Dr Oliver Mathematics
Mathematics: Advanced Higher
2014 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. (a) Given (3)

$$f(x) = \frac{x^2 - 1}{x^2 + 1},$$

obtain $f'(x)$ and simplify your answer.

Solution

$$u = x^2 - 1,\ v = x^2 + 1 \Rightarrow \frac{du}{dx} = \frac{dv}{dx} = 2x.$$

Now,

$$\begin{aligned} f(x) = \frac{x^2 - 1}{x^2 + 1} \Rightarrow f'(x) &= \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2} \\ \Rightarrow f'(x) &= \frac{(2x^3 + 2x) - (2x^3 - 2x)}{(x^2 + 1)^2} \\ \Rightarrow f'(x) &= \frac{4x}{(x^2 + 1)^2}. \end{aligned}$$

(b) Differentiate (3)

$$y = \tan^{-1}(3x^2).$$

Solution

$$\begin{aligned} y = \tan^{-1}(3x^2) \Rightarrow \frac{dy}{dx} &= \frac{1}{1 + (3x)^2} \cdot 6x \\ \Rightarrow \frac{dy}{dx} &= \frac{6x}{1 + 9x^2}. \end{aligned}$$

2. (a) Write down and simplify the general term in the expression (1)

$$\left(\frac{2}{x} + \frac{1}{4x^2}\right)^{10}.$$

Solution

$$\text{General term} = \underline{\underline{\binom{10}{r}\left(\frac{2}{x}\right)^r\left(\frac{1}{4x^2}\right)^{n-r}}}.$$

(b) Hence, or otherwise, obtain the term in $\frac{1}{x^{13}}$. (4)

Solution

$$\begin{aligned}\binom{10}{r}\left(\frac{2}{x}\right)^r\left(\frac{1}{4x^2}\right)^{10-r} &= \binom{10}{r}\left(\frac{2^r}{x^r}\right)\left(\frac{1}{4^{10-r}x^{2(10-r)}}\right)\\ &= \binom{10}{r}\left(\frac{2^r}{4^{10-r}x^r x^{2(10-r)}}\right)\\ &= \binom{10}{r}\left(\frac{2^r}{4^{10-r}x^{20-r}}\right)\end{aligned}$$

and

$$20 - r = 13 \Rightarrow r = 7.$$

Hence, the specific term is

$$\begin{aligned}\binom{10}{7}\left(\frac{2^7}{4^{10-7}x^{20-7}}\right) &= 120\left(\frac{128}{64x^{13}}\right)\\ &= \underline{\underline{\frac{240}{x^{13}}}}.\end{aligned}$$

3. (a) Use Gaussian elimination on the system of equations below to give an expression for z in terms of λ: (4)

$$\begin{aligned}x + y + z &= 2\\ 4x + 3y - \lambda z &= 4\\ 5x + 6y + 8z &= 11.\end{aligned}$$

Solution

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 4 & 3 & -\lambda & 4 \\ 5 & 6 & 8 & 11 \end{array}\right)$$

Do $R_2 + 4R_1$ and $R_3 - 5R_1$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & \lambda + 4 & 4 \\ 0 & 1 & 3 & 1 \end{array}\right)$$

Do $R_2 - R_3$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & \lambda + 4 & 4 \\ 0 & 0 & \lambda + 1 & 3 \end{array}\right)$$

Now,

$$\begin{aligned}
(\lambda + 1)z = 3 &\Rightarrow \underline{\underline{z = \frac{3}{\lambda + 1}}} \\
&\Rightarrow y + (\lambda + 4)z = 4 \\
&\Rightarrow y + \frac{3(\lambda + 4)}{\lambda + 1} = 4 \\
&\Rightarrow y = 4 - \frac{3(\lambda + 4)}{\lambda + 1} \\
&\Rightarrow y = \frac{4(\lambda + 1) - 3(\lambda + 4)}{\lambda + 1} \\
&\Rightarrow \underline{\underline{y = \frac{\lambda - 8}{\lambda + 1}}} \\
&\Rightarrow x + \frac{\lambda - 8}{\lambda + 1} + \frac{3}{\lambda + 1} = 2 \\
&\Rightarrow x + \frac{\lambda - 5}{\lambda + 1} = 2 \\
&\Rightarrow x = 2 - \frac{\lambda - 5}{\lambda + 1} \\
&\Rightarrow x = \frac{2(\lambda + 1) - (\lambda - 5)}{\lambda + 1} \\
&\Rightarrow \underline{\underline{x = \frac{\lambda + 7}{\lambda + 1}}}.
\end{aligned}$$

(b) For what values of λ does this system have a solution? (1)

Solution

There is no solution for $\underline{\underline{\lambda = -1}}$.

(c) Determine the solution to this system of equations when $\lambda = 2$. (1)

Solution

$\underline{\underline{x = 3,\ y = -2,\ z = 1}}$.

4. Given (3)

$$x = \ln(1 + t^2) \text{ and } y = \ln(1 + 2t^2),$$

use parametric differentiation to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t.

Solution

$$x = \ln(1 + t^2) \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2t}{1 + t^2}$$
$$y = \ln(1 + 2t^2) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{4t}{1 + 2t^2}.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\frac{4t}{1+2t^2}}{\frac{2t}{1+t^2}} = \underline{\underline{\frac{2(1 + t^2)}{1 + 2t^2}}}.$$

5. Three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are given by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ where

$$\mathbf{u} = 5\mathbf{i} + 13\mathbf{j},\ \mathbf{v} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k},\ \text{ and } \mathbf{w} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}.$$

(a) Calculate (3)

$$\mathbf{u}.(\mathbf{v} \times \mathbf{w}).$$

Solution

$$\mathbf{u}.(\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 5 & 13 & 0 \\ 2 & 1 & 3 \\ 1 & 4 & -1 \end{vmatrix}$$
$$= 5(-1-12) - 13(-2-3) + 0$$
$$= -65 + 65$$
$$= \underline{\underline{0}}.$$

(b) Interpret your result geometrically. (1)

Solution

$\mathbf{u}$ is perpendicular to the vector $\mathbf{v} \times \mathbf{w}$.

6. Given (3)

$$\mathrm{e}^y = x^3 \cos^2 x, \; x > 0,$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{x} + b \tan x,$$

for some constants a and b and state the values of a and b.

Solution

$$\begin{aligned} \mathrm{e}^y = x^3 \cos 2x &\Rightarrow \ln(\mathrm{e}^y) = \ln(x^3 \cos^2 x) \\ &\Rightarrow y = \ln x^3 + \ln \cos^2 x \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{x^3} + \frac{(-2 \sin x \cos x)}{\cos^2 x} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{x} - \frac{2 \sin x}{\cos x} \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{x} - 2 \tan x}}; \end{aligned}$$

hence, $\underline{\underline{a = 3}}$ and $\underline{\underline{b = -2}}$.

7. Given $\mathbf{A}$ is the matrix (4)

$$\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix},$$

prove by induction that

$$\mathbf{A}^n = \begin{pmatrix} 2^n & a(2^n - 1) \\ 0 & 1 \end{pmatrix}, \; n \geqslant 1.$$

Solution

$\underline{n = 1}$:

$$\begin{aligned} \mathbf{A}^1 &= \begin{pmatrix} 2^1 & a(2^1 - 1) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix} \end{aligned}$$

and the result is true for $n = 1$.

Suppose that the result is true for $n = k$, i.e.,

$$\mathbf{A}^k = \begin{pmatrix} 2^k & a(2^k - 1) \\ 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{A}^{k+1} &= \mathbf{A}\mathbf{A}^k \\ &= \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2^k & a(2^k - 1) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \cdot 2^k & 2a(2^k - 1) + a \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & a[2(2^k - 1) + 1] \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & a[(2^{k+1} - 2) + 1] \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & a[(2^{k+1} - 1)] \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

and the result is true for $n = k + 1$.

Hence, by mathematical induction, the result is $\underline{\underline{\text{true}}}$ for all $n \geqslant 1$.

8. Find the solution $y = \mathrm{f}(x)$ to the differential equation (6)

$$4\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0,$$

given that $y = 4$ and $\dfrac{dy}{dx} = 3$ when $x = 0$.

Solution

Complementary function:

$$4m^2 - 4m + 1 = 0 \Rightarrow (2m-1)^2 = 0 \Rightarrow m = \tfrac{1}{2} \text{ (repeated)}$$

and hence the complementary function is

$$y = e^{\frac{1}{2}x}(A + Bx).$$

Now,

$$y = e^{\frac{1}{2}x}(A + Bx) \Rightarrow \frac{dy}{dx} = \tfrac{1}{2}e^{\frac{1}{2}x}(A + Bx) + Be^{\frac{1}{2}x}.$$

Next,

$$x = 0, y = 4 \Rightarrow 4 = A$$

and

$$\begin{aligned} x = 0, \frac{dy}{dx} = 3 \Rightarrow 3 &= \tfrac{1}{2}A + B \\ &\Rightarrow B = 1. \end{aligned}$$

Hence, the particular solution is

$$\underline{\underline{y = e^{\frac{1}{2}x}(4 + x)}}.$$

9. (a) Give the first three non-zero terms of the Maclaurin series for $\cos 3x$. (2)

Solution

Well,

$$\cos x = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 + \dots$$

and so

$$\begin{aligned} \cos 3x &= 1 - \tfrac{1}{2!}(3x)^2 + \tfrac{1}{4!}(3x)^4 + \dots \\ &= \underline{\underline{1 - \tfrac{9}{2}x^2 + \tfrac{27}{8}x^4 + \dots}}. \end{aligned}$$

(b) Write down the first four terms of the Maclaurin series for e^{2x}. (1)

Solution

Well,

$$e^x = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \ldots$$

and so

$$\begin{aligned} e^{2x} &= 1 + (2x) + \tfrac{1}{2!}(2x)^2 + \tfrac{1}{3!}(2x)^3 + \ldots \\ &= \underline{\underline{1 + 2x + 2x^2 + \tfrac{4}{3}x^3 + \ldots}}. \end{aligned}$$

(c) Hence, or otherwise, determine the Maclaurin series for $e^{2x}\cos 3x$ up to, and including, the term in x^3. (3)

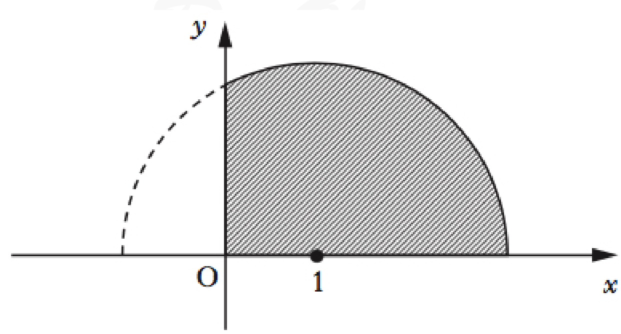
Solution

$\times$	1	$-\frac{9}{2}x^2$
1	1	$-\frac{9}{2}x^2$
$+2x$	$+2x$	$-9x^3$
$+2x^2$	$+2x^2$	$\ldots$
$+\frac{4}{3}x^3$	$+\frac{4}{3}x^3$	$\ldots$

Hence

$$\begin{aligned} e^{2x}\cos 3x &= (1 - \tfrac{9}{2}x^2 + \tfrac{27}{8}x^4 + \ldots)(1 + 2x + 2x^2 + \tfrac{4}{3}x^3 + \ldots) \\ &= \underline{\underline{1 + 2x - \tfrac{5}{2}x^2 - \tfrac{23}{3}x^3 + \ldots}}. \end{aligned}$$

10. A semi-circle with centre $(1, 0)$ and radius 2, lies on the x-axis as shown. (5)



Find the volume of the solid of revolution formed when the shaded region is rotated completely about the x-axis.

Solution

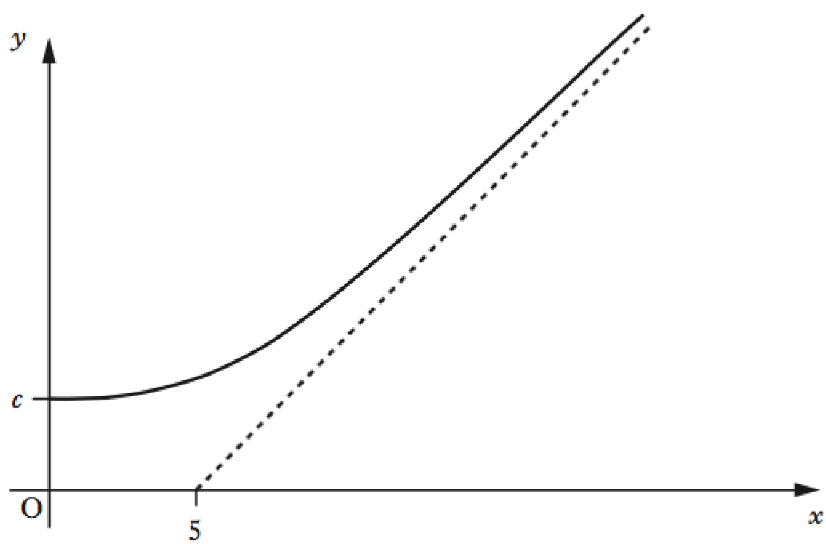
The equation of the (whole) circle is

$$(x-1)^2 + y^2 = 4$$

and

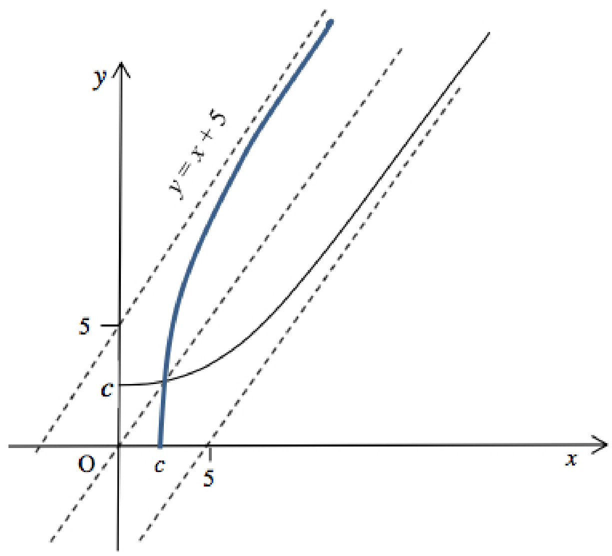
$$\begin{aligned}\text{volume} &= \int_0^3 \pi y^2 \, \mathrm{d}x \\ &= \pi \int_0^3 \left[4 - (x-1)^2\right] \mathrm{d}x \\ &= \pi \left[4x - \tfrac{1}{3}(x-1)^3\right]_{x=0}^3 \\ &= \pi \left[\left(12 - 2\tfrac{3}{3}\right) - \left(0 + \tfrac{1}{3}\right)\right] \\ &= \underline{\underline{9\pi}}.\end{aligned}$$

11. The function $\mathrm{f}(x)$ is defined for all $x \geqslant 0$.
The graph of $y = \mathrm{f}(x)$ intersects the y-axis at $(0, c)$, where $0 < c < 5$.
The graph of the function and its asymptote, $y = x - 5$, are shown below.



(a) Sketch the graph of $y = \mathrm{f}^{-1}(x)$. (4)
Clearly show any points of intersection and any asymptotes.

Solution



It has symmetry, the correct shape and behaviour approaching when it is approaching the asymptote, 5 marked on y-axis at asymptote, and $0 < c < 5$ on x-axis

(b) What is the equation of the asymptote of the graph of $y = \mathrm{f}(x + 2)$? (1)

Solution

$y = x - 3$.

(c) Why does your diagram show that the equation $x = \mathrm{f}(\mathrm{f}(x))$ has at least one solution? (1)

Solution

Well,

$$\mathrm{f}(x) = \mathrm{f}^{-1}(x) \Rightarrow x = \mathrm{f}(\mathrm{f}(x))$$

has at least one solution.

12. Use the substitution $x = \tan\theta$ to determine the exact value of (6)

$$\int_0^1 \frac{1}{(1 + x^2)^{\frac{3}{2}}} \, \mathrm{d}x.$$

Solution

$$\begin{aligned} x = \tan\theta &\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta \\ &\Rightarrow \mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta \end{aligned}$$

and

$$\begin{aligned} x = 0 &\Rightarrow \theta = 0, \\ x = 1 &\Rightarrow \theta = \tfrac{1}{4}\pi. \end{aligned}$$

Now,

$$\begin{aligned} \int_0^1 \frac{1}{(1+x^2)^{\frac{3}{2}}}\,\mathrm{d}x &= \int_0^{\frac{1}{4}\pi} \frac{\sec^2\theta}{(1+\tan^2\theta)^{\frac{3}{2}}}\,\mathrm{d}\theta \\ &= \int_0^{\frac{1}{4}\pi} \frac{\sec^2\theta}{(\sec^2\theta)^{\frac{3}{2}}}\,\mathrm{d}\theta \\ &= \int_0^{\frac{1}{4}\pi} \frac{\sec^2\theta}{\sec^3\theta}\,\mathrm{d}\theta \\ &= \int_0^{\frac{1}{4}\pi} \frac{1}{\sec\theta}\,\mathrm{d}\theta \\ &= \int_0^{\frac{1}{4}\pi} \cos\theta\,\mathrm{d}\theta \\ &= \left[\sin\theta\right]_{\theta=0}^{\frac{1}{4}\pi} \\ &= \tfrac{\sqrt{2}}{2} - 0 \\ &= \underline{\underline{\tfrac{\sqrt{2}}{2}}}. \end{aligned}$$

13. The fuel efficiency, $F(x)$, in km per litre, of a vehicle varies with its speed, s km per hour, and for a particular vehicle the relationship is thought to be (10)

$$F(x) = 15 + \mathrm{e}^x(\sin x - \cos x - \sqrt{2}),$$

where

$$x = \frac{\pi(s-40)}{80},$$

for speeds in the range $40 \leqslant s \leqslant 120$ km per hour.

What is the greatest and least efficiency over the range and at what speeds do they occur?

Solution

$$\begin{aligned} & F(x) = 15 + \mathrm{e}^x(\sin x - \cos x - \sqrt{2}) \\ \Rightarrow\quad & F'(x) = \mathrm{e}^x(\sin x - \cos x - \sqrt{2}) + \mathrm{e}^x(\cos x + \sin x) \\ \Rightarrow\quad & F'(x) = \mathrm{e}^x[(\sin x - \cos x - \sqrt{2}) + (\cos x + \sin x)] \\ \Rightarrow\quad & F'(x) = \mathrm{e}^x(2\sin x - \sqrt{2}) \end{aligned}$$

and

$$\begin{aligned} F'(x) = 0 &\Rightarrow \mathrm{e}^x(2\sin x - \sqrt{2}) = 0 \\ &\Rightarrow \sin x = \tfrac{\sqrt{2}}{2} \\ &\Rightarrow x = \tfrac{1}{4}\pi,\ \tfrac{3}{4}\pi. \end{aligned}$$

Now,

$$F(\tfrac{1}{4}\pi) = 11.898\,233\,61 \text{ (FCD)}$$

and

$$F(\tfrac{3}{4}\pi) = 15 \text{ (exactly!)}.$$

Next,

$$\begin{aligned} s = 40 &\Rightarrow x = 0 \\ &\Rightarrow F(0) = 14 - \sqrt{2} \end{aligned}$$

and

$$\begin{aligned} s = 120 &\Rightarrow x = \pi \\ &\Rightarrow F(\pi) = 5.414\,811\,269 \text{ (FCD)}. \end{aligned}$$

Now,

$$\begin{aligned} x = \tfrac{3}{4}\pi &\Rightarrow \tfrac{3}{4}\pi = \frac{\pi(s-40)}{80} \\ &\Rightarrow \tfrac{3}{4} = \frac{s-40}{80} \\ &\Rightarrow 60 = s - 40 \\ &\Rightarrow s = 100. \end{aligned}$$

Hence, the least efficiency 5.41 km/litre (3 sf) at 120 km/h and the greatest efficiency 15 km/litre at 100 km/h.

14. (a) (i) Given the series (1)

$$1 + r + r^2 + r^3 + \dots,$$

write down the sum to infinity when $|r| < 1$.

Solution

$$1 + r + r^2 + r^3 + \dots = \underline{\underline{\frac{1}{1-r}}}.$$

(ii) Hence obtain an infinite geometric series for (2)

$$\frac{1}{2-3r}.$$

Solution

$$\begin{aligned}\frac{1}{2-3r} &= \frac{1}{2}(1 - \tfrac{3}{2}r) \\ &= \tfrac{1}{2}\left[1 + (\tfrac{3}{2}r) + (\tfrac{3}{2}r)^2 + (\tfrac{3}{2}r)^3 + \dots\right] \\ &= \tfrac{1}{2}\left[1 + \tfrac{3}{2}r + \tfrac{9}{4}r^2 + \tfrac{27}{8}r^3 + \dots\right] \\ &= \underline{\underline{\tfrac{1}{2} + \tfrac{3}{4}r + \tfrac{9}{8}r^2 + \tfrac{27}{16}r^3 + \dots}}.\end{aligned}$$

(iii) For what values of r is this series valid? (1)

Solution

$$|\tfrac{3}{2}r| < 1 \Rightarrow \underline{\underline{|r| < \tfrac{2}{3}}}.$$

(b) (i) Express (3)

$$\frac{1}{3r^2 - 5r + 2}$$

in partial fractions.

Solution

$$\left.\begin{array}{ll}\text{add to:} & -5 \\ \text{multiply to:} & (+3) \times (+2) = +6\end{array}\right\} -3,\ -2$$

$$\begin{aligned}3r^2 - 5r + 2 &= 3r^2 - 3r - 2r + 2 \\ &= 3r(r-1) - 2(r-1) \\ &= (3r-2)(r-1).\end{aligned}$$

Now,

$$\begin{aligned}\frac{1}{3r^2-5r+2} &\equiv \frac{1}{(3r-2)(r-1)}\\ &\equiv \frac{A}{3r-2}+\frac{B}{r-1}\\ &\equiv \frac{A(r-1)+B(3r-2)}{(3r-2)(r-1)}\end{aligned}$$

for some A and B and hence

$$1 \equiv A(r-1)+B(3r-2).$$

$\underline{r=1}$: $1=B$.
$\underline{r=\frac{2}{3}}$: $1=-\frac{1}{3}A \Rightarrow A=-3$.
Hence,

$$\frac{1}{3r^2-5r+2} = \underline{\underline{-\frac{3}{3r-2}+\frac{1}{r-1}}}.$$

(ii) Hence, or otherwise, determine the first three terms of an infinite series for (2)

$$\frac{1}{3r^2-5r+2}.$$

Solution

$$\begin{aligned}\frac{1}{3r^2-5r+2} &= -\frac{3}{3r-2}+\frac{1}{r-1}\\ &= \frac{3}{2-3r}-\frac{1}{1-r}\\ &= 3\left(\tfrac{1}{2}+\tfrac{3}{4}r+\tfrac{9}{8}r^2+\ldots\right)-(1+r+r^2+\ldots)\\ &= \left(\tfrac{3}{2}+\tfrac{9}{4}r+\tfrac{27}{8}r^2+\ldots\right)-(1+r+r^2+\ldots)\\ &= \underline{\underline{\tfrac{1}{2}+\tfrac{5}{4}r+\tfrac{19}{8}r^2+\ldots}}.\end{aligned}$$

(iii) For what values of r does the series converge? (1)

Solution
Well,

$$|r|<1 \textbf{ and } |r|<\tfrac{2}{3}$$

which means $\underline{\underline{|r| < \frac{2}{3}}}$.

15. (a) Use integration by parts to obtain an expression for (4)

$$\int \mathrm{e}^x \cos x \, \mathrm{d}x.$$

Solution

$$u = \mathrm{e}^x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x \Rightarrow v = \sin x.$$

Now,

$$\int \mathrm{e}^x \cos x \, \mathrm{d}x = \mathrm{e}^x \sin x - \int \mathrm{e}^x \sin x \, \mathrm{d}x$$

$$u = \mathrm{e}^x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \Rightarrow v = -\cos x$$

$$= \mathrm{e}^x \sin x - (-\mathrm{e}^x \cos x + \int \mathrm{e}^x \cos x \, \mathrm{d}x)$$
$$= \mathrm{e}^x \sin x + \mathrm{e}^x \cos x - \int \mathrm{e}^x \cos x \, \mathrm{d}x.$$

Next,

$$\int \mathrm{e}^x \cos x \, \mathrm{d}x = \mathrm{e}^x \sin x + \mathrm{e}^x \cos x - \int \mathrm{e}^x \cos x \, \mathrm{d}x$$
$$\Rightarrow \quad 2\int \mathrm{e}^x \cos x \, \mathrm{d}x = \mathrm{e}^x \sin x + \mathrm{e}^x \cos x$$
$$\Rightarrow \quad \int \mathrm{e}^x \cos x \, \mathrm{d}x = \underline{\underline{\tfrac{1}{2}\mathrm{e}^x(\sin x + \cos x) + c}}.$$

(b) Similarly, given (4)

$$I_n = \int e^x \cos nx \, dx, \text{ where } n \neq 0,$$

obtain an expression for I_n.

Solution

$$u = e^x \Rightarrow \frac{du}{dx} = e^x$$
$$\frac{dv}{dx} = \cos nx \Rightarrow v = \tfrac{1}{n} \sin nx.$$

Now,

$$\int e^x \cos x \, dx = \tfrac{1}{n} e^x \sin nx - \tfrac{1}{n} \int e^x \sin nx \, dx$$

$$u = e^x \Rightarrow \frac{du}{dx} = e^x$$
$$\frac{dv}{dx} = \sin nx \Rightarrow v = -\tfrac{1}{n} \cos nx$$

$$= \tfrac{1}{n} e^x \sin nx - \tfrac{1}{n} \left(-\tfrac{1}{n} e^x \cos nx + \tfrac{1}{n} \int e^x \cos nx \, dx \right)$$
$$= \tfrac{1}{n} e^x \sin nx + \tfrac{1}{n^2} e^x \cos nx - \tfrac{1}{n^2} \int e^x \cos nx \, dx.$$

Next,

$$\int e^x \cos nx \, dx = \tfrac{1}{n} e^x \sin nx + \tfrac{1}{n^2} e^x \cos nx - \tfrac{1}{n^2} \int e^x \cos nx \, dx$$
$$\Rightarrow \quad (1 + \tfrac{1}{n^2}) \int e^x \cos nx \, dx = \tfrac{1}{n} e^x \sin nx + \tfrac{1}{n^2} e^x \cos nx$$
$$\Rightarrow \quad \tfrac{n^2+1}{n^2} \int e^x \cos nx \, dx = \tfrac{1}{n^2} e^x (n \sin nx + \cos nx)$$
$$\Rightarrow \quad \int e^x \cos nx \, dx = \underline{\underline{\tfrac{1}{n^2+1} e^x (n \sin nx + \cos nx) + c}}.$$

(c) Hence evaluate (2)

$$\int_0^{\frac{1}{2}\pi} e^x \cos 8x \, dx.$$

Solution

In the case, $n = 8$:

$$\begin{aligned}\int_0^{\frac{1}{2}\pi} \mathrm{e}^x \cos 8x \,\mathrm{d}x &= \tfrac{1}{65}\left[\mathrm{e}^x(8\sin 8x + \cos 8x)\right]_{x=0}^{\frac{1}{2}\pi} \\ &= \tfrac{1}{65}\left[\left(\mathrm{e}^{\frac{1}{2}\pi}(0+1)\right) - (0+1)\right] \\ &= \underline{\underline{\tfrac{1}{65}\left(\mathrm{e}^{\frac{1}{2}\pi} - 1\right)}}.\end{aligned}$$

16. (a) Express -1 as a complex number in polar form and hence determine the solutions to the equation $z^4 + 1 = 0$. (3)

Solution

$$-1 = \underline{\underline{1(\cos\pi + \mathrm{i}\sin\pi)}}.$$

Now,

$$\begin{aligned}z^4 = -1 &\Rightarrow z^4 = \cos\pi + \mathrm{i}\sin\pi \\ &\Rightarrow z^4 = \cos(\pi + 2n\pi) + \mathrm{i}\sin(\pi + 2n\pi) \\ &\Rightarrow z = \cos\tfrac{(2n+1)}{4}\pi + \mathrm{i}\sin\tfrac{(2n+1)}{4}\pi.\end{aligned}$$

$\underline{n = 0}$:

$$\underline{\underline{z = \cos\tfrac{1}{4}\pi + \mathrm{i}\sin\tfrac{1}{4}\pi}}.$$

$\underline{n = 1}$:

$$\underline{\underline{z = \cos\tfrac{3}{4}\pi + \mathrm{i}\sin\tfrac{3}{4}\pi}}.$$

$\underline{n = 2}$:

$$z = \cos\tfrac{5}{4}\pi + \mathrm{i}\sin\tfrac{5}{4}\pi = \underline{\underline{\cos\tfrac{3}{4}\pi - \mathrm{i}\sin\tfrac{3}{4}\pi}}.$$

$\underline{n = 3}$:

$$z = \cos\tfrac{7}{4}\pi + \mathrm{i}\sin\tfrac{7}{4}\pi = \underline{\underline{\cos\tfrac{1}{4}\pi - \mathrm{i}\sin\tfrac{1}{4}\pi}}.$$

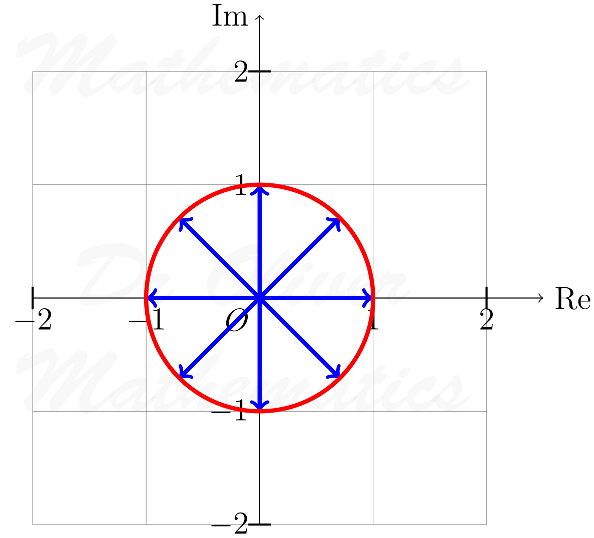
(b) Write down the four solutions to the equation $z^4 - 1 = 0$. (2)

Solution

1, -1, i, and $-\text{i}$.

(c) Plot the solutions of both equations on an Argand diagram. (1)

Solution



(d) Show that the solutions of $z^4 + 1 = 0$ and the solutions of $z^4 - 1 = 0$ are also solutions of the equation $z^8 - 1 = 0$. (2)

Solution

$$\begin{aligned} z^8 - 1 = 0 &\Rightarrow z^8 - z^4 + z^4 - 1 = 0 \\ &\Rightarrow z^4(z^4 - 1) + (z^4 - 1) = 0 \\ &\Rightarrow (z^4 + 1)(z^4 - 1) = 0; \end{aligned}$$

hence, the solutions to $z^4 + 1 = 0$ and $z^4 - 1 = 0$ are also the solutions to $z^8 - 1 = 0$.

(e) Hence identify all the solutions to the equation (2)

$$z^6 + z^4 + z^2 + 1 = 0.$$

Solution

$$\begin{aligned} z^6 + z^4 + z^2 + 1 = 0 &\Rightarrow z^4(z^2+1) + z^2 + 1 = 0 \\ &\Rightarrow (z^4+1)(z^2+1) = 0; \end{aligned}$$

hence, the solutions are

$$\underline{\underline{\pm \mathrm{i},\ \cos \tfrac{1}{4}\pi \pm \mathrm{i} \sin \tfrac{1}{4}\pi,\ \cos \tfrac{3}{4}\pi \pm \mathrm{i} \sin \tfrac{3}{4}\pi}}.$$