Dr Oliver Mathematics
Further Mathematics
Conic Sections: Ellipses
Past Examination Questions

This booklet consists of 17 questions across a variety of examination topics.
The total number of marks available is 155.

1. The ellipse C has equation
$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$

(a) Find an equation of the normal to C at the point $P(5\cos\theta, 3\sin\theta)$. (5)

Solution

$$\begin{aligned} \frac{x^2}{25} + \frac{y^2}{9} = 1 &\Rightarrow \frac{2x}{25} + \frac{2y}{9}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \\ &\Rightarrow \frac{2y}{9}\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{25} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{9x}{25y}, \end{aligned}$$

and, at the point $P(5\cos\theta, 3\sin\theta)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{45\cos\theta}{75\sin\theta} = -\frac{3\cos\theta}{5\sin\theta};$$

hence, the gradient of the normal is

$$m = \frac{5\sin\theta}{3\cos\theta}.$$

Now,

$$\begin{aligned} & y - 3\sin\theta = \frac{5\sin\theta}{3\cos\theta}(x - 5\cos\theta) \\ \Rightarrow\ & 3\cos\theta(y - 3\sin\theta) = 5\sin\theta(x - 5\cos\theta) \\ \Rightarrow\ & 3y\cos\theta - 9\sin\theta\cos\theta = 5x\sin\theta - 25\sin\theta\cos\theta \\ \Rightarrow\ & \underline{\underline{5x\sin\theta - 3y\cos\theta = 16\sin\theta\cos\theta}}. \end{aligned}$$

The normal to C at P meets the coordinate axes at Q and R.
Given that $ORSQ$ is a rectangle, where O is the origin,

(b) show that, as θ varies, the locus of S is an ellipse and find its equation in Cartesian form. (8)

Solution

$$x = 0 \Rightarrow -3y\cos\theta = 16\sin\theta\cos\theta \Rightarrow y = -\tfrac{16}{3}\sin\theta,$$

$$y = 0 \Rightarrow 5x\sin\theta = 16\sin\theta\cos\theta \Rightarrow x = \tfrac{16}{5}\cos\theta,$$

and $S(\tfrac{16}{5}\cos\theta, -\tfrac{16}{3}\sin\theta)$. So, it is of the $(a\cos\theta, b\sin\theta)$, and it is an ellipse. Finally,

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \frac{25x^2}{256} + \frac{9y^2}{256} = 1.$$

2. The ellipse E has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are positive constants and $a > b$.

(a) Find an equation of the normal to C at the point $P(a\cos\theta, b\sin\theta)$. (5)

Solution

$$\begin{aligned}\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 &\Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0\\ &\Rightarrow \frac{2y}{b^2}\frac{dy}{dx} = -\frac{2x}{a^2}\\ &\Rightarrow \frac{dy}{dx} = -\frac{b^2x}{a^2y},\end{aligned}$$

and, at the point $P(a\cos\theta, b\sin\theta)$,

$$\frac{dy}{dx} = -\frac{ab^2\cos\theta}{a^2b\sin\theta} = -\frac{b\cos\theta}{a\sin\theta};$$

hence, the gradient of the normal is

$$m = \frac{a\sin\theta}{b\cos\theta}.$$

Now,

$$\begin{aligned}& y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)\\ \Rightarrow\ & b\cos\theta(y - b\sin\theta) = a\sin\theta(x - a\cos\theta)\\ \Rightarrow\ & by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta\\ \Rightarrow\ & ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta.\end{aligned}$$

The normal to C at P meets the x-axis at Q. R is the foot of the perpendicular from P to the x-axis.

(b) Show that $\dfrac{OQ}{OR} = e^2$, where e is the eccentricity of C. (7)

Solution

$\underline{Q}$:

$$y = 0 \Rightarrow ax \sin\theta = (a^2 - b^2)\sin\theta\cos\theta \Rightarrow x = \frac{(a^2 - b^2)\cos\theta}{a}.$$

$\underline{R}$:

$$x = a\cos\theta.$$

Now,

$$\begin{aligned}\frac{OQ}{OR} &= \frac{\frac{(a^2-b^2)\cos\theta}{a}}{a\cos\theta} \\ &= \frac{a^2 - b^2}{a^2} \\ &= 1 - \frac{b^2}{a^2} \\ &= \underline{\underline{e^2}},\end{aligned}$$

as required.

3. The line with equation $y = mx + c$ is a tangent to the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(a) Show that $c^2 = a^2m^2 + b^2$. (8)

Solution

$$\begin{aligned}y = mx + c &\Rightarrow \frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1 \\ &\Rightarrow b^2x^2 + a^2(mx+c)^2 = a^2b^2 \\ &\Rightarrow b^2x^2 + a^2(m^2x^2 + 2cmx + c^2) = a^2b^2 \\ &\Rightarrow b^2x^2 + a^2m^2x^2 + 2a^2cmx + a^2c^2 = a^2b^2 \\ &\Rightarrow (b^2 + a^2m^2)x^2 + 2a^2cmx + (a^2c^2 - a^2b^2) = 0.\end{aligned}$$

Now, the tangent to the ellipse will have '$b^2 - 4ac = 0$':

$$\begin{aligned}
& (2a^2cm)^2 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2) = 0 \\
\Rightarrow\ & 4a^4c^2m^2 - 4(a^2b^2c^2 - a^2b^4 + a^4c^2m^2 - a^4b^2m^2) = 0 \\
\Rightarrow\ & a^4c^2m^2 - a^2b^2c^2 + a^2b^4 - a^4c^2m^2 + a^4b^2m^2 = 0 \\
\Rightarrow\ & -b^2c^2 + b^4 + a^2b^2m^2 = 0 \\
\Rightarrow\ & -c^2 + b^2 + a^2m^2 = 0 \\
\Rightarrow\ & \underline{\underline{c^2 = b^2 + a^2m^2}}.
\end{aligned}$$

(b) Hence, or otherwise, find the equations of the tangents from the point $(3, 4)$ to the ellipse with equation (7)

$$\frac{x^2}{16} + \frac{y^2}{25} = 1.$$

Solution

We form equations

$$4 = 3m + c \text{ and } 16m^2 + 25 = c^2 :$$

$$\begin{aligned}
c = 4 - 3m &\Rightarrow 16m^2 + 25 = (4 - 3m)^2 \\
&\Rightarrow 16m^2 + 25 = 16 - 24m + 9m^2 \\
&\Rightarrow 7m^2 + 24m + 9 = 0 \\
&\Rightarrow (7m + 3)(m + 3) = 0 \\
&\Rightarrow m = -3 \text{ or } m = -\tfrac{3}{7}
\end{aligned}$$

and we finish with either

$$\underline{\underline{y = -3x + 13}}$$

or

$$\underline{\underline{y = -\tfrac{3}{7}x + \tfrac{37}{7}}}.$$

4. The ellipse C has parametric equations (5)

$$x = 4\cos\theta,\ y = \sqrt{7}\sin\theta,\ 0 \leqslant \theta < 2\pi.$$

Show that the foci of C are at the points $(3, 0)$ and $(-3, 0)$.

Solution

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{7} = 1;$$

$a^2 = 16$, $b^2 = 7$, and

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{3}{4}$$

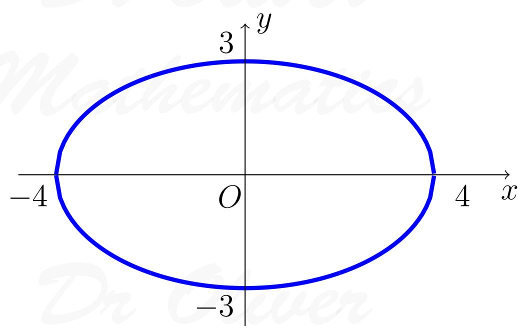
and the foci are at $\underline{\underline{(3, 0)}}$ and $\underline{\underline{(-3, 0)}}$.

5. An ellipse has equation

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

(a) Sketch the ellipse. (1)

Solution



(b) Find the eccentricity e. (2)

Solution

$a = 4$, $b = 3$, and

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \underline{\underline{\frac{\sqrt{7}}{4}}}.$$

(c) State the coordinates of the foci of the ellipse. (2)

Solution

$x = ae$; therefore, the foci are at $\underline{\underline{(\sqrt{7}, 0)}}$ and $\underline{\underline{(-\sqrt{7}, 0)}}$.

6. An ellipse, with equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1,$$

has foci S and S'.

(a) Find the coordinates of the foci of the ellipse. (4)

Solution

$a = 3$, $b = 2$, and

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{\sqrt{5}}{3}.$$

Hence, we have $\underline{\underline{(-\sqrt{5}, 0)}}$ and $\underline{\underline{(\sqrt{5}, 0)}}$.

(b) Using the focus-directrix property of the ellipse, show that, for any point P on the ellipse, (3)

$$SP + S'P = 6.$$

Solution

$$\begin{aligned} SP + S'P &= eNP + eN'P \\ &= e\,(NP + N'P) \\ &= eNN' \\ &= \frac{\sqrt{5}}{3} \times \frac{18\sqrt{5}}{3} \\ &= \underline{\underline{6}}. \end{aligned}$$

7. The point S, which lies on the positive x-axis, is a focus of the ellipse with equation

$$\frac{x^2}{4} + y^2 = 1.$$

Given that S is also the focus of a parabola P, with vertex at the origin, find

(a) a cartesian equation for P, (4)

Solution

$a = 2$, $b = 1$, and

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{\sqrt{3}}{2}.$$

and we have the focus at

$$y^2 = 4(ae)x \Rightarrow y^2 = 4 \times 2 \times \frac{\sqrt{3}}{2} \times x$$
$$\Rightarrow \underline{\underline{y^2 = 4\sqrt{3}x}}.$$

(b) an equation for the directrix of P. (1)

Solution
$\underline{\underline{x = -\sqrt{3}}}$.

8. The ellipse D has equation

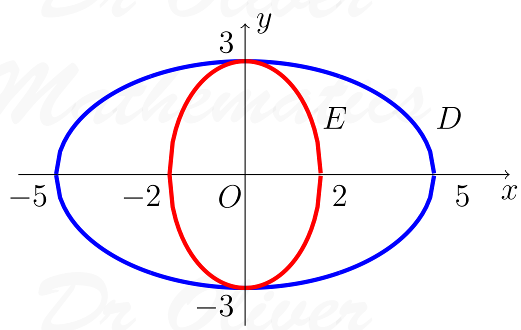
$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

and the ellipse E has equation

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

(a) Sketch D and E on the same diagram, showing the coordinates of the point where each curve crosses the axes. (3)

Solution



The point S is a focus for D and the point T is a focus for E.

(b) Find the length of ST. (5)

Solution
$\underline{S}$:

$$a = 5, b = 3, e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{4}{5};$$

the foci are at $(-4, 0)$ and $(4, 0)$.
$\underline{T}$:

$$\text{`}a = 3\text{'}, \text{`}b = 2\text{'}, e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{\sqrt{5}}{3};$$

the foci are at $(0, -\sqrt{5})$ and $(0, \sqrt{5})$.
Then

$$ST = \sqrt{4^2 + (\sqrt{5})^2} = \underline{\underline{\sqrt{21}}}.$$

9. The ellipse E has equation

$$\frac{x^2}{a^2} + \frac{y^2}{8} = 1,$$

where $a > 2\sqrt{2}$. The eccentricity of E is $\dfrac{1}{\sqrt{2}}$.

(a) Calculate the value of a. (2)

Solution

$$\begin{aligned} 8 = a^2(1 - e^2) &\Rightarrow 8 = \tfrac{1}{2}a^2 \\ &\Rightarrow a^2 = 16 \\ &\Rightarrow \underline{\underline{a = 4}}. \end{aligned}$$

The ellipse E cuts the y-axis at the points D and D'. The foci of E are S and S'.

(b) Calculate the area of the quadrilateral $SDS'D'$. (3)

Solution
D and D' are $(0, 2\sqrt{2})$ and $(0, -2\sqrt{2})$.
S and S' are $(2\sqrt{2}, 0)$ and $(-2\sqrt{2}, 0)$.
Now,

$$\begin{aligned} \text{area of the quadrilateral } SDS'D' &= 4 \times \text{area of } OSD \\ &= 4 \times \tfrac{1}{2} \times 2\sqrt{2} \times 2\sqrt{2} \\ &= \underline{\underline{16}}. \end{aligned}$$

10. The line $x = 8$ is a directrix of the ellipse E with equation (5)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > 0, b > 0,$$

and the point $(2, 0)$ is the corresponding focus.
Find the value of a and the value of b.

Solution

We have

$$ae = 2 \text{ and } \frac{a}{e} = 8.$$

Solve:

$$a^2 = 16 \Rightarrow \underline{\underline{a = 4}},$$

$$e^2 = \frac{1}{4},$$

and

$$b = \sqrt{a^2(1 - e^2)} = \underline{\underline{2\sqrt{3}}}.$$

11. The ellipse E has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The line l_1 is a tangent to E at the point $P(a\cos\theta, b\sin\theta)$.

(a) Using calculus, show that an equation for l_1 is (4)

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1.$$

Solution

$$\begin{aligned}\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 &\Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \\ &\Rightarrow \frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{a^2} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{b^2x}{a^2y},\end{aligned}$$

and, at the point $P(a\cos\theta, b\sin\theta)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{ab^2\cos\theta}{a^2b\sin\theta} = -\frac{b\cos\theta}{a\sin\theta}.$$

Now,

$$\begin{aligned} y - b\sin\theta &= -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta) \\ \Rightarrow \quad a\sin\theta(y - b\sin\theta) &= -b\cos\theta(x - a\cos\theta) \\ \Rightarrow \quad ay\sin\theta - ab\sin^2\theta &= -bx\cos\theta + ab\cos^2\theta \\ \Rightarrow \quad bx\cos\theta + ay\sin\theta &= ab\sin^2\theta + ab\cos^2\theta \\ \Rightarrow \quad bx\cos\theta + ay\sin\theta &= ab \\ \Rightarrow \quad \underline{\underline{\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1}}. \end{aligned}$$

The circle C has equation

$$x^2 + y^2 = a^2.$$

The line l_2 is a tangent to C at the point $Q(a\cos\theta, a\sin\theta)$.

(b) Find an equation for the line l_2. (2)

Solution

$$\begin{aligned} x^2 + y^2 = a^2 \Rightarrow 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{y}, \end{aligned}$$

and, at the point $P(a\cos\theta, b\sin\theta)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\cos\theta}{\sin\theta}.$$

Now,

$$\begin{aligned} y - a\sin\theta &= -\frac{\cos\theta}{\sin\theta}(x - a\cos\theta) \\ \Rightarrow \quad \sin\theta(y - a\sin\theta) &= -\cos\theta(x - a\cos\theta) \\ \Rightarrow \quad y\sin\theta - a\sin^2\theta &= -x\cos\theta + a\cos^2\theta \\ \Rightarrow \quad \underline{\underline{x\cos\theta + y\sin\theta = a}}. \end{aligned}$$

Given that l_1 and l_2 meet at the point R,

(c) find, in terms of a, b, and θ, the coordinates of R. (3)

Solution

From part (a),

$$bx\cos\theta + ay\sin\theta = ab,$$

and, from part (b),

$$x\cos\theta + y\sin\theta = a \Rightarrow bx\cos\theta + by\sin\theta = ab.$$

Subtract:

$$\begin{aligned} y(a\sin\theta - b\sin\theta) = 0 &\Rightarrow y = 0 \\ &\Rightarrow x\cos\theta = a \\ &\Rightarrow x = \frac{a}{\cos\theta}; \end{aligned}$$

hence, $\underline{\underline{R(a\sec\theta, 0)}}$.

(d) Find the locus of R, as θ varies. (2)

Solution

$\underline{\underline{|x| \geqslant 1, y = 0}}$.

12. The ellipse E has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a > b > 0$. The line l is a normal to E at the point $P(a\cos\theta, b\sin\theta)$, $0 < \theta < \frac{\pi}{2}$.

(a) Using calculus, show that an equation for l is (5)

$$ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta.$$

Solution

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 &\Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \\ &\Rightarrow \frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{a^2} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{b^2x}{a^2y}, \end{aligned}$$

and, at the point $P(a\cos\theta, b\sin\theta)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{ab^2\cos\theta}{a^2b\sin\theta} = -\frac{b\cos\theta}{a\sin\theta};$$

hence, the gradient of the normal is

$$m = \frac{a\sin\theta}{b\cos\theta}.$$

Now,

$$\begin{aligned} & y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta) \\ \Rightarrow\ & b\cos\theta(y - b\sin\theta) = a\sin\theta(x - a\cos\theta) \\ \Rightarrow\ & by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta \\ \Rightarrow\ & \underline{\underline{ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta}}. \end{aligned}$$

The line l meets the x-axis at A and the y-axis at B.

(b) Show that the are of the triangle OAB, where O is the origin, may be written as $k\sin 2\theta$, giving the value of the constant k in terms of a and b. (4)

Solution

$$x = 0 \Rightarrow -by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta \Rightarrow y = -\frac{(a^2 - b^2)\sin\theta}{b}$$

and

$$y = 0 \Rightarrow ax\sin\theta = (a^2 - b^2)\sin\theta\cos\theta \Rightarrow x = \frac{(a^2 - b^2)\cos\theta}{a}.$$

Now,

$$\begin{aligned} \text{area} &= \frac{1}{2} \times \frac{(a^2 - b^2)\cos\theta}{a} \times \frac{(a^2 - b^2)\sin\theta}{b} \\ &= \frac{(a^2 - b^2)^2\sin\theta\cos\theta}{2ab} \\ &= \underline{\underline{\frac{(a^2 - b^2)^2\sin^2\theta}{4ab}}}. \end{aligned}$$

(c) Find, in terms of a and b, the exact coordinates of the point P, for which the area of the triangle OAB is a maximum. (3)

Solution

For the biggest area, we need

$$\sin 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

and the coordinates of the point P are $\underline{\underline{(\frac{1}{\sqrt{2}}a, \frac{1}{\sqrt{2}}b)}}$.

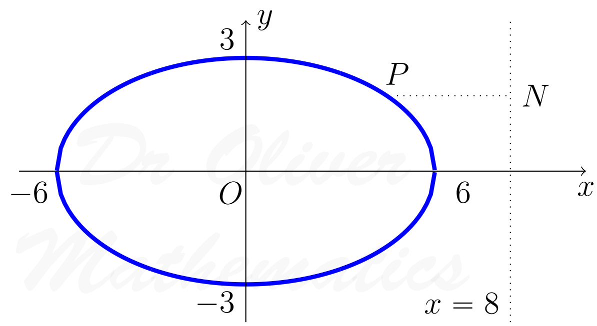
13. The point P lies on the ellipse E with equation

$$\frac{x^2}{36} + \frac{y^2}{9} = 1.$$

N is the foot of the perpendicular from the point P to the line $x = 8$. M is the midpoint of PN.

(a) Sketch the graph of the ellipse E, showing also the line $x = 8$ and a possible position for the line PN. (1)

Solution



(b) Find an equation of the locus of M as the point P moves around the ellipse. (4)

Solution

$M\left(\frac{x+8}{2}, y\right)$ — call this (X, Y). Now,

$$X = \frac{x+8}{2} \Rightarrow 2X = x + 8 \Rightarrow x = 2X - 8$$

and we have

$$\underline{\underline{\frac{(2X-8)^2}{36} + \frac{Y^2}{9} = 1.}}$$

(c) Show that this locus is a circle and state its centre and radius. (3)

Solution

$$\frac{(2X-8)^2}{36} + \frac{Y^2}{9} = 1 \Rightarrow \frac{(X-4)^2}{9} + \frac{Y^2}{9} = 1$$
$$\Rightarrow (X-4)^2 + Y^2 = 9;$$

it is a circle, with centre $(4, 0)$ and radius 3.

14. The points $P(3\cos\alpha, 2\sin\alpha)$ and $Q(3\cos\beta, 2\sin\beta)$, where $\alpha \neq \beta$, lie on the ellipse with equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

(a) Show the equation of the chord PQ is (4)

$$\frac{x}{3}\cos\frac{(\alpha+\beta)}{2} + \frac{y}{2}\sin\frac{(\alpha+\beta)}{2} = \cos\frac{(\alpha-\beta)}{2}.$$

Solution

$$\begin{aligned}\text{gradient} &= \frac{2\sin\alpha - 2\sin\beta}{3\cos\alpha - 3\cos\beta} \\ &= \frac{2\cos\frac{(\alpha+\beta)}{2}\sin\frac{(\alpha-\beta)}{2}}{-3\sin\frac{(\alpha+\beta)}{2}\sin\frac{(\alpha-\beta)}{2}} \\ &= -\frac{2\cos\frac{(\alpha+\beta)}{2}}{3\sin\frac{(\alpha+\beta)}{2}}.\end{aligned}$$

Now,

$$\begin{aligned}
& y - 2\sin\alpha = -\frac{2\cos\frac{(\alpha+\beta)}{2}}{3\sin\frac{(\alpha+\beta)}{2}}(x - 3\cos\alpha) \\
\Rightarrow\quad & 3\sin\frac{(\alpha+\beta)}{2}(y - 2\sin\alpha) = -2\cos\frac{(\alpha+\beta)}{2}(x - 3\cos\alpha) \\
\Rightarrow\quad & 3y\sin\frac{(\alpha+\beta)}{2} - 6\sin\alpha\sin\frac{(\alpha+\beta)}{2} \\
& \qquad = -2x\cos\frac{(\alpha+\beta)}{2} + 6\cos\alpha\cos\frac{(\alpha+\beta)}{2} \\
\Rightarrow\quad & 2x\cos\frac{(\alpha+\beta)}{2} + 3y\sin\frac{(\alpha+\beta)}{2} \\
& \qquad = 6\left(\sin\alpha\sin\frac{(\alpha+\beta)}{2} + \cos\alpha\cos\frac{(\alpha+\beta)}{2}\right) \\
\Rightarrow\quad & 2x\cos\frac{(\alpha+\beta)}{2} + 3y\sin\frac{(\alpha+\beta)}{2} = 6\cos\frac{(\alpha-\beta)}{2} \\
\Rightarrow\quad & \underline{\underline{\frac{x}{3}\cos\frac{(\alpha+\beta)}{2} + \frac{y}{2}\sin\frac{(\alpha+\beta)}{2} = \cos\frac{(\alpha-\beta)}{2}}}.
\end{aligned}$$

(b) Write down the coordinates of the mid-point of PQ. (1)

Solution

$$\begin{aligned}
\text{The midpoint of } x &= \frac{3\cos\alpha + 3\cos\beta}{2} \\
&= \frac{3(\cos\alpha + \cos\beta)}{2} \\
&= 3\cos\frac{(\alpha+\beta)}{2}\cos\frac{(\alpha-\beta)}{2}
\end{aligned}$$

and

$$\begin{aligned}
\text{the midpoint of } y &= \frac{2\sin\alpha + 2\sin\beta}{2} \\
&= \sin\alpha + \sin\beta \\
&= 2\sin\frac{(\alpha+\beta)}{2}\cos\frac{(\alpha-\beta)}{2}.
\end{aligned}$$

Hence, the midpoint is

$$\underline{\underline{\left(3\cos\frac{(\alpha+\beta)}{2}\cos\frac{(\alpha-\beta)}{2}, 2\sin\frac{(\alpha+\beta)}{2}\cos\frac{(\alpha-\beta)}{2}\right)}}.$$

Given the that gradient, m, of the chord PQ is a constant,

(c) show that the centre of the chord lies on a line $y = -km$, expressing k in terms of m. (5)

Solution

$$m = -\frac{2\cos\frac{(\alpha+\beta)}{2}}{3\sin\frac{(\alpha+\beta)}{2}} \Rightarrow \frac{2}{3m} = -\frac{\sin\frac{(\alpha+\beta)}{2}}{\cos\frac{(\alpha+\beta)}{2}}$$
$$\Rightarrow -\frac{2}{3m} = \frac{\sin\frac{(\alpha+\beta)}{2}}{\cos\frac{(\alpha+\beta)}{2}}$$
$$\Rightarrow y = \frac{2}{3}\left(-\frac{2}{3m}\right)x$$
$$\Rightarrow y = -\frac{4}{9m}x;$$

hence,

$$\underline{\underline{k = \frac{4}{9m}}}.$$

15. The ellipse E has equation

$$x^2 + 9y^2 = 9.$$

The point $P(3\cos\theta, 2\sin\theta)$ is a general point on the ellipse E.

(a) Write down the value of a and the value of b. (1)

Solution

$$x^2 + 9y^2 = 9 \Rightarrow \frac{x^2}{9} + y^2 = 1;$$

hence $\underline{\underline{a = 3}}$ and $\underline{\underline{b = 1}}$.

The line L is a tangent to E at the point P.

(b) Show that an equation of the line L is given by (3)

$$3y\sin\theta + x\cos\theta = 3.$$

Solution

$$\frac{x^2}{9} + y^2 = 1 \Rightarrow \frac{2x}{9} + 2y\,\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\Rightarrow 2y\,\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{9}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{9y},$$

and, at the point $P(3\cos\theta, \sin\theta)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3\cos\theta}{9\sin\theta} = -\frac{\cos\theta}{3\sin\theta}.$$

Now,

$$\begin{aligned} & y - \sin\theta = -\frac{\cos\theta}{3\sin\theta}(x - 3\cos\theta) \\ \Rightarrow\quad & 3\sin\theta(y - \sin\theta) = -\cos\theta(x - 3\cos\theta) \\ \Rightarrow\quad & 3y\sin\theta - 3\sin^2\theta = -x\cos\theta + 3\cos^2\theta \\ \Rightarrow\quad & 3y\sin\theta + x\cos\theta = 3(\sin^2\theta + \cos^2\theta) \\ \Rightarrow\quad & \underline{\underline{3y\sin\theta + x\cos\theta = 3}}. \end{aligned}$$

The line L meets the x-axis at the point Q and meets the y-axis at the point R.

(c) Show that the area of the triangle OQR, where O is the origin, is given by $k \operatorname{cosec} 2\theta$, where k is a constant to be found. (3)

Solution

$$x = 0 \Rightarrow 3y\sin\theta = 3 \Rightarrow y = \operatorname{cosec}\theta$$

and

$$y = 0 \Rightarrow x\cos\theta = 3 \Rightarrow x = 3\sec\theta.$$

Now,

$$\begin{aligned} \text{area} &= \frac{1}{2} \times \operatorname{cosec}\theta \times 3\sec\theta \\ &= \frac{3}{2\sin\theta\cos\theta} \\ &= \frac{3}{\sin^2\theta} \\ &= \underline{\underline{3\operatorname{cosec}^2\theta}}. \end{aligned}$$

The point M is the midpoint of QR.

(d) Find a cartesian equation of the locus of M, giving your answer in the form $y^2 = \mathrm{f}(x)$. (4)

Solution

$$x = \frac{3}{2\cos\theta} \Rightarrow \cos\theta = \frac{3}{2x}$$

and

$$y = \frac{1}{2\sin\theta} \Rightarrow \sin\theta = \frac{1}{2y}.$$

Now,

$$\begin{aligned}\sin^2\theta + \cos^2\theta = 1 &\Rightarrow \left(\frac{1}{2y}\right)^2 + \left(\frac{3}{2x}\right)^2 = 1\\ &\Rightarrow \frac{1}{4y^2} = 1 - \frac{9}{4x^2}\\ &\Rightarrow \frac{1}{4y^2} = \frac{4x^2-9}{4x^2}\\ &\Rightarrow 4y^2 = \frac{4x^2}{4x^2-9}\\ &\Rightarrow \underline{\underline{y^2 = \frac{x^2}{4x^2-9}}}.\end{aligned}$$

16. The ellipse E has equation $x^2 + 4y^2 = 4$.

(a) (i) Find the coordinates of the foci, F_1 and F_2, of E. (4)

Solution
The standard form of the ellipse is

$$\frac{x^2}{4} + y^2 = 1$$

and hence $a = 2$, $b = 1$, and $b^2 = a^2(1 - e^2)$ gives $e = \frac{\sqrt{3}}{2}$. So the foci are $\underline{\underline{(\sqrt{3}, 0)}}$ and $\underline{\underline{(-\sqrt{3}, 0)}}$.

(ii) Write down the equations of the directrices of E.

Solution

The equations are $x = \pm\frac{a}{e}$ and so $\underline{\underline{x = \pm\frac{4\sqrt{3}}{3}}}$.

(b) Given that the point P lies of the ellipse, show that (4)

$$|PF_1| + |PF_2| = 4.$$

Solution

For any ellipse, $|PF_1|+|PF_2| = 2a$ and, since $a = 2$, we have $\underline{\underline{|PF_1| + |PF_2| = 4}}$, as required. If, however, you do not feel that this gives enough detail for the four marks that are available the you can proceed as follows. Let N_1 and N_2 be the points on the corresponding directrix so that PN_1 and PN_2 are perpendicular to the directrices. Then

$$\begin{aligned}|PF_1| + |PF_2| &= e\,|PN_1| + e\,|PN_2| \\ &= e\,(|PN_1| + |PN_2|) \\ &= e\,|N_1N_2| \text{ (since } N_1PN_2 \text{ is a straight line)} \\ &= \tfrac{\sqrt{3}}{2} \times \tfrac{8\sqrt{3}}{3} \text{ (the distance between the directrices is } \tfrac{8\sqrt{3}}{3}) \\ &= \underline{\underline{4}},\end{aligned}$$

as required.

A chord of an ellipse is a line segment joining two points on the ellipse. The set of midpoints of the parallel chords of E with gradient m, where m is a constant, lie on the straight line l.

(c) Find an equation of l. (6)

Solution

Let A and B be the points where the chord with equation $y = mx + c$ cuts the ellipse E. Then

$$\begin{aligned}x^2 + 4\,(mx + c)^2 = 4 &\Rightarrow x^2 + 4\left(m^2x^2 + 2cmx + c^2\right) = 4 \\ &\Rightarrow (4m^2 + 1)x^2 + 8cmx + (4c^2 - 4) = 0.\end{aligned}$$

If x_A and x_B are the x-coordinates of A and B respectively then

$$x_A + x_B = -\frac{8cm}{4m^2 + 1}$$

and hence x_M, the x-coordinate of the midpoint of the chord AB, is

$$x_M = -\frac{4cm}{4m^2+1}.$$

So y_M, the y-coordinate of the midpoint of the chord AB, is

$$y_M = -\frac{4cm^2}{4m^2+1} + c = \frac{-4cm^2 + c(4m^2+1)}{4m^2+1} = \frac{c}{4m^2+1}.$$

So an equation of the line l is

$$\underline{\underline{y = -\tfrac{1}{4m}x}}.$$

17. The ellipse E has equation

$$\frac{x^2}{36} + \frac{y^2}{25} = 1.$$

The line l is the normal to E at the point $P(6\cos\theta, 5\sin\theta)$, where $0 < \theta < \frac{\pi}{2}$.

(a) Use calculus to show that an equation of l is (5)

$$6x\sin\theta - 5y\cos\theta = 11\sin\theta\cos\theta.$$

Solution

$$\begin{aligned}\frac{x^2}{36} + \frac{y^2}{25} = 1 &\Rightarrow \frac{2x}{36} + \frac{2y}{25}\frac{\mathrm{d}y}{\mathrm{d}x} = 0\\ &\Rightarrow \frac{2y}{25}\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{36}\\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{25x}{36y},\end{aligned}$$

and, at the point $P(6\cos\theta, 5\sin\theta)$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{150\cos\theta}{180\sin\theta} = -\frac{5\cos\theta}{6\sin\theta};$$

hence, the gradient of the normal is

$$m = \frac{6\sin\theta}{5\cos\theta}.$$

Now,

$$\begin{aligned} & y - 5\sin\theta = \frac{6\sin\theta}{5\cos\theta}(x - 6\cos\theta) \\ \Rightarrow\ & 5\cos\theta(y - 5\sin\theta) = 6\sin\theta(x - 6\cos\theta) \\ \Rightarrow\ & 5y\cos\theta - 25\sin\theta\cos\theta = 6x\sin\theta - 36\sin\theta\cos\theta \\ \Rightarrow\ & \underline{\underline{6x\sin\theta - 5y\cos\theta = 11\sin\theta\cos\theta}}. \end{aligned}$$

The line l meets the x-axis at the point Q. The point R is the foot of the perpendicular from P to the x-axis.

(b) Show that $\dfrac{OQ}{OR} = e^2$, where e is the eccentricity of the ellipse E. (4)

Solution

$\underline{Q}$:

$$y = 0 \Rightarrow 6x\sin\theta = 11\sin\theta\cos\theta \Rightarrow x = \tfrac{11}{6}\cos\theta.$$

$\underline{R}$:

$$x = 6\cos\theta.$$

Now,

$$\begin{aligned} \frac{OQ}{OR} &= \frac{\frac{11}{6}\cos\theta}{6\cos\theta} \\ &= \frac{11}{36} \\ &= 1 - \frac{25}{36} \\ &= \underline{\underline{e^2}}, \end{aligned}$$

as required.