Dr Oliver Mathematics
AQA Further Maths Level 2
June 2022 Paper 1
1 hour 45 minutes

The total number of marks available is 80.
You must write down all the stages in your working.
You are **not** permitted to use a scientific or graphical calculator in this paper.

1. $(x + 1)$ is increased by 20%. (3)

   Its value is now the same as $(x + 6)$.

   Work out the value of x.

   Solution

   Well,

   $$\begin{aligned} 1.2(x+1) = x+6 &\Rightarrow 1.2x + 1.2 = x + 6 \\ &\Rightarrow 0.2x = 4.8 \\ &\Rightarrow x = 5 \times 4.8 \\ &\Rightarrow \underline{\underline{x = 24}}. \end{aligned}$$

2. The point $(-6, -4)$ lies on a straight line with gradient $\frac{3}{2}$. (2)

   Work out the coordinates of the point where the line crosses the y-axis.

   Solution

   The equation of the straight line is

   $$y + 4 = \tfrac{3}{2}(x + 6) \Rightarrow y + 4 = \tfrac{3}{2}x + 9.$$

   Now,

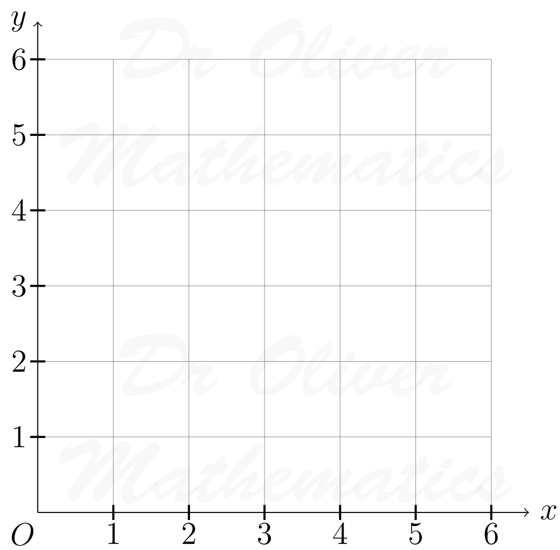
   $$\begin{aligned} x = 0 &\Rightarrow y + 4 = 9 \\ &\Rightarrow y = 5; \end{aligned}$$

   so, the coordinates of the point are $\underline{\underline{(0, 5)}}$.

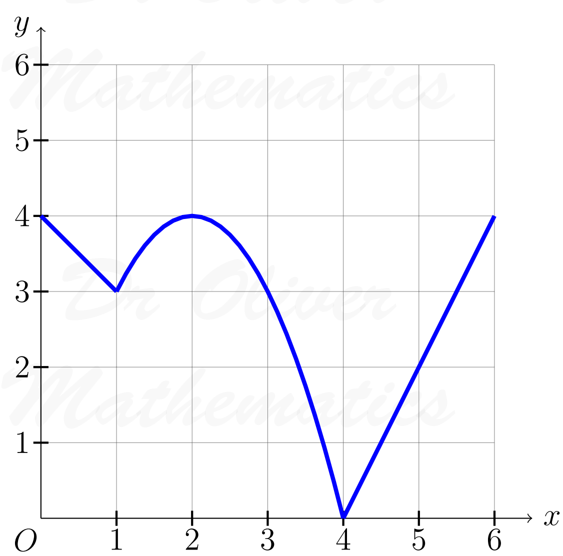
3. (a) (4)

$$f(x) = \begin{cases} 4 - x & \text{for } 0 \leqslant x < 1, \\ 4x - x^2 & \text{for } 1 \leqslant x < 4, \\ 2x - 8 & \text{for } 4 \leqslant x \leqslant 6. \end{cases}$$

On the grid, draw the graph of $y = f(x)$.



Solution



(b) (2)

$$g(x) = 6 - 3x.$$

Work out $g^{-1}(x)$.

Solution

$$\begin{aligned} y = 6 - 3x \Rightarrow 3x &= 6 - y \\ \Rightarrow x &= 2 - \tfrac{1}{3}y; \end{aligned}$$

so,

$$\underline{\underline{y = 2 - \tfrac{1}{3}x}}.$$

4. (a) Circle the value of (1)

$$\tan^2 30°.$$

$\frac{1}{4}$ $\frac{1}{3}$ $\frac{1}{2}$ $\frac{3}{4}$

Solution

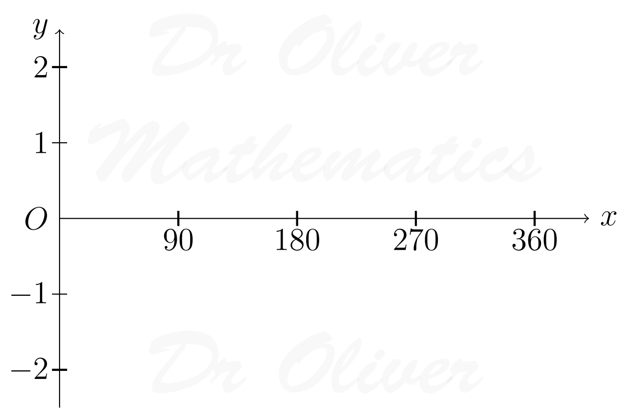
$$\begin{aligned} \tan^2 30° &= (\tan 30°)^2 \\ &= (\tfrac{1}{\sqrt{3}})^2 \\ &= \tfrac{1}{3} \end{aligned}$$

so

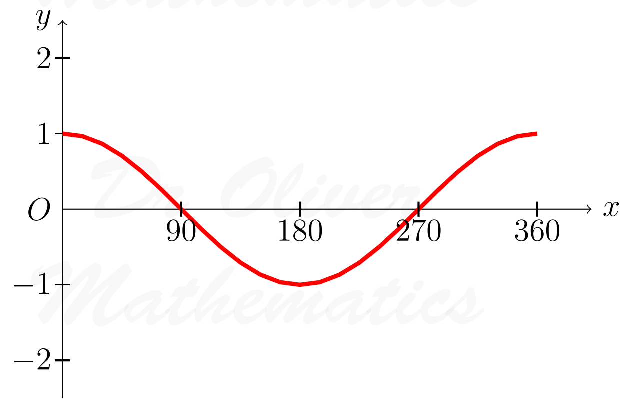
$\frac{1}{4}$ $\underline{\underline{\frac{1}{3}}}$ $\frac{1}{2}$ $\frac{3}{4}$

(b) On the axes, sketch (2)

$$y = \cos x \text{ for } 0° \leqslant x \leqslant 360°.$$



Solution



5. $$(3x + a)(5x - 4) \equiv 15x^2 - 2x + b.$$ (3)

Work out the values of a and b.

Solution

$\times$	$3x$	$+a$
$5x$	$15x^2$	$+5ax$
-4	$-12x$	$-4a$

so

$$(3x + a)(5x - 4) \equiv 15x^2 + (5a - 12)b - 4b.$$

$\underline{x}$:

$$5a - 12 = -2 \Rightarrow 5a = 10$$
$$\Rightarrow \underline{\underline{a = 2}}.$$

Constant term :

$$b = -4(2) = \underline{\underline{-8}}.$$

6. $$y = 2x^4 \left(x^3 + 2 - \frac{3}{x} \right).$$ (3)

Work out $\frac{dy}{dx}$.

Solution

Well,

$$\begin{aligned} y = 2x^4\left(x^3 + 2 - \frac{3}{x}\right) &\Rightarrow y = 2x^4\left(x^3 + 2 - 3x^{-1}\right) \\ &\Rightarrow y = 2x^7 + 4x^4 - 6x^3 \\ &\Rightarrow \underline{\underline{\frac{dy}{dx} = 14x^6 + 16x^3 - 18x^2}}. \end{aligned}$$

7. ABC is a right-angled triangle with vertices $A(-1, 5)$, $B(-2, 5)$, and $C(-1, 5\frac{3}{4})$. (3)

Work out the length of BC.

Solution

$$\begin{aligned} BC &= \sqrt{(5\tfrac{3}{4} - 5)^2 + [-1 - (-2)]^2} \\ &= \sqrt{(\tfrac{3}{4})^2 + 1^2} \\ &= \sqrt{\tfrac{9}{16} + 1} \\ &= \sqrt{\tfrac{25}{16}} \\ &= \underline{\underline{1\tfrac{1}{4}}}. \end{aligned}$$

8. Use **matrix multiplication** to show that, in the x-y plane, (3)

- a rotation, 90° anticlockwise about the origin, followed by
- a reflection in the line $y = x$

is equivalent to a reflection in the x-axis.

Solution

Well,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

the result is equivalent to a reflection in the x-axis.

9. (a) A quadratic sequence starts (3)

$$-2 \quad -1 \quad 4 \quad 13.$$

Work out an expression for the nth term.

Solution

Let the

$$n\text{th term} = an^2 + bn + c.$$

We only need the second line of differences (why?):

$$\begin{array}{ccccc} -2 & & -1 & & 4 \\ & 1 & & 5 & \\ & & 4 & & \end{array}$$

$$\begin{array}{ccccc} a+b+c & & 4a+2b+c & & 9a+3b+c \\ & 3a+b & & 5a+b & \\ & & 2a & & \end{array}$$

We compare terms:

$$2a = 4 \Rightarrow a = 2,$$

$$\begin{aligned} 3a + b = 6 &\Rightarrow 3 \times 2 + b = 1 \\ &\Rightarrow b = -5, \end{aligned}$$

and

$$\begin{aligned} a + b + c = -2 &\Rightarrow 2 - 5 + c = -2 \\ &\Rightarrow c = 1; \end{aligned}$$

hence,

$$n\text{th term} = \underline{\underline{2n^2 - 5n + 1}}.$$

(b) A different quadratic sequence has nth term (3)

$$n^2 + 10n.$$

Use an algebraic method to work out how many terms in the sequence are less than 2 000.

Do **not** use trial and improvement.

You **must** show your working.

Solution

$$n^2 + 10n < 2\,000 \Rightarrow n^2 + 10n - 2\,000 < 0$$

$$\left.\begin{array}{ll}\text{add to:} & +10 \\ \text{multiply to:} & -2\,000\end{array}\right\} +50,\ -40$$

$$\Rightarrow (n+50)(n-40) < 0.$$

We need a 'table of signs':

	$n < -50$	$n = -50$	$-50 < n < 40$	$n = 40$	$n > 40$
$n + 50$	$-$	0	$+$	$+$	$+$
$n - 40$	$-$	$-$	$-$	0	$+$
$(n+50)(n-40)$	$+$	0	$-$	0	$+$

Cleary, $-50 < n < 40$. So, $\underline{\underline{n = 39}}$.

10. Rationalise and simplify fully (3)

$$\frac{3}{3+\sqrt{3}}.$$

Solution

Well,

$$\frac{3}{3+\sqrt{3}} = \left(\frac{3}{3+\sqrt{3}}\right) \times \left(\frac{3-\sqrt{3}}{3-\sqrt{3}}\right)$$

$\times$	3	$+\sqrt{3}$
3	9	$+3\sqrt{3}$
$-\sqrt{3}$	$-3\sqrt{3}$	-3

$$\begin{aligned} &= \frac{3(3-\sqrt{3})}{9-3} \\ &= \frac{9-3\sqrt{3}}{6} \\ &= \underline{\underline{\tfrac{3}{2} - \tfrac{1}{2}\sqrt{3}}}. \end{aligned}$$

11. Expand and simplify fully (4)

$$(3+2x)^5.$$

Solution

$$\begin{aligned} & (3+2x)^5 \\ = \; & (3)^5 + \binom{5}{1}(3)^4(2x) + \binom{5}{2}(3)^3(2x)^2 + \binom{5}{3}(3)^2(2x)^3 + \binom{5}{4}(3)(2x)^4 + (2x)^5 \\ = \; & 243 + (5)(81)(2x) + (10)(27)(4x^2) + (10)(9)(8x^3) + (5)(3)(16x^4) + 32x^5 \\ = \; & \underline{\underline{243 + 810x + 1\,080x^2 + 720x^3 + 240x^4 + 32x^5}}. \end{aligned}$$

12. The nth term of a sequence is

$$\frac{3n^2}{n^2+2}.$$

(a) One term in the sequence is (2)

$$\tfrac{32}{11}.$$

Work out the value of n.

Solution

Cross-multiply:

$$\frac{3n^2}{n^2+2} = \tfrac{32}{11} \Rightarrow (3n^2)(11) = (32)(n^2+2)$$
$$\Rightarrow 33n^2 = 32n^2 + 64$$
$$\Rightarrow n^2 = 64$$
$$\Rightarrow \underline{\underline{n = 8}},$$

as $n \geqslant 1$.

(b) Write down the limiting value of the sequence as $n \to \infty$. (1)

Solution

Well,

$$\frac{3n^2}{n^2+2} = \frac{3}{1+\frac{2}{n^2}}$$
$$\to \frac{3}{1+0}$$
$$= \underline{\underline{3}},$$

as $n \to \infty$

13. Simplify fully (3)

$$(6x^3y^{-2} + 9x^5y) \div 3x^2y^{-3}.$$

Solution

Well,

$$(6x^3y^{-2} + 9x^5y) \div 3x^2y^{-3} = \frac{6x^3y^{-2} + 9x^5y}{3x^2y^{-3}}$$
$$= \underline{\underline{2xy + 3x^3y^4}}.$$

14. Rearrange (3)

$$ef = \frac{5e+4}{3}$$

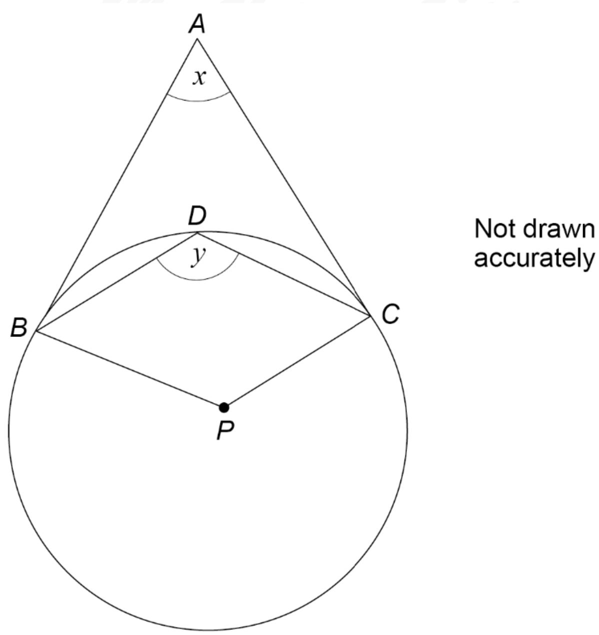
to make e the subject.

Solution

Now,

$$\begin{aligned} ef = \frac{5e+4}{3} &\Rightarrow 3ef = 5e + 4 \\ &\Rightarrow 3ef - 5e = 4 \\ &\Rightarrow e(3f-5) = 4 \\ &\Rightarrow e = \frac{4}{3f-5}. \end{aligned}$$

15. B, C, and D are points on a circle, centre P. (5)
AB and AC are tangents to the circle.



Prove that

$$y = 90 + \tfrac{1}{2}x.$$

Solution

$\angle ABP = \angle ACP = 90^\circ$ (right-angles)
$\angle BPC = 180 - x$ (complete the quadrilateral)

Let E be a point on the circle which is directly below P, so that $BPCE$ is a 'kite' shape.

Now, opposite angles in a cyclic quadrilateral add up to 180° so $\angle BEC = 180 - y$.

$\angle BEC = \frac{1}{2}\angle BPC$ (angle at the centre is twice the angle at the circumference) Finally,

$$\begin{aligned}\tfrac{1}{2}(180 - x) = 180 - y &\Rightarrow 90 - \tfrac{1}{2}x = 180 - y \\ &\Rightarrow \underline{\underline{y = 90 + \tfrac{1}{2}x}},\end{aligned}$$

as required.

16. Solve the simultaneous equations (6)

$$\begin{aligned} x - y &= \tfrac{19}{4} \\ xy &= -3. \end{aligned}$$

Do **not** use trial and improvement.
You **must** show your working.

Solution

Now,

$$xy = -3 \Rightarrow y = -\frac{3}{x}$$

and

$$x - y = \tfrac{19}{4} \Rightarrow x + \frac{3}{x} = \tfrac{19}{4}$$

multiply by $4x$:

$$\begin{aligned}&\Rightarrow 4x^2 + 12 = 19x \\ &\Rightarrow 4x^2 - 19x + 12 = 0\end{aligned}$$

$$\begin{array}{ll} \text{add to:} & -19 \\ \text{multiply to:} & (+4) \times (+12) = +48 \end{array} \Bigg\} -16,\ -3$$

e.g.,

$$\begin{aligned} &\Rightarrow 4x^2 - 16x - 3x + 12 = 0 \\ &\Rightarrow 4x(x-4) + 3(x-4) = 0 \\ &\Rightarrow (4x+3)(x-4) = 0 \\ &\Rightarrow x = -\tfrac{3}{4} \text{ or } x = 4 \\ &\Rightarrow y = 4 \text{ or } y = -\tfrac{3}{4}; \end{aligned}$$

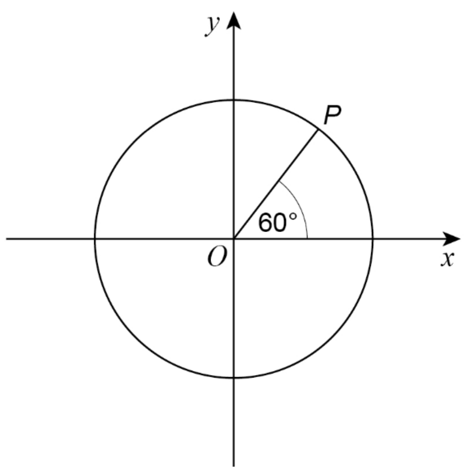
hence,

$$\underline{\underline{x = -\tfrac{3}{4}, y = 4 \text{ or } x = 4, y = -\tfrac{3}{4}}}.$$

17. The point P lies on the circle

$$x^2 + y^2 = 16.$$

The line OP is at an angle of 60° to the positive x-axis.



Not drawn accurately

(a) Show that the coordinates of point P are $(2, 2\sqrt{3})$. (2)

Solution
Well,

$$OP = \sqrt{16} = 4$$

so the coordinates are

$$(4\cos 60^\circ, 4\sin 60^\circ) = \underline{\underline{(2, 2\sqrt{3})}}.$$

(b) Work out the equation of the tangent to the circle at P. (4)

Write your answer in the form

$$x + ay = b,$$

where a and b are constants.

Solution

Well,

$$m = \frac{2\sqrt{3} - 0}{2 - 0} = \sqrt{3}$$

so the gradient of the tangent is

$$-\frac{1}{\sqrt{3}}.$$

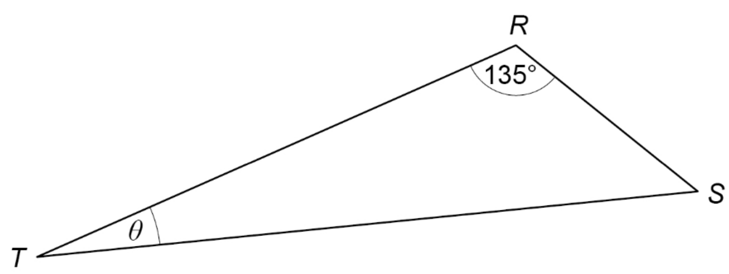
Finally, the equation of the tangent is

$$y - 2\sqrt{3} = -\frac{1}{\sqrt{3}}(x - 2) \Rightarrow \sqrt{3}(y - 2\sqrt{3}) = -x + 2$$
$$\Rightarrow y\sqrt{3} - 6 = -x + 2$$
$$\Rightarrow \underline{\underline{x + y\sqrt{3} = 8}};$$

so, $\underline{\underline{a = \sqrt{3}}}$ and $\underline{\underline{b = 8}}$.

18. In triangle RST, (3)

$$RS : ST = 1 : 4.$$



Work out the exact value of $\sin\theta$.

Solution

Sine rule:

$$\frac{\sin\theta}{RS} = \frac{\sin 135^\circ}{ST} \Rightarrow \frac{\sin\theta}{1} = \frac{\sin 135^\circ}{4}$$
$$\Rightarrow \sin\theta = \frac{\frac{1}{\sqrt{2}}}{4}$$
$$\Rightarrow \underline{\underline{\sin\theta = \tfrac{1}{4\sqrt{2}}}}$$
$$\Rightarrow \sin\theta = \tfrac{1}{4\sqrt{2}} \times \tfrac{\sqrt{2}}{\sqrt{2}}$$
$$\Rightarrow \underline{\underline{\sin\theta = \tfrac{\sqrt{2}}{8}}}.$$

19. Write (3)

$$6x^2 - 24x + 17$$

in the form

$$a(x+b)^2 + c,$$

where a, b, and c are integers.

Solution

$$\begin{aligned} 6x^2 - 24x + 17 &= 6[x^2 - 4x] + 17 \\ &= 6[(x^2 - 4x + 4) - 4] + 17 \\ &= 6[(x-2)^2 - 4] + 17 \\ &= 6(x-2)^2 - 24 + 17 \\ &= \underline{\underline{6(x-2)^2 - 7}}; \end{aligned}$$

hence, $\underline{\underline{a = 6}}$, $\underline{\underline{b = -2}}$, and $\underline{\underline{c = -7}}$.

20. The curve (6)

$$y = x^4 - 18x^2$$

has three stationary points.

Work out the coordinates of the three stationary points and determine their nature. You **must** show your working.

Solution

$$\begin{aligned} y = x^4 - 18x^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 4x^3 - 36x \\ \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 12x^2 - 36. \end{aligned}$$

Now,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 4x^3 - 36x = 0 \\ &\Rightarrow 4x(x^2 - 9) = 0 \\ &\Rightarrow x = -3,\ x = 0, \text{ or } x = 3 \\ &\Rightarrow y = -81,\ y = 0, \text{ or } y = -81. \end{aligned}$$

Next,

$$\begin{aligned} x = -3 &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 72 > 0 \\ x = 0 &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -36 < 0 \\ x = 3 &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 72 > 0. \end{aligned}$$

Hence, the three stationary points are

$(-3, 94)$ which is minimum point,

$(0, 0)$ which is maximum point, and

$(3, 94)$ which is minimum point.

21. Show that (3)

$$\frac{4\cos^2 x + 3\sin^2 x - 4}{\cos^2 x} \equiv -\tan^2 x.$$

Solution

Well,

$$\begin{aligned}\frac{4\cos^2 x + 3\sin^2 x - 4}{\cos^2 x} &\equiv \frac{3(\cos^2 x + \sin^2 x) + \cos^2 x - 4}{\cos^2 x}\\ &\equiv \frac{3 + \cos^2 x - 4}{\cos^2 x}\\ &\equiv \frac{\cos^2 x - 1}{\cos^2 x}\\ &\equiv \frac{-(1-\cos^2 x)}{\cos^2 x}\\ &\equiv \frac{-\sin^2 x}{\cos^2 x}\\ &\equiv \underline{\underline{-\tan^2 x}},\end{aligned}$$

as required.