Dr Oliver Mathematics
Further Mathematics
Numerical Solutions
Past Examination Questions

This booklet consists of 28 questions across a variety of examination topics.
The total number of marks available is 227.

There are three examples of this technique.

1 Interval bisection

We take, for example, $(a, \mathrm{f}(a)) < 0$ and $(b, \mathrm{f}(b)) > 0$. We then take

$$\left(\frac{a+b}{2}, \mathrm{f}\left(\frac{a+b}{2}\right)\right)$$

and it depends on the sign of

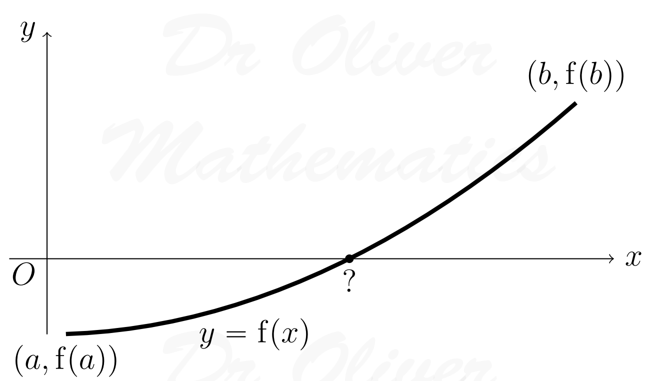
$$\mathrm{f}\left(\frac{a+b}{2}\right):$$

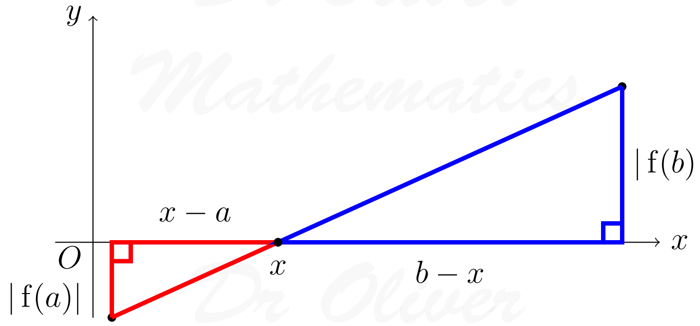
$$\left(\frac{a+b}{2}, \mathrm{f}\left(\frac{a+b}{2}\right)\right) < 0 \text{ then we take } \left(\frac{a+b}{2}, b\right)$$

and

$$\left(\frac{a+b}{2}, \mathrm{f}\left(\frac{a+b}{2}\right)\right) > 0 \text{ then we take } \left(a, \frac{a+b}{2}\right).$$

2 Linear interpolation





$$
\begin{aligned}
&\frac{x-a}{b-x} = \frac{|\,f(a)|}{|\,f(b)|}\\
\Rightarrow\quad & |\,f(b)|(x-a) = |\,f(a)|(b-x)\\
\Rightarrow\quad & |\,f(b)|x - |\,f(b)|a = |\,f(a)|b - |\,f(a)|x\\
\Rightarrow\quad & |\,f(a)|x + |\,f(b)|x = a|\,f(b)| + b|\,f(a)|\\
\Rightarrow\quad & x\,(|\,f(a)| + |\,f(b)|) = a|\,f(b)| + b|\,f(a)|\\
\Rightarrow\quad & \boxed{x = \frac{a|\,f(b)| + b|\,f(a)|}{|\,f(a)| + |\,f(b)|}}.
\end{aligned}
$$

3 Newton-Raphson

We take as a starting point

$$y - f(a) = f'(a)(x - a).$$

Set $y = 0$:

$$
\begin{aligned}
-\,f(a) = f'(a)(x-a) \Rightarrow x - a &= -\frac{f(a)}{f'(a)}\\
\Rightarrow x &= a - \frac{f(a)}{f'(a)}.
\end{aligned}
$$

That is, if a is an first approximation to a root of $f(x) = 0$, a better approximation is, in general,

$$a - \frac{f(a)}{f'(a)};$$

i.e.,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}.$$

4 Questions

1. Figure 1 shows part of the graph of $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x \sin x + 2x - 3.$$

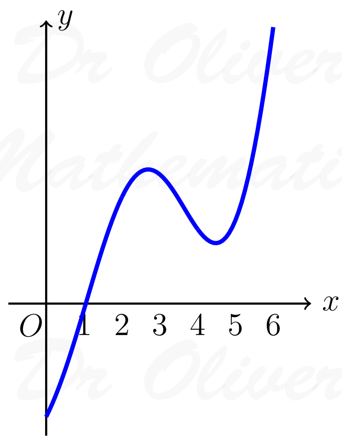


Figure 1: $\mathrm{f}(x) = x \sin x + 2x - 3$

The equation $\mathrm{f}(x) = 0$ has a single root α.

(a) Taking $x_1 = 1$ as a first approximation to α, apply the Newton-Raphson procedure once to $\mathrm{f}(x)$ to find a second approximation to α, to 3 significant figures. (5)

(b) Given $x_1 = 5$ as taken as a first approximation to α, apply the Newton-Raphson procedure,

(i) use Figure 1 to produce a rough sketch of $y = \mathrm{f}(x)$ for $3 \leqslant x \leqslant 6$ (1)

and by drawing suitable tangents, and without any further calculations,

(ii) show the approximate positions of x_2 and x_3, the second and third approximations to α. (1)

2.

$$\mathrm{f}(x) = 1 - \mathrm{e}^x + 3 \sin 2x.$$

The equation $\mathrm{f}(x) = 0$ has a root α in $1.0 < x < 1.4$.

(a) Starting with the interval $(1.0, 1.4)$, use interval bisection three times to find of value of α to one decimal place. (3)

(b) Taking your answer to part (a) as a first approximation to α, apply the Newton-Raphson procedure once to $\mathrm{f}(x)$ to find a second approximation to α. (4)

(c) By considering the change in sign of $\mathrm{f}(x)$ over an appropriate interval, show your your to part (b) is accurate to 2 decimal places. (2)

3.
$$f(x) = 0.25x - 2 + 4\sin\sqrt{x}.$$

(a) Show that the equation $f(x) = 0$ has a root α between $x = 0.24$ and $x = 0.28$. (2)

(b) Starting with the interval $[0.24, 0.28]$, use interval bisection three times to find an interval of width 0.005 which contains α. (3)

The equation $f(x) = 0$ also has a root β between $x = 10.75$ and $x = 11.25$.

(c) Taking 11 as a first approximation to β, apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to β. Give your answer to 2 decimal places. (6)

4.
$$f(x) = \ln x + x - 3, x > 0.$$

(a) Find $f(2.0)$ and $f(2.5)$, each to 4 decimal places, and show that the root α of the equation $f(x) = 0$ satisfies $2.0 < \alpha < 2.5$. (3)

(b) Use linear interpolation with your values of $f(2.0)$ and $f(2.5)$ to estimate α, giving your answer to 3 decimal places. (2)

(c) Taking 2.25 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places. (5)

(d) Show that your answer in part (d) gives α correct to 3 decimal places. (2)

5.
$$f(x) = x^3 + 8x - 19.$$

(a) Show that the equation $f(x) = 0$ has only one real root. (2)

(b) Show that the equation $f(x) = 0$ lies between 1 and 2. (2)

(c) Obtain an approximation to the real root of $f(x) = 0$ by performing two applications of Newton-Raphson procedure tof(x), using $x = 2$ as the first approximation. Give your answer to 3 decimal places. (4)

(d) By considering the change in sign of $f(x)$ over an appropriate interval, show that your answer to part (c) is accurate to 3 decimal places. (2)

6.
$$f(x) = 3x^2 + x - \tan\left(\frac{x}{2}\right) - 2, -\pi < x < \pi.$$

The equation $f(x) = 0$ has a root α in the interval $[0.7, 0.8]$.

(a) Use linear interpolation, on the values at the end points of this interval, to obtain an approximation to α. Give your answer to 3 decimal places. (4)

(b) Taking 0.75 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α. Give your answer to 3 decimal places. (4)

7.

$$f(x) = 4\cos x + e^{-x}.$$

(a) Show that the equation $f(x) = 0$ has a root α between 1.6 and 1.7. (2)

(b) Taking 1.6 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α. Give your answer to 3 significant figures. (4)

8.

$$f(x) = 3\sqrt{x} + \frac{18}{\sqrt{x}} - 20.$$

(a) Show that the equation $f(x) = 0$ has a root α between $[1.1, 1.2]$. (2)

(b) Find $f'(x)$. (3)

(c) Using $x_0 = 1.1$ as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α, giving your answer to 3 significant figures. (4)

9. Figure 2 shows part of the curve with equation $y = f(x)$, where

$$f(x) = 1 - x - \sin(x^2).$$

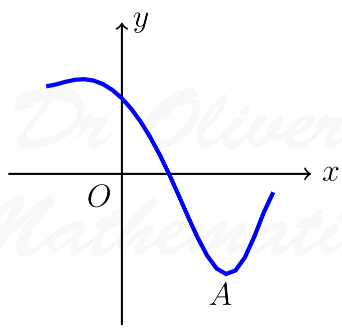


Figure 2: $f(x) = 1 - x - \sin(x^2)$

The point A, with x-coordinate p, is a stationary point on the curve. The equation $f(x) = 0$ has a root α in the interval $0.6 < \alpha < 0.7$.

(a) Explain why $x_0 = p$ is not suitable to use as a first approximation to α when applying the Newton-Raphson process to $f(x)$. (1)

(b) Using $x_0 = 0.6$ as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places. (5)

(c) By considering the change in sign of $f(x)$ over an appropriate interval, show your your to part (b) is accurate to 3 decimal places. (2)

10. Given that α is the only real root of the equation

$$\sin 2x - \ln 3x = 0,$$

(a) show that $0.8 < \alpha < 0.9$. (2)

(b) Taking 0.9 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places. (5)

(c) Use linear interpolation once on the interval $[0.8, 0.9]$ to find another approximation to α, giving your answer to 3 decimal places. (3)

11. Given that α is the only real root of the equation

$$x^3 - x^2 - 6 = 0,$$

(a) show that $2.2 < \alpha < 2.3$. (2)

(b) Taking 2.2 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places. (5)

(c) Use linear interpolation once on the interval $[2.2, 2.3]$ to find another approximation to α, giving your answer to 3 decimal places. (3)

12.

$$f(x) = x\cos x - 2x + 5.$$

(a) Show that $f(x) = 0$ has a root α in the interval $[2, 2.1]$. (2)

(b) Taking 2 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α, giving your answer to 2 decimal places. (5)

(c) Show that your answer to part (b) gives α to 2 decimal places. (2)

13.

$$f(x) = 3x^2 - \frac{11}{x^2}.$$

(a) Write down, to 3 decimal places, the value of $f(1.3)$ and the value of $f(1.4)$. (1)

The equation $f(x) = 0$ has a root α between 1.3 and 1.4.

(b) Starting with the interval $[1.3, 1.4]$, use interval bisection three times to find an interval of width 0.025 which contains α. (3)

(c) Taking 1.4 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places. (5)

14.

$$f(x) = x^3 - \frac{7}{x} + 2, x > 0.$$

(a) Show that the equation $f(x) = 0$ has a root α between 1.4 and 1.5. (2)

(b) Starting with the interval $[1.4, 1.5]$, use interval bisection three times to find an interval of width 0.025 which contains α. (3)

(c) Taking 1.45 as a first approximation to α, apply the Newton-Raphson process once to (5)

$$f(x) = x^3 - \frac{7}{x} + 2$$

to find a second approximation to α, giving your answer to 3 decimal places.

15.

$$f(x) = 5x^2 - 4x^{\frac{3}{2}} - 6, x \geqslant 0.$$

The root α the equation $f(x) = 0$ lies in the interval $[1.6, 1.8]$.

(a) Use linear interpolation once on the interval $[1.6, 1.8]$ to find to approximation to α. Give your answer to 3 decimal places. (4)

(b) Differentiate $f(x)$ to find $f'(x)$. (2)

(c) Taking 1.7 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α. Give your answer to 3 decimal places. (4)

16.

$$f(x) = 3^x + 3x - 7.$$

(a) Show that $f(x) = 0$ has a root α between $x = 1$ and $x = 2$. (2)

(b) Starting with the interval $[1, 2]$, use interval bisection three times to find an interval of width 0.25 which contains α. (3)

17.

$$f(x) = x^2 + \frac{5}{2x} - 3x - 1, x \neq 0.$$

(a) Differentiate $f(x)$ to find $f'(x)$. (2)

The root α the equation $f(x) = 0$ lies in the interval $[0.7, 0.9]$.

(b) Taking 0.8 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α. Give your answer to 3 decimal places. (4)

18. (a) Show that $f(x) = x^4 + x - 1$ has a real root α in the interval $[0.5, 1.0]$. (2)

(b) Starting with the interval $[0.5, 1.0]$, use interval bisection three times to find an interval of width 0.125 which contains α. (3)

(c) Taking 0.75 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α. Give your answer to 3 decimal places. (5)

19. (6)

$$f(x) = x^2 + \frac{3}{4\sqrt{x}} - 3x - 7, x > 0.$$

The root α the equation $f(x) = 0$ lies in the interval $[3, 5]$.
Taking 4 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α. Give your answer to 2 decimal places.

20.

$$f(x) = \tan\left(\frac{x}{2}\right) + 3x - 6, -\pi < x < \pi.$$

(a) Show that $f(x) = 0$ has a root α in the interval $[1, 2]$. (2)

(b) Use linear interpolation once on the interval $[2.2, 2.3]$ to find another approximation to α, giving your answer to 3 decimal places. (3)

21.

$$f(x) = 2x^{\frac{1}{2}} + x^{-\frac{1}{2}} - 5, x > 0.$$

(a) Find $f'(x)$. (2)

The equation $f(x) = 0$ has a root α in the interval $[4.5, 5.5]$.

(b) Using $x_0 = 5$ as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α, giving your answer to 3 significant figures. (4)

22.

$$f(x) = \cos(x^2) - x + 3, 0 < x < \pi.$$

(a) Show that the equation $f(x) = 0$ has a root α in the interval $[2.5, 3]$. (2)

(b) Use linear interpolation once on the interval $[2.5, 3]$ to find to approximation to α, giving your answer to 2 decimal places. (3)

23.

$$f(x) = \tfrac{1}{2}x^4 - x^3 + x - 3.$$

(a) Show that $f(x) = 0$ has a root α in the interval $x = 2$ and $x = 2.5$. (2)

(b) Starting with the interval $[2, 2.5]$, use interval bisection twice to find an interval of width 0.125 which contains α. (3)

The equation $f(x) = 0$ has a root β in the interval $[-2, -1]$.

(c) Taking -1.5 as a first approximation to β, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to β. Give your answer to 2 decimal places. (5)

24.

$$f(x) = x^3 - \frac{5}{2x^{\frac{3}{2}}} + 2x - 3, x > 0.$$

(a) Show that $f(x) = 0$ has a root α in the interval $[1.1, 1.5]$. (2)

(b) Find $f'(x)$. (2)

(c) Using $x_0 = 1.1$ as a first approximation to α, apply the Newton-Raphson process once to $\mathrm{f}(x)$ to find a second approximation to α, giving your answer to 3 decimal places. (3)

25.

$$\mathrm{f}(x) = 3\cos 2x + x - 2, -\pi \leqslant x \leqslant \pi.$$

(a) Show that $\mathrm{f}(x) = 0$ has a root α in the interval $[2, 3]$. (2)

(b) Use linear interpolation once on the interval $[2, 3]$ to find to approximation to α. Give your answer to 3 decimal places. (3)

(c) The equation $\mathrm{f}(x) = 0$ has a root β in the interval $[-1, -0]$. Starting with the interval $[-1, 0]$, use interval bisection to find an interval of width 0.25 which contains β. (4)

26. In the interval $13 < x < 14$, the equation

$$3 + x\sin\left(\frac{x}{4}\right) = 0,$$

where x is measured in radians, has exactly one root.

(a) Starting with the interval $[13, 14]$, use interval bisection twice to find an interval of width 0.25 which contains α. (3)

(b) Use linear interpolation once on the interval $[2, 3]$ to find to approximation to α. Give your answer to 3 decimal places. (4)

27.

$$\mathrm{f}(x) = 3x^{\frac{3}{2}} - 25x^{-\frac{1}{2}} - 125, x > 0.$$

(a) Find $\mathrm{f}'(x)$. (2)

The equation $\mathrm{f}(x) = 0$ has a root α in the interval $[12, 13]$.

(b) Using $x_0 = 12.5$ as a first approximation to α, apply the Newton-Raphson process once to $\mathrm{f}(x)$ to find a second approximation to α, giving your answer to 3 decimal places. (4)

28.

$$\mathrm{f}(x) = \tfrac{1}{3}x^2 + \frac{4}{x^2} - 2x - 1, \; x > 0.$$

(a) Show that the equation $\mathrm{f}(x) = 0$ has a root α in the interval $[6, 7]$. (2)

(b) Taking 6 as a first approximation to α, apply the Newton-Raphson process once to $\mathrm{f}(x)$ to obtain a second approximation to α. Give your answer to 2 decimal places. (5)