Dr Oliver Mathematics
Advance Level Further Mathematics
Core Pure Mathematics 2: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.

1. (a) Prove that (5)

$$\tanh^{-1} x = \tfrac{1}{2} \ln \left(\frac{1+x}{1-x} \right), \ -k < x < k,$$

stating the value of the constant k.

(b) Hence, or otherwise, solve the equation (5)

$$2x = \tanh(\ln \sqrt{2-3x}).$$

2. The roots of the equation (8)

$$x^3 - 2x^2 + 4x - 5 = 0$$

are p, q, and r.

Without solving the equation, find the value of

(a) $\dfrac{2}{p} + \dfrac{2}{q} + \dfrac{2}{r}$,

(b) $(p-4)(q-4)(r-4)$,

(c) $p^3 + q^3 + r^3$.

3.

$$f(x) = \frac{1}{\sqrt{4x^2+9}}.$$

(a) Using a substitution, that should be stated clearly, show that (4)

$$\int f(x)\,dx = A \sinh^{-1}(Bx) + c,$$

where c is an arbitrary constant and A and B are constants to be found.

(b) Hence find, in exact form in terms of natural logarithms, the mean value of $f(x)$ over the interval $[0, 3]$. (2)

4. The infinite series C and S are defined by

$$C = \cos\theta + \tfrac{1}{2}\cos 5\theta + \tfrac{1}{4}\cos 9\theta + \tfrac{1}{8}\cos 13\theta + \dots$$
$$S = \sin\theta + \tfrac{1}{2}\sin 5\theta + \tfrac{1}{4}\sin 9\theta + \tfrac{1}{8}\sin 13\theta + \dots$$

Given that the series C and S are both convergent,

(a) show that (4)

$$C + \mathrm{i}S = \frac{2\mathrm{e}^{\mathrm{i}\theta}}{2 - \mathrm{e}^{4\mathrm{i}\theta}}.$$

(b) Hence show that (4)

$$S = \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta}.$$

5. An engineer is investigating the motion of a sprung diving board at a swimming pool. Let E be the position of the end of the diving board when it is at rest in its equilibrium position and when there is no diver standing on the diving board.
A diver jumps from the diving board.
The vertical displacement, h cm, of the end of the diving board above E is modelled by the differential equation

$$4\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + 4\frac{\mathrm{d}h}{\mathrm{d}t} + 37h = 0,$$

where t seconds is the time after the diver jumps.

(a) Find a general solution of the differential equation. (2)

When $t = 0$, the end of the diving board is 20 cm below E and is moving upwards with a speed of 55 $\mathrm{cm\,s^{-1}}$.

(b) Find, according to the model, the maximum vertical displacement of the end of the diving board above E. (8)

(c) Comment on the suitability of the model for large values of t. (2)

6. In an Argand diagram, the points A, B, and C are the vertices of an equilateral triangle with its centre at the origin. The point A represents the complex number $6 + 2\mathrm{i}$.

(a) Find the complex numbers represented by the points B and C, giving your answers in the form $x + \mathrm{i}y$, where x and y are real and exact. (6)

The points D, E, and F are the midpoints of the sides of triangle ABC.

(b) Find the exact area of triangle DEF. (3)

7.

$$\mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & k & 4 \\ 3 & 2 & -1 \end{pmatrix},$$

where k is a constant.

(a) Find the values of k for which the matrix $\mathbf{M}$ has an inverse. (2)

(b) Find, in terms of p, the coordinates of the point where the following planes intersect (5)

$$\begin{aligned} 2x - y + z &= p \\ 3x - 6y + 4z &= 1 \\ 3x + 2y - z &= 0. \end{aligned}$$

(c) (i) Find the value of q for which the set of simultaneous equations (4)

$$\begin{aligned} 2x - y + z &= 1 \\ 3x - 5y + 4z &= q \\ 3x + 2y - z &= 0 \end{aligned}$$

can be solved.

(ii) For this value of q, interpret the solution of the set of simultaneous equations geometrically.

8. Figure 1 shows the central vertical cross section $ABCD$ of a paddling pool that has a circular horizontal cross section.

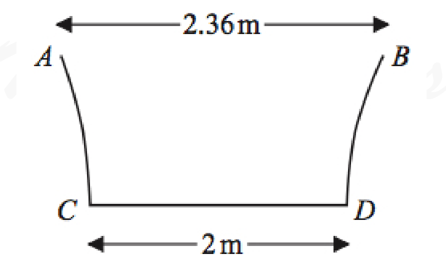

Figure 1: the central vertical cross section $ABCD$

Measurements of the diameters of the top and bottom of the paddling pool have been taken in order to estimate the volume of water that the paddling pool can contain.

Using these measurements, the curve BD is modelled by the equation

$$y = \ln(3.6x - k),\ 1 \leqslant x \leqslant 1.18,$$

as shown in Figure 2.

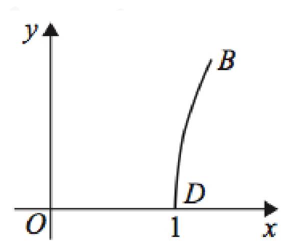

Figure 2: $y = \ln(3.6x - k)$

(a) Find the value of k. (1)

(b) Find the depth of the paddling pool according to this model. (2)

The pool is being filled with water from a tap.

(c) Find, in terms of h, the volume of water in the pool when the pool is filled to a depth of h m. (5)

Given that the pool is being filled at a constant rate of 15 litres every minute,

(d) find, in cm h^{-1}, the rate at which the water level is rising in the pool when the depth of the water is 0.2 m. (3)