x = 0 \text{ or } \sin x = \tfrac{-3\pm\sqrt{17}}{4}.
\end{aligned}
$$

$\underline{\tan x = 0}$:

$$\tan x = 0 \Rightarrow \underline{\underline{x = 0,\ 180}}.$$

$\underline{\sin x = \frac{-3+\sqrt{17}}{4}}$:

$$\begin{aligned} \sin x = \tfrac{-3+\sqrt{17}}{4} &\Rightarrow x = 16.306\,548\,52,\ 163.693\,451\,5 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 16.3,\ 163.7 \text{ (1 dp)}}}. \end{aligned}$$

$\underline{\sin x = \frac{-3-\sqrt{17}}{4}}$: No solutions since

$$\sin x = -1.78\ldots$$