Dr Oliver Mathematics
OCR FMSQ Additional Mathematics
2015 Paper
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.
Final answers should be given correct to three significant figures where appropriate.

Section A

1. Find the equation of the line which is perpendicular to the line (3)

$$2x + 3y = 5$$

and which passes through the point $(3, 4)$.

Solution

$$\begin{aligned} 2x + 3y = 5 &\Rightarrow 3y = -2x - 5 \\ &\Rightarrow y = -\tfrac{2}{3}x - \tfrac{5}{3} \end{aligned}$$

and so the gradient of the perpendicular is

$$-\frac{1}{-\frac{2}{3}} = \tfrac{3}{2}.$$

Hence, the equation of the line is

$$\begin{aligned} y - 4 = \tfrac{3}{2}(x - 3) &\Rightarrow y - 4 = \tfrac{3}{2}x - \tfrac{9}{2} \\ &\Rightarrow \underline{\underline{y = \tfrac{3}{2}x - \tfrac{1}{2}}}. \end{aligned}$$

2. (a) Find α in the range $0^\circ \leqslant \alpha \leqslant 180^\circ$ such that (2)

$$\tan \alpha = -1.5.$$

Solution

$$\tan\alpha = -1.5 \Rightarrow \alpha = -56.309\ldots \text{ (out of range)}, 123.690\,067\,5 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\alpha = 124 \text{ (3 sf)}}}.$$

(b) Find β in the range $0^\circ \leqslant \beta \leqslant 180^\circ$ such that (2)

$$\sin\beta = 0.2.$$

Solution

$$\sin\beta = 0.2 \Rightarrow \beta = 11.536\,959\,03,\ 168.463\,041 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\beta = 11.5,\ 168 \text{ (3 sf)}}}.$$

3. Find the equation of the tangent to the curve (5)

$$y = x^3 + 3x - 5$$

at the point $(2, 9)$.

Solution

$$y = x^3 + 3x - 5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 3$$

and

$$x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 15.$$

Hence, the equation of the tangent is

$$y - 9 = 15(x - 2) \Rightarrow y - 9 = 15x - 30$$
$$\Rightarrow \underline{\underline{y = 15x - 21}}.$$

4. (a) Find (4)

$$\int_1^2 (x^2 + 2x + 3)\,\mathrm{d}x.$$

Solution

$$\int_1^2 (x^2 + 2x + 3)\,\mathrm{d}x = \left[\tfrac{1}{3}x^3 + x^2 + 3x\right]_{x=1}^{2}$$
$$= \left(\tfrac{8}{3} + 4 + 6\right) - \left(\tfrac{1}{3} + 1 + 3\right)$$
$$= \underline{\underline{8\tfrac{1}{3}}}.$$

(b) Interpret your answer geometrically. (1)

Solution

There is $\underline{\underline{8\frac{1}{3}\text{ units}^2}}$ of area between the curve, the x-axis, the $x = 1$, and the $x = 2$.

5. A train accelerates from rest from a point O such that at t seconds the displacement, s metres from O, is given by the formula

$$s = \tfrac{3}{2}t^2 - 2t + 3.$$

(a) Show by calculus that the acceleration is constant. (3)

Solution

$$s = \tfrac{3}{2}t^2 - 2t + 3 \Rightarrow v = 3t - 2$$
$$\Rightarrow \underline{\underline{a = 3\text{ ms}^{-2}}},$$

as required.

(b) Find the velocity after 5 seconds. (2)

Solution

$$t = 5 \Rightarrow v = \underline{\underline{13\text{ ms}^{-1}}}.$$

6. You are given that n is a positive integer and $(n - 1)$, n, $(n + 1)$ are three consecutive integers.

In each of the following cases form an equation in n and solve it.

(a) The three integers add up to 99. (2)

Solution

$$\begin{aligned}(n-1)+n+(n+1)99 &\Rightarrow 3n = 99 \\ &\Rightarrow \underline{\underline{n = 33}}.\end{aligned}$$

(b) When the product of the first integer and third integer is added to 5 times the second integer the sum is 203. (4)

Solution

×	n	-1
n	n^2	$-n$
$+1$	$+n$	-1

$$\begin{aligned}(n-1)(n+1)+5n = 203 &\Rightarrow (n^2-1)+5n = 203 \\ &\Rightarrow n^2+5n-204 = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & +5 \\ \text{multiply to:} & -204\end{array}\right\} -12,\ +17$$

$$\begin{aligned}&\Rightarrow (n-12)(n+17) = 0 \\ &\Rightarrow n = 12 \text{ or } n = -17;\end{aligned}$$

clearly, $n > 0$ and so we have $\underline{\underline{n = 12}}$.

7. (a) Solve algebraically the simultaneous equations (4)

$$y = 3+5x-x^2 \text{ and } andy = x+7.$$

Solution

$$3+5x-x^2 = x+7 \Rightarrow x^2-4x+4 = 0$$

$$\left.\begin{array}{ll}\text{add to:} & -4 \\ \text{multiply to:} & +4\end{array}\right\} -2, -2$$

$$\begin{aligned} &\Rightarrow (x-2)^2 = 0 \\ &\Rightarrow x = 2 \text{ (repeated)} \\ &\Rightarrow y = 9; \end{aligned}$$

hence,

$$\underline{\underline{x = 2,\ y = 9}}.$$

(b) Interpret your answer geometrically. (1)

Solution
The line is $\underline{\underline{\text{tangent to the curve}}}$ at $\underline{\underline{(2, 9)}}$.

8. The cubic polynomial

$$f(x) = x^3 + ax + 6,$$

where a is a constant, has a factor of $(x + 3)$.

(a) Find the value of a. (2)

Solution
We use synthetic division:

-3	1	0	a	6
	$\downarrow$	-3	9	$-3(a+9)$
	1	-3	$a+9$	$6-3(a+9)$

Now,

$$\begin{aligned} f(-3) = 0 &\Rightarrow 6 - 3(a+9) = 0 \\ &\Rightarrow 6 = 3(a+9) \\ &\Rightarrow 2 = a + 9 \\ &\Rightarrow \underline{\underline{a = -7}}. \end{aligned}$$

(b) Hence or otherwise, solve the equation $f(x) = 0$ for this value of a. (4)

Solution

$$\text{f}(x) = 0 \Rightarrow (x+3)(x^2 - 3x + 2) = 0$$

$$\left.\begin{array}{ll} \text{add to:} & -3 \\ \text{multiply to:} & +2 \end{array}\right\} -2,\ -1$$

$$\begin{aligned} &\Rightarrow (x+3)(x-2)(x-1) = 0 \\ &\Rightarrow \underline{\underline{x = -3,\ x = 1,\ \text{or } x = 2}}. \end{aligned}$$

9. The equation of the circle C is

$$x^2 + y^2 - 8x + 2y - 19 = 0.$$

(a) Express the equation of C in the form (4)

$$(x - a)^2 + (y - b)^2 = r^2.$$

Solution

$$\begin{aligned} x^2 + y^2 - 8x + 2y - 19 = 0 &\Rightarrow x^2 - 8x + y^2 + 2y = 19 \\ &\Rightarrow (x^2 - 8x + 16) + (y^2 + 2y + 1) = 19 + 16 + 1 \\ &\Rightarrow (x-4)^2 + (y+1)^2 = 36 \\ &\Rightarrow \underline{\underline{(x-4)^2 + (y+1)^2 = 6^2}}; \end{aligned}$$

hence, $\underline{\underline{a = 4}}$, $\underline{\underline{b = -1}}$, and $\underline{\underline{c = 6}}$.

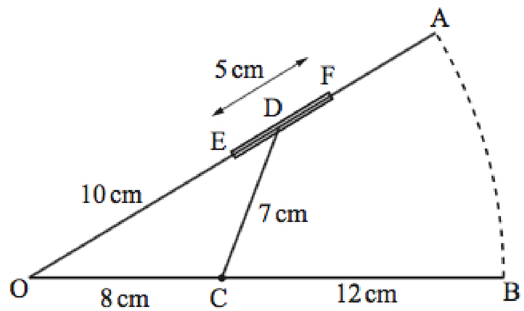
(b) Hence or otherwise, use an algebraic method to decide whether the point $(8, 3)$ lies inside, outside or on the circumference of the circle. (2)
Show all your working.

Solution

$$\begin{aligned} (8-4)^2 + (3+1)^2 &= 4^2 + 4^2 \\ &= 32 \\ &> 36 \\ &= 6^2; \end{aligned}$$

hence, the point $(8, 3)$ lies inside the circle.

10. The figure shows a partly open window OA, viewed from above. The window is hinged at O. When the window is closed, the end A is at point B. The window is kept open by a rod CD, where C is a fixed point on the line OB.



The point D slides along a fixed bar EF. When the window is closed, D is at F. When the window is fully open, D is at E.

$OA = OB = 20$ cm, $OC = 8$ cm, $CD = 7$ cm, $EF = 5$ cm, and $OE = 10$ cm.

Find

(a) angle EOC when the window is fully open, (3)

Solution

Let $\theta°$ when the window is fully open. We use the cosine rule:

$$\cos\theta = \frac{10^2 + 8^2 - 7^2}{2 \times 10 \times 8} \Rightarrow \cos\theta = \tfrac{23}{32}$$
$$\Rightarrow \theta = 44.048\,625\,67 \text{ (FCD)}$$
$$\Rightarrow \theta = 44.0° \text{ (3 sf)}.$$

(b) the distance OD when angle EOC is $30°$. (4)

Solution

We use $x = OD$. We use the cosine rule:

$$7^2 = x^2 + 8^2 - 2 \cdot x \cdot 8 \cdot \cos 30° \Rightarrow 49 = x^2 + 64 - 8\sqrt{3}\,x$$
$$\Rightarrow x^2 - 8\sqrt{3}\,x + 15 = 0$$

$a = 1$, $b = -8\sqrt{3}$, $c = 15$

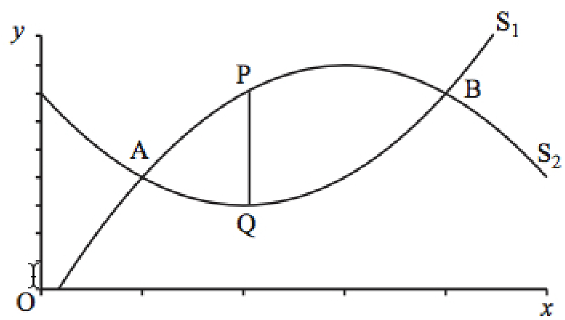
$$\begin{aligned} &\Rightarrow x = \frac{8\sqrt{3} \pm \sqrt{132}}{2} \\ &\Rightarrow x = 1.183\ldots \text{ (no!)}, 12.672\,765\,88 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 12.7 \text{ cm (3 sf)}}}. \end{aligned}$$

Section B

11. Two curves, S_1 and S_2 have equations

$$y = x^2 - 4x + 7 \text{ and } y = 6x - x^2 - 1$$

respectively.



The curves meet at A and at B.

(a) Show that the coordinates of A and B are $(1, 4)$ and $(4, 7)$ respectively. (2)

Solution

$$\begin{aligned} x^2 - 4x + 7 = 6x - x^2 - 1 &\Rightarrow 2x^2 - 10x + 8 = 0 \\ &\Rightarrow 2(x^2 - 5x + 4) = 0 \end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -5 \\ \text{multiply to:} & -4\end{array}\right\} -4, -1$$

$$\begin{aligned}&\Rightarrow 2(x-1)(x-4) = 0\\ &\Rightarrow x = 1 \text{ or } x = 4\\ &\Rightarrow y = 4 \text{ or } y = 7;\end{aligned}$$

hence, A and B are $\underline{\underline{(1, 4)}}$ and $\underline{\underline{(4, 7)}}$ respectively.

Points P and Q lie on S_1 and S_2 between A and B. P and Q have the same x-coordinate so that PQ is parallel to the y-axis, as shown in the above figure.

(b) Find an expression, in its simplest form, for the length PQ as a function of x. (2)

Solution

$$\begin{aligned}PQ &= (6x - x^2 - 1) - (x^2 - 4x + 7)\\ &= \underline{\underline{-2x^2 + 10x - 8}}.\end{aligned}$$

(c) Use calculus to find the greatest length of PQ. (4)

Solution

$$PQ = -2x^2 + 10x - 8 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(PQ) = -4x + 10$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(PQ) = 0 &\Rightarrow -4x + 10 = 0\\ &\Rightarrow 4x = 10\\ &\Rightarrow x = 2.5\\ &\Rightarrow \underline{\underline{PQ = 4.5 \text{ cm}}}\end{aligned}$$

(d) Find the area between the two curves. (4)

Solution

$$\begin{aligned}\text{Area} &= \int_1^4 (-2x^2 + 10x - 8)\,\mathrm{d}x \\ &= \left[-\tfrac{2}{3}x^3 + 5x^2 - 8x\right]_{x=1}^4 \\ &= \left(-42\tfrac{2}{3} + 80 - 32\right) - \left(-\tfrac{2}{3} + 5 - 8\right) \\ &= \underline{\underline{9}}.\end{aligned}$$

12. A distributor of flower bulbs has a large number of tulip bulbs and daffodil bulbs, mixed in the ratio 1 : 3 respectively. He packs the bulbs in boxes. He puts 10 bulbs, chosen at random, into each box.

(a) Find the probability that a box, chosen at random, contains

(i) exactly 4 daffodil bulbs, (4)

Solution

$$\begin{aligned}\text{P(exactly 4 daffodil bulbs)} &= \binom{10}{4}(0.75)^4(0.25)^6 \\ &= 0.016\,222\,000\,12 \text{ (FCD)} \\ &= \underline{\underline{0.016\,2 \text{ (3 sf)}}}.\end{aligned}$$

(ii) at least 1 tulip bulb. (3)

Solution

$$\begin{aligned}\text{P(at least 1 tulip bulb)} &= 1 - \text{P(0 tulip bulbs)} \\ &= 1 - (0.75)^{10} \\ &= 0.943\,686\,485\,3 \text{ (FCD)} \\ &= \underline{\underline{0.944 \text{ (3 sf)}}}.\end{aligned}$$

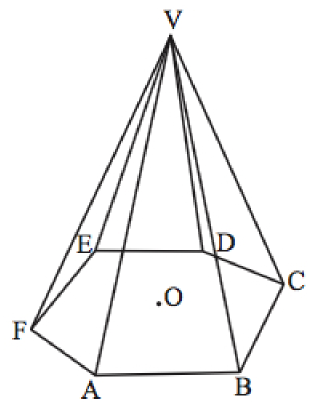
Two boxes of bulbs are chosen at random.

(b) Find the probability that there is a total of 3 tulip bulbs in the two boxes. (5)

Solution

$$\begin{aligned} \text{P(3 tulip bulbs)} &= \binom{20}{3}(0.25)^3(0.75)^{17} \\ &= 0.133\,895\,615\,2 \text{ (FCD)} \\ &= \underline{\underline{0.134 \text{ (3 sf)}}}. \end{aligned}$$

13. A gardener marks out a regular hexagon $ABCDEF$ on his horizontal garden. Each side of the hexagon is 0.5 m. The gardener sticks a cane in the ground at each point of the hexagon. He joins the six canes at V where V is vertically above the centre, O, of the hexagon, as shown below. Each cane has a length of 2.4 m from the ground to V.



Calculate, giving your answers to 3 significant figures,

(a) the vertical height of V above the ground, (3)

Solution

Well, $\triangle OAB$ is an equilateral triangle and so

$$\begin{aligned} AV^2 = OA^2 + OV^2 &\Rightarrow 2.4^2 = 0.5^2 + OV^2 \\ &\Rightarrow OV^2 = 5.51 \\ &\Rightarrow OV = 2.347\,338\,919 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{OV = 2.35 \text{ m (3 sf)}}}. \end{aligned}$$

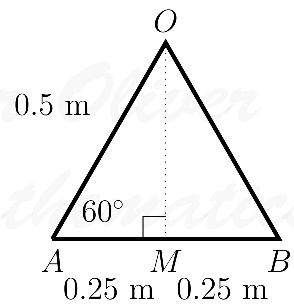
(b) the angle between each cane and the ground, (2)

Solution

$$\cos = \frac{\text{adj}}{\text{hyp}} \Rightarrow \cos OAV = \frac{0.5}{2.4}$$
$$\Rightarrow \angle OAV = 77.975\,300\,82 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\angle OAV = 78.0^\circ \text{ (3 sf)}}}.$$

(c) the angle between the plane VAB and the ground. (4)

Solution



$$OA^2 = AM^2 + MO^2 \Rightarrow 0.5^2 = 0.25^2 + MO^2$$
$$\Rightarrow MO^2 = 0.187\,5$$
$$\Rightarrow MO = 0.433\,012\,701\,9 \text{ (FCD)}$$

and

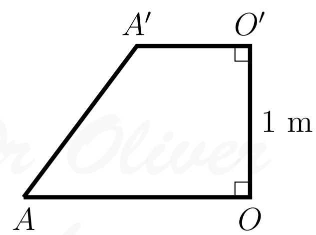
$$\tan = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan VMO = \frac{2.347\ldots}{0.433\ldots}$$
$$\Rightarrow \angle VMO = 79.548\,167\,94 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\angle VMO = 79.5^\circ \text{ (3 sf)}}}.$$

The gardener stretches a horizontal wire around the structure to strengthen it. He fixes the wire to each cane at a point 1 m vertically above the ground.

(d) Find the length of the wire. (3)

Solution

Let O' and A' be the points O and A moved 1 m up.



We use similar shapes:

$$\begin{aligned}\text{LSF} &= \frac{OV'}{OV} \\ &= \frac{OV-1}{OV} \\ &= \frac{2.347\ldots-1}{2.347\ldots} \\ &= 0.573\,985\,677\,2 \text{ (FCD)}\end{aligned}$$

and

$$\begin{aligned}OA' &= OA \times \text{LSF} \\ &= 0.5 \times 0.286\ldots \\ &= 0.286\,992\,838\,6 \text{ (FCD)}.\end{aligned}$$

Finally,

$$\begin{aligned}\text{length} &= 6 \times 0.286\ldots \\ &= 1.721\,957\,031 \text{ (FCD)} \\ &= \underline{\underline{1.72 \text{ m (3 sf)}}}.\end{aligned}$$

14. A company produces bottles of two liquids, X and Y. There are two ingredients, A and B, in each liquid.

The table shows the quantities, in centilitres (cl), of A and B needed for each bottle of liquid.

	A	B
X	4	2
Y	3	5

Each day the company can use 84 cl of A and 90 cl of B.

From this information an analyst writes down the inequality

$$4x + 3y \leqslant 84.$$

(a) Explain what x and y stand for in this inequality and explain what the inequality models. (2)

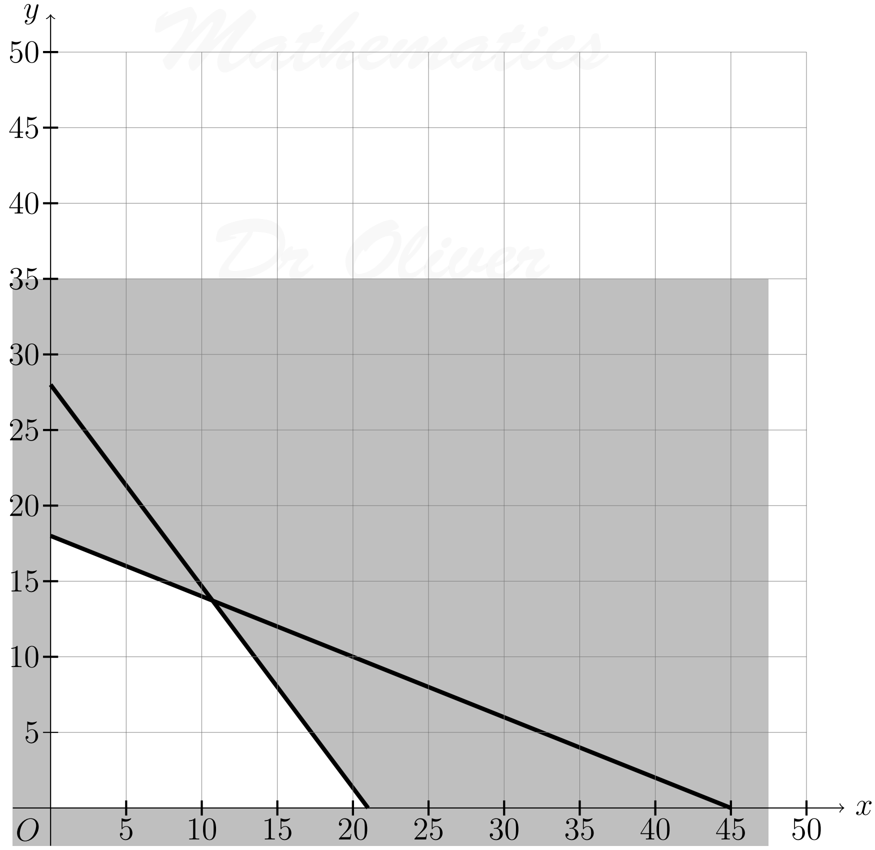
Solution
x is the number of units of X produced, y is the number of units of Y produced, and $4x + 3y \leqslant 84$ models the quantity of A.

(b) Use the information given to write down another inequality, other than $x \geqslant 0$ and $y \geqslant 0$. (1)

Solution
$2x + 5y \leqslant 90$.

(c) Illustrate your two inequalities. Shade the region that is **not** required. (3)

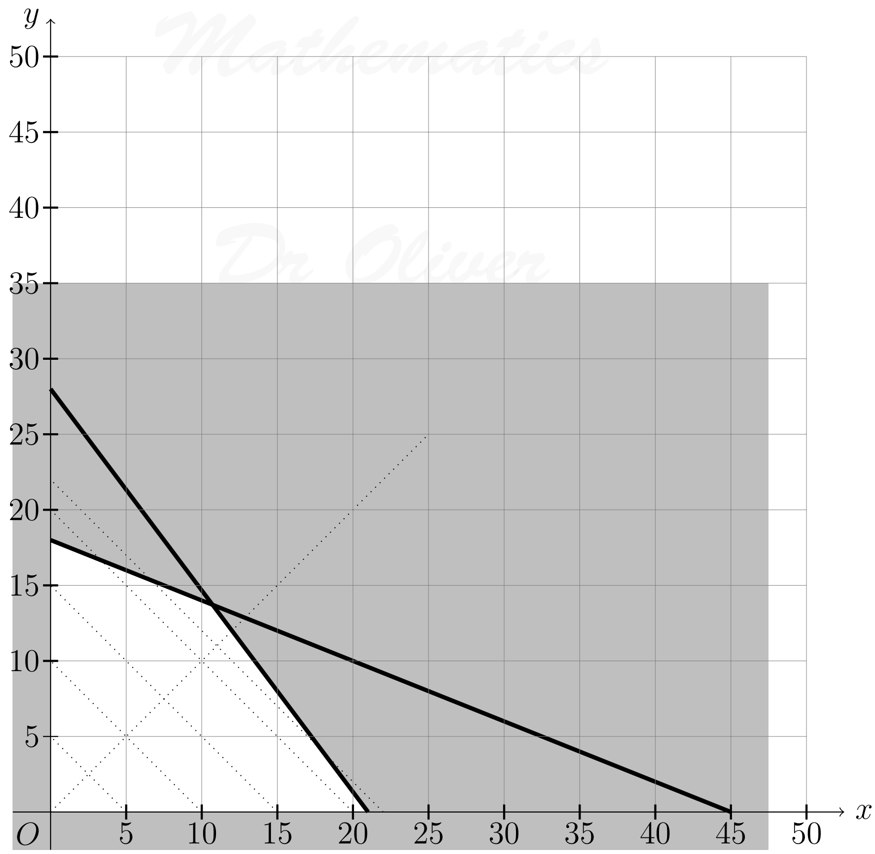
Solution



The company needs to produce the same number of bottles of X and of Y each day.

(d) Find the maximum number of bottles of X and of Y that the company can produce. (2)

Solution



Hence, $\underline{\underline{x = 12,\ y = 12}}$.

On one day the company does not have to produce the same numbers of bottles of X and of Y.

(e) Write down the maximum number of bottles that can be produced and all the combinations that will give this maximum. (4)

Solution

The maximum is $\underline{\underline{24}}$: $\underline{\underline{x = 10,\ y = 14}}$, $\underline{\underline{x = 11,\ y = 13}}$, or $\underline{\underline{x = 12,\ y = 12}}$.