Dr Oliver Mathematics
Advanced Level: Pure Mathematics 2
June 2022: Calculator
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.
Inexact answers should be given to three significant figures unless otherwise stated.

1. **In this question you must show all stages of your working.** (4)
   Solutions relying entirely on calculator technology are not acceptable.

   Figure 1 shows a sketch of the graph with equation

$$y = |3 - 2x|.$$

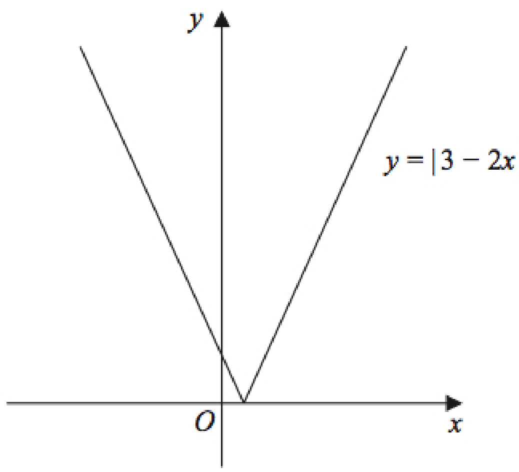


Figure 1: $y = |3 - 2x|$

Solve

$$|3 - 2x| = 7 + x.$$

Solution

$$\begin{aligned} 3 - 2x = 7 + x &\Rightarrow -3x = 4 \\ &\Rightarrow x = -1\tfrac{1}{3} \end{aligned}$$

and

$$-(3-2x) = 7+x \Rightarrow -3+2x = 7+x$$
$$x = 10;$$

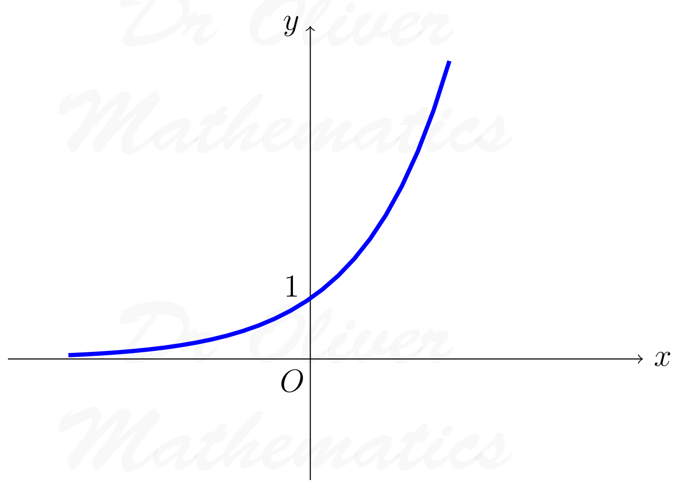
hence, the solutions are $\underline{\underline{x = -1\frac{1}{3}}}$ or $\underline{\underline{x = 10}}$.

2. (a) Sketch the curve with equation (2)

$$y = 4^x,$$

stating any points of intersection with the coordinate axes.

Solution



(b) Solve (2)

$$4^x = 100,$$

giving your answer to 2 decimal places.

Solution

Well, you can use any log that you like:

$$\begin{aligned} 4^x = 100 &\Rightarrow \log 4^x = \log 100 \\ &\Rightarrow x \log 4 = \log 100 \\ &\Rightarrow x = \frac{\log 100}{\log 4} \\ &\Rightarrow x = 3.321\,928\,095 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 3.32 \text{ (2 dp)}}}. \end{aligned}$$

3. A sequence of terms a_1, a_2, a_3, ... is defined by

$$\begin{aligned} a_1 &= 3 \\ a_{n+1} &= 8 - a_n. \end{aligned}$$

(a) (i) Show that this sequence is periodic. (2)

Solution
Well,

$$\begin{aligned} a_1 &= 3 \\ a_2 &= 8 - 3 = 5 \\ a_3 &= 8 - 5 = 3, \end{aligned}$$

so it is periodic ...

(ii) State the order of this periodic sequence.

Solution
... of order $\underline{\underline{2}}$.

(b) Find the value of (2)

$$\sum_{n=1}^{85} a_n.$$

Solution
Well, we have

$$\underbrace{(3+5) + (3+5) + \ldots + (3+5)}_{42 \text{ times}} + 3$$

so

$$\sum_{n=1}^{85} a_n = 42(3+5) + 3$$
$$= 336 + 3$$
$$= \underline{\underline{339}}.$$

4. Given that (3)

$$y = 2x^2,$$

use differentiation from first principles to show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x.$$

Solution

$\times$	x	$+h$
x	x^2	$+hx$
$+h$	$+hx$	$+h^2$

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0} \frac{2(x+h)^2 - 2x^2}{h}$$
$$= \lim_{h\to 0} \frac{2(x^2+2hx+h^2) - 2x^2}{h}$$
$$= \lim_{h\to 0} \frac{(2x^2+4hx+2h^2) - 2x^2}{h}$$
$$= \lim_{h\to 0} \frac{4hx+2h^2}{h}$$
$$= \lim_{h\to 0} (4x+2h)$$
$$= \underline{\underline{4x}},$$

as required.

5. The table below shows corresponding values of x and y for

$$y = \log_3 2x.$$

The values of y are given to 2 decimal places as appropriate.

x	3	4.5	6	7.5	9
y	1.63	2	2.26	2.46	2.63

(a) Using the trapezium rule with all the values of y in the table, find an estimate for (3)

$$\int_3^9 \log_3 2x \, \mathrm{d}x.$$

Solution

$$\begin{aligned} \text{Area} &= \tfrac{1}{2}(1.5)\left[1.63 + 2(2 + 2.26 + 2.46) + 2.63\right] \\ &= 0.75 \times 17.7 \\ &= 13.275 \text{ (exact!)} \\ &= \underline{\underline{13.28 \text{ (2 dp)}}}. \end{aligned}$$

Using your answer to part (a) and making your method clear, estimate

(b) (i) $\displaystyle\int_3^9 \log_3 (2x)^{10} \, \mathrm{d}x$, (3)

Solution

$$\begin{aligned} \int_3^9 \log_3 (2x)^{10} \, \mathrm{d}x &= \int_3^9 10 \log_3 (2x) \, \mathrm{d}x \\ &= 10 \times 13.275 \\ &= 132.75 \text{ (exact!)} \\ &= \underline{\underline{132.8 \text{ (1 dp)}}}. \end{aligned}$$

(ii) $\displaystyle\int_3^9 \log_3 18x \, \mathrm{d}x$

Solution

$$\begin{aligned}\int_3^9 \log_3 18x \,\mathrm{d}x &= \int_3^9 \log_3[9(2x)] \,\mathrm{d}x \\ &= \int_3^9 \log_3 9 \,\mathrm{d}x + \int_3^9 \log_3(2x) \,\mathrm{d}x \\ &= \int_3^9 \log_3 3^2 \,\mathrm{d}x + 13.275 \\ &= \int_3^9 2 \,\mathrm{d}x + 13.275 \\ &= 2(9-3) + 13.275 \\ &= 12 + 13.275 \\ &= 25.275 \text{ (exact!)} \\ &= \underline{\underline{25.28 \text{ (2 dp)}}}.\end{aligned}$$

6. Figure 2 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 8\sin(\tfrac{1}{2}x) - 3x + 9,\ x > 0,$$

and x is measured in radians.

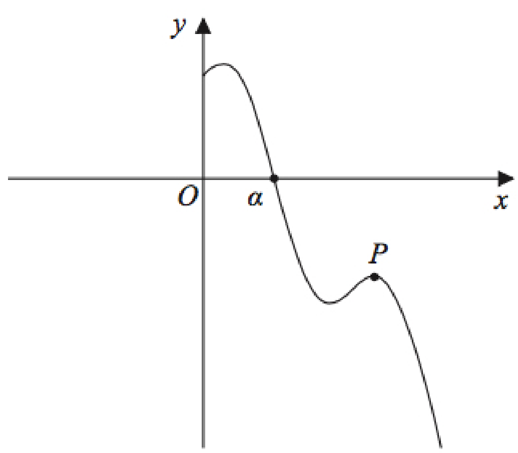


Figure 2: $\mathrm{f}(x) = 8\sin(\frac{1}{2}x) - 3x + 9$, $x > 0$

The point P, shown in Figure 2, is a local maximum point on the curve.

Using calculus and the sketch in Figure 2,

(a) find the x-coordinate of P, giving your answer to 3 significant figures. (4)

Solution
Well,

$$f(x) = 8\sin(\tfrac{1}{2}x) - 3x + 9 \Rightarrow f'(x) = 4\cos(\tfrac{1}{2}x) - 3$$

and

$$\begin{aligned} f'(x) = 0 &\Rightarrow 4\cos(\tfrac{1}{2}x) - 3 = 0 \\ &\Rightarrow 4\cos(\tfrac{1}{2}x) = 3 \\ &\Rightarrow \cos(\tfrac{1}{2}x) = \tfrac{3}{4} \\ &\Rightarrow \tfrac{1}{2}x = 0.722\,734\,247\,8,\ 5.560\,451\,059,\ 7.005\,919,555\ \text{(FCD)} \\ &\Rightarrow x = 1.445\,468\,496,\ 11.129\,902\,12,\ 14.011\,839,11\ \text{(FCD)} \end{aligned}$$

but it is the third such point (why?)

$$\Rightarrow \underline{\underline{x = 14.0\ \text{(3 sf)}}}.$$

The curve crosses the x-axis at $x = \alpha$, as shown in Figure 2.

Given that, to 3 decimal places,

$$f(4) = 4.274 \text{ and } f(5) = -1.212,$$

(b) explain why α must lie in the interval $[4, 5]$. (1)

Solution
Well, $f(4) > 0$, $f(5) < 0$, and the function is continuous so α must lie in the interval $[4, 5]$.

(c) Taking $x_0 = 5$ as a first approximation to α, apply the Newton-Raphson method once to $f(x)$ to obtain a second approximation to α.
Show your method and give your answer to 3 significant figures. (2)

Solution

Newton-Raphson:

$$\begin{aligned} x_1 &= 5 - \frac{8\sin(\frac{1}{2}(5)) - 3(5) + 9}{4\cos(\frac{1}{2}(5)) - 3} \\ &= 4.804\,624\,337 \text{ (FCD)} \\ &= \underline{\underline{4.80 \text{ (3 sf)}}}. \end{aligned}$$

7. (a) Find the first four terms, in ascending powers of x, of the binomial expansion of (4)

$$\sqrt{4-9x},$$

writing each term in simplest form.

Solution

Well,

$$\begin{aligned} \sqrt{4-9x} &= [4 + (-9x)]^{\frac{1}{2}} \\ &= \left[4(1 + (-\tfrac{9}{4}x))\right]^{\frac{1}{2}} \\ &= 2[1 + (-\tfrac{9}{4}x)]^{\frac{1}{2}} \\ &= 2\left[1 + (-\tfrac{9}{4}x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(-\tfrac{9}{4}x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}(-\tfrac{9}{4}x)^3 + \dots\right] \\ &= 2\left[1 - \tfrac{9}{4}x - \tfrac{81}{128}x^2 - \tfrac{729}{1\,024}x^3 + \dots\right] \\ &= \underline{\underline{2 - \tfrac{9}{2}x - \tfrac{81}{64}x^2 - \tfrac{729}{512}x^3 + \dots}}. \end{aligned}$$

A student uses this expansion with $x = \frac{1}{9}$ to find an approximation for $\sqrt{3}$.

Using the answer to part (a) and without doing any calculations,

(b) state whether this approximation will be an overestimate or an underestimate of $\sqrt{3}$, giving a brief reason for your answer. (1)

Solution

As $x > 0$, the approximation will be an <u>overestimate</u> since the second and succeeding terms are all negative.

8. **In this question you must show all stages of your working.** (6)
Solutions relying on calculator technology are not acceptable.

Figure 3 shows a sketch of part of a curve with equation

$$y = \frac{(x-2)(x-4)}{4\sqrt{x}}, \; x > 0.$$

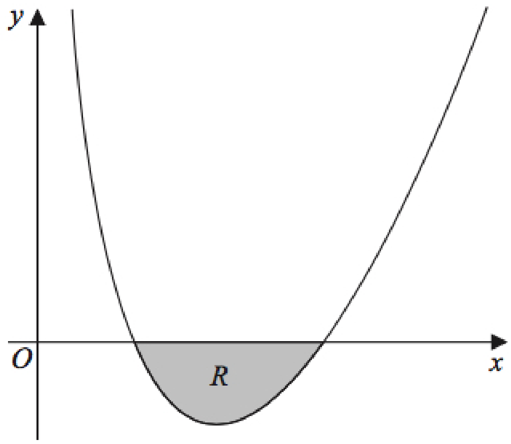


Figure 3: $y = \dfrac{(x-2)(x-4)}{4\sqrt{x}}$, $x > 0$

The region R, shown shaded in Figure 3, is bounded by the curve, and the x-axis.

Find the exact area of R, writing your answer in the form $a\sqrt{2} + b$, where a and b are constants to be found.

Solution

Well,

$$\begin{aligned} y = 0 &\Rightarrow \frac{(x-2)(x-4)}{4\sqrt{x}} = 0 \\ &\Rightarrow (x-2)(x-4) = 0 \\ &\Rightarrow x - 2 = 0 \text{ or } x - 4 = 0 \\ &\Rightarrow x = 2 \text{ or } x = 4. \end{aligned}$$

and

$$y = \frac{(x-2)(x-4)}{4\sqrt{x}}$$

$\times$	x	-4
x	x^2	$-4x$
-2	$-2x$	$+8$

$$\begin{aligned} &= \frac{x^2-6x+8}{4x^{\frac{1}{2}}} \\ &= \tfrac{1}{4}x^{\frac{3}{2}} - \tfrac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}. \end{aligned}$$

Next,

$$\begin{aligned} \int_2^4 (\tfrac{1}{4}x^{\frac{3}{2}} - \tfrac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}})\,\mathrm{d}x &= \left[\tfrac{1}{10}x^{\frac{5}{2}} - x^{\frac{3}{2}} + 4x^{\frac{1}{2}}\right]_{x=2}^4 \\ &= \left(\tfrac{16}{5} - 8 + 8\right) - \left(\tfrac{2}{5}\sqrt{2} - 2\sqrt{2} + 4\sqrt{2}\right) \\ &= \tfrac{16}{5} - \tfrac{12}{5}\sqrt{2}. \end{aligned}$$

Finally,

$$\begin{aligned} \text{area of } R &= -(\tfrac{16}{5} - \tfrac{12}{5}\sqrt{2}) \\ &= \underline{\underline{\tfrac{12}{5}\sqrt{2} - \tfrac{16}{5}}}. \end{aligned}$$

9. Figure 4 shows a sketch of a Ferris wheel.

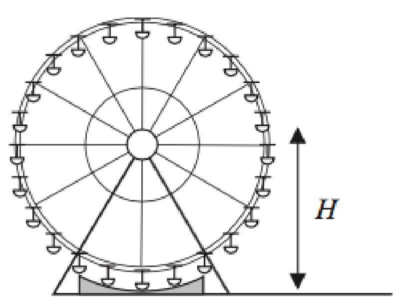


Figure 4: a sketch of a Ferris wheel

The height above the ground, H m, of a passenger on the Ferris wheel, t seconds after the wheel starts turning, is modelled by the equation

$$H = |A \sin(bt + \alpha)^\circ|,$$

where A, b, and α are constants.

Figure 5 shows a sketch of the graph of H against t, for one revolution of the wheel.

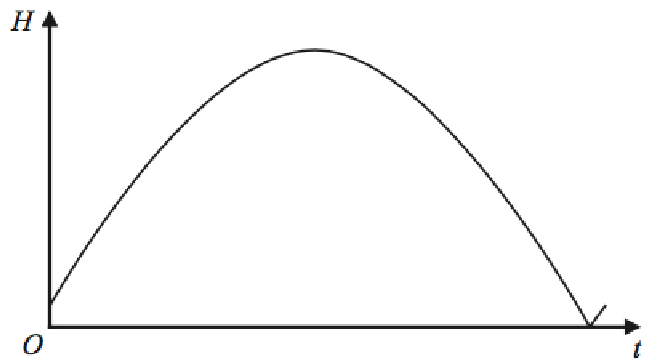


Figure 5: after the wheel starts turning

Given that

- the maximum height of the passenger above the ground is 50 m,
- the passenger is 1 m above the ground when the wheel starts turning, and
- the wheel takes 720 seconds to complete one revolution,

(a) find a complete equation for the model, giving the exact value of A, the exact value of b and the value of α to 3 significant figures. (4)

Solution
Well,

$$\underline{\underline{A = 50}}.$$

b? Well, normally, the sine graph takes 180 s to go up and down (half of the 360 that it takes to do a full cycle) but we are told the graph takes 720 s. Consequently,

$$\underline{\underline{b = \frac{180}{720} = \tfrac{1}{4}}}.$$

Now,

$$t = 0, H = 1 \Rightarrow 1 = |50\sin(\alpha)^\circ|$$
$$\Rightarrow \sin(\alpha)^\circ = \tfrac{1}{50}$$
$$\Rightarrow \alpha = 1.145\,991\,998 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\alpha = 1.15 \text{ (3 sf)}}}.$$

So,

$$\underline{\underline{H = |50\sin(\tfrac{1}{4}t + 1.15)^\circ| \text{ (3 sf)}}}.$$

(b) Explain why an equation of the form (1)

$$H = |A\sin(bt + \alpha)^\circ| + d,$$

where d is a positive constant, would be a more appropriate model.

Solution
E.g., adding d means the passenger does not touch the ground.

10. The function f is defined by

$$\mathrm{f}(x) = \frac{8x+5}{2x+3},\ x > -\tfrac{3}{2}.$$

(a) Find $\mathrm{f}^{-1}(\frac{3}{2})$. (2)

Solution
Well,

$$\tfrac{3}{2} = \frac{8x+5}{2x+3} \Rightarrow 3(2x+3) = 2(8x+5)$$
$$\Rightarrow 6x + 9 = 16x + 10$$
$$\Rightarrow 10x = -1$$
$$\Rightarrow \underline{\underline{x = -\tfrac{1}{10}}}.$$

(b) Show that (2)

$$\mathrm{f}(x) = A + \frac{B}{2x+3},$$

where A and B are constants to be found.

Solution

$$\begin{aligned}\mathrm{f}(x) &= \frac{8x+5}{2x+3} \\ &= \frac{4(2x+3)-7}{2x+3} \\ &= \underline{\underline{4-\frac{7}{2x+3}}};\end{aligned}$$

hence, $\underline{\underline{A=4}}$ and $\underline{\underline{B=-7}}$.

The function $\mathrm{g}(x)$ is defined by

$$\mathrm{g}(x) = 16 - x^2,\ 0 \leqslant x \leqslant 4.$$

(c) State the range of g^{-1}. (1)

Solution

$$\underline{\underline{0 \leqslant \mathrm{g}^{-1}(x) 4}}.$$

$$\begin{aligned} x = 16 - y^2 \Rightarrow y^2 &= 16 - x \\ \Rightarrow y &= \sqrt{16-x};\end{aligned}$$

hence, $\underline{\underline{\{y \in \mathbb{R} : 0 \leqslant y \leqslant 16\}}}$.

(d) Find the range of $\mathrm{f}\,\mathrm{g}^{-1}$. (3)

Solution

Well,

$$\mathrm{f}(0) = \frac{8(0)+5}{2(0)+3} = \tfrac{5}{3}$$

and

$$\mathrm{f}(4) = \frac{8(4)+5}{2(4)+3} = \tfrac{37}{11}$$

and

$$\underline{\underline{\tfrac{5}{3} \leqslant \mathrm{f}\,\mathrm{g}^{-1}(x) \leqslant \tfrac{37}{11}}}.$$

11. Prove, using algebra, that (4)

$$n(n^2 + 5)$$

is even for all $n \in \mathbb{N}$.

Solution

$$n(n^2 + 5) = n^3 + 5n.$$

$\underline{n \text{ even}}$: if $n = 2k$,

$$(2k)[(2k)^2 + 5] = (2k)(4k^2 + 5),$$

which is even.

$\underline{n \text{ odd}}$: if $n = 2k + 1$,

$$\begin{aligned}(2k+1)[(2k+1)^2 + 5] &= (2k+1)[(4k^2 + 4k + 1) + 5] \\ &= (2k+1)(4k^2 + 4k + 6) \\ &= 2(2k+1)(2k^2 + 2k + 3),\end{aligned}$$

which is even.

Hence, $n(n^2 + 5)$ is $\underline{\underline{\text{even}}}$ for all $n \in \mathbb{N}$.

12. The function f is defined by

$$\mathrm{f}(x) = \frac{\mathrm{e}^{3x}}{4x^2 + k},$$

where k is a positive constant.

(a) Show that (3)

$$\mathrm{f}'(x) = (12x^2 - 8x + 3k)\,\mathrm{g}(x),$$

where $\mathrm{g}(x)$ is a function to be found.

Solution

$$u = \mathrm{e}^{3x} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 3\mathrm{e}^{3x}$$

$$v = 4x^2 + k \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 8x$$

Now,

$$\begin{aligned}
f'(x) &= \frac{(4x^2+k)(3e^{3x}) - (8x)(e^{3x})}{(4x^2+k)^2} \\
&= \frac{12x^2e^{3x} - 8xe^{3x} + 3ke^{3x}}{(4x^2+k)^2} \\
&= \frac{e^{3x}(12x^2 - 8x + 3k)}{(4x^2+k)^2} \\
&= \underline{\underline{(12x^2 - 8x + 3k)\frac{e^{3x}}{(4x^2+k)^2}}};
\end{aligned}$$

hence,

$$\underline{\underline{g(x) = \frac{e^{3x}}{(4x^2+k)^2}}}.$$

Given that the curve with equation $y = f(x)$ has at least one stationary point,

(b) find the range of possible values of k. (3)

Solution

If $f(x)$ has at least one stationary point, then

$$12x^2 - 8x + 3k = 0$$

has at least one root. So:

$$\begin{aligned}
b^2 - 4ac \geqslant 0 &\Rightarrow (-8)^2 - 4(12)(3k) \geqslant 0 \\
&\Rightarrow 64 \geqslant 144k \\
&\Rightarrow k \leqslant \tfrac{4}{9}.
\end{aligned}$$

Hence,

$$\underline{\underline{0 < k \leqslant \tfrac{4}{9}}}.$$

13. Relative to a fixed origin O,

- the point A has position vector $4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$,
- the point B has position vector $4\mathbf{j} + 6\mathbf{k}$,
- the point C has position vector $-16\mathbf{i} + p\mathbf{j} + 10\mathbf{k}$,

where p is a constant.

Given that A, B, and C lie on a straight line,

(a) find the value of p. (3)

Solution

Well,

$$\overrightarrow{AB} = k\overrightarrow{AC},$$

for some vale of k:

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= -(4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) + (4\mathbf{j} + 6\mathbf{k}) \\ &= -4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k} + 4\mathbf{j} + 6\mathbf{k} \\ &= -4\mathbf{i} + 7\mathbf{j} + \mathbf{k}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC} \\ &= -\overrightarrow{OA} + \overrightarrow{OC} \\ &= -(4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) + (-16\mathbf{i} + p\mathbf{j} + 10\mathbf{k}) \\ &= -4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k} - 16\mathbf{i} + p\mathbf{j} + 10\mathbf{k} \\ &= -20\mathbf{i} + (3 + p)\mathbf{j} + 5\mathbf{k}.\end{aligned}$$

Well, we can see from the **i** or **k** components that

$$k = \frac{-20}{-4} = 5:$$

$$\begin{aligned}5(7) = 3 + p &\Rightarrow 35 = 3 + p \\ &\Rightarrow \underline{\underline{p = 32}}.\end{aligned}$$

The line segment OB is extended to a point D so that $\overrightarrow{CD}$ is parallel to $\overrightarrow{OA}$,

(b) Find $|\overrightarrow{OD}|$, writing your answer as a fully simplified surd. (3)

Solution

Well,

$$\begin{aligned}\overrightarrow{OD} &= k\overrightarrow{OB} \\ &= k(4\mathbf{j} + 6\mathbf{k}) \\ &= 4k\mathbf{j} + 6k\mathbf{k},\end{aligned}$$

for some value of k. Now,

$$\begin{aligned}\overrightarrow{CD} &= \overrightarrow{CO} + \overrightarrow{OD} \\ &= -\overrightarrow{OC} + \overrightarrow{OD} \\ &= -(-16\mathbf{i} + 32\mathbf{j} + 10\mathbf{k}) + (4k\mathbf{j} + 6k\mathbf{k}) \\ &= 16\mathbf{i} - 32\mathbf{j} - 10\mathbf{k}) + 4k\mathbf{j} + 6k\mathbf{k}) \\ &= 16\mathbf{i} + (4k - 32)\mathbf{j} + (6k - 10)\mathbf{k}.\end{aligned}$$

Next,

$$\overrightarrow{OA} = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k} \Rightarrow 4\overrightarrow{OA} = 16\mathbf{i} - 12\mathbf{j} + 20\mathbf{k}.$$

We will take the **j** component:

$$\begin{aligned}4k - 32 = -12 &\Rightarrow 4k = 20 \\ &\Rightarrow k = 5.\end{aligned}$$

Finally,

$$\begin{aligned}|\overrightarrow{OD}| &= |5(4\mathbf{j} + 6\mathbf{k})| \\ &= 5\sqrt{4^2 + 6^2} \\ &= 5\sqrt{52} \\ &= \underline{\underline{10\sqrt{13}}}.\end{aligned}$$

14. (a) Express (3)

$$\frac{3}{(2x - 1)(x + 1)}$$

in partial fractions.

Solution

Let

$$\frac{3}{(2x-1)(x+1)} = \frac{A}{(2x-1)} + \frac{B}{(x+1)}$$
$$= \frac{A(x+1)+B(2x-1)}{(2x-1)(x+1)}.$$

$\underline{x=-1}$: $3 = -3B \Rightarrow B = -1$.
$\underline{x=\frac{1}{2}}$: $3 = \frac{3}{2}A \Rightarrow A = 2$.
Hence,

$$\frac{3}{(2x-1)(x+1)} = \underline{\underline{\frac{2}{(2x-1)} - \frac{1}{(x+1)}}}.$$

When chemical A and chemical B are mixed, oxygen is produced.

A scientist mixed these two chemicals and measured the total volume of oxygen produced over a period of time.

The total volume of oxygen produced, V m^3, t hours after the chemicals were mixed, is modelled by the differential equation

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{3V}{(2t-1)(t+1)},$$

where k is a constant.

Given that exactly 2 hours after the chemicals were mixed, a total volume of 3 m^3 of oxygen had been produced,

(b) solve the differential equation to show that (5)

$$V = \frac{3(2t-1)}{t+1}.$$

Solution

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{3V}{(2t-1)(t+1)} \Rightarrow \frac{1}{V}\,\mathrm{d}V = \frac{3}{(2t-1)(t+1)}\,\mathrm{d}t$$
$$\Rightarrow \int \frac{1}{V}\,\mathrm{d}V = \int \frac{3}{(2t-1)(t+1)}\,\mathrm{d}t$$
$$\Rightarrow \int \frac{1}{V}\,\mathrm{d}V = \int \left(\frac{2}{2t-1} - \frac{1}{t+1}\right)\mathrm{d}t$$
$$\Rightarrow \ln V = \ln(2t-1) - \ln(t+1) + c,$$

for some constant c. Now,

$$t = 2, V = 3 \Rightarrow \ln 3 = \ln 3 - \ln 3 + c$$
$$\Rightarrow c = \ln 3.$$

Finally,

$$\ln V = \ln(2t-1) - \ln(t+1) + \ln 3 \Rightarrow \ln V = \ln\left[\frac{3(2t-1)}{t+1}\right]$$
$$\Rightarrow \underline{\underline{V = \frac{3(2t-1)}{t+1}}},$$

as required.

The scientist noticed that

- there was a **time delay** between the chemicals being mixed and oxygen being produced and
- there was a **limit** to the total volume of oxygen produced.

Deduce from the model

(c) (i) the **time delay** giving your answer in minutes, (2)

Solution
Well,

$$\frac{3(2t-1)}{t+1} = 0 \Rightarrow 2t - 1 = 0$$
$$\Rightarrow t = \tfrac{1}{2};$$

there is a delay of $\underline{\underline{\text{30 minutes}}}$.

(ii) the **limit** giving your answer in m^3.

Solution
Well,

$$V = \frac{3(2t-1)}{t+1} = \frac{3(2-\frac{1}{t})}{1+\frac{1}{t}}$$

and

$$t \to \infty \Rightarrow V \to \frac{3(2-0)}{1+0} \\ \Rightarrow V \to 6;$$

the limit is $\underline{\underline{6 \text{ m}^3}}$.

15. **In this question you must show all stages of your working. Solutions relying on calculator technology are not acceptable.**

Given that the first three terms of a geometric series are

$$12\cos\theta,\ 5+2\sin\theta,\ \text{and}\ 6\tan\theta,$$

(a) show that (3)

$$4\sin^2\theta - 52\sin\theta + 25 = 0.$$

Solution
Well,

$$\frac{5+2\sin\theta}{12\cos\theta} = \frac{6\tan\theta}{5+2\sin\theta}$$
$$\Rightarrow\ (5+2\sin\theta)^2 = (6\tan\theta)(12\cos\theta)$$

$\times$	x	$+2\sin\theta$
5	25	$+10\sin\theta$
$+2\sin\theta$	$+10\sin\theta$	$+4\sin^2\theta$

$$\Rightarrow\ 25 + 20\sin\theta + 4\sin^2\theta = 72\sin^2\theta$$
$$\Rightarrow\ \underline{\underline{4\sin^2\theta - 52\sin\theta + 25 = 0}},$$

as required.

Given that θ is an obtuse angle measured in radians,

(b) solve the equation in part (a) to find the exact value of θ, (2)

Solution

$$\left.\begin{array}{ll}\text{add to:} & -52 \\ \text{multiply to:} & (+4)\times(+25) = +100\end{array}\right\} -50,\ -2$$

Now,

$$\begin{aligned}4\sin^2\theta - 52\sin\theta + 25 = 0 &\Rightarrow 4\sin^2\theta - 50\sin\theta - 2\sin\theta + 25 = 0 \\ &\Rightarrow 2\sin\theta(2\sin\theta - 25) - (2\sin\theta - 25) = 0 \\ &\Rightarrow (2\sin\theta - 1)(2\sin\theta - 25) = 0 \\ &\Rightarrow 2\sin\theta - 1 = 0 \text{ or } 2\sin\theta - 25 = 0 \\ &\Rightarrow \sin\theta = \tfrac{1}{2} = 0 \text{ or } \sin\theta = \tfrac{25}{2} \text{ (no!)}.\end{aligned}$$

As θ is an obtuse angle,

$$\underline{\underline{\theta = \tfrac{5}{6}\pi}}.$$

(c) show that the sum to infinity of the series can be expressed in the form (5)

$$k(1 - \sqrt{3}),$$

where k is a constant to be found.

Solution

The first term, a, is

$$12\cos(\tfrac{5}{6}\pi) = -6\sqrt{3}$$

and the common ratio,

$$r = \frac{5 + 2\sin(\frac{5}{6}\pi)}{12\cos(\frac{5}{6}\pi)} = -\tfrac{1}{3}\sqrt{3}.$$

Hence,

$$\begin{aligned} S_\infty &= \frac{-6\sqrt{3}}{1-(-\frac{1}{3}\sqrt{3})} \\ &= \frac{-6\sqrt{3}}{1+\frac{1}{3}\sqrt{3}} \\ &= \frac{-18\sqrt{3}}{3+\sqrt{3}} \\ &= \frac{-18\sqrt{3}}{3+\sqrt{3}} \times \frac{3-\sqrt{3}}{3-\sqrt{3}} \\ &= \frac{54-54\sqrt{3}}{9-3} \\ &= \frac{54(1-\sqrt{3})}{6} \\ &= \underline{\underline{9(1-\sqrt{3})}}; \end{aligned}$$

hence, $\underline{\underline{k=9}}$.

16. Figure 6 shows a sketch of the curve C with parametric equations

$$x = 2\tan t + 1,\ y = 2\sec^2 t + 3,\ -\tfrac{1}{4}\pi \leqslant t \leqslant \tfrac{1}{3}\pi.$$

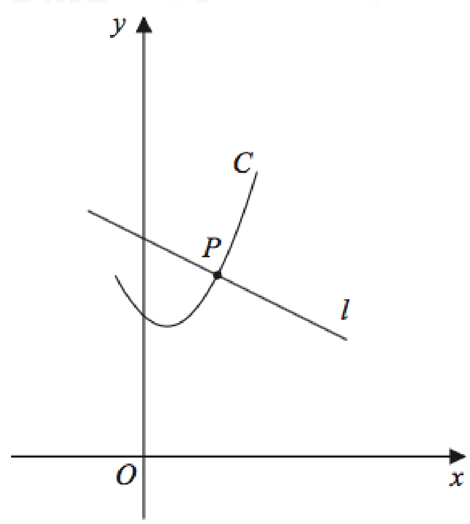


Figure 6: $x = 2\tan t + 1$, $y = 2\sec^2 t + 3$

The line l is the normal to C at the point P where $t = \frac{1}{4}\pi$.

(a) Using parametric differentiation, show that an equation for l is (5)

$$y = -\tfrac{1}{2}x + \tfrac{17}{2}.$$

Solution

$$x = 2\tan t + 1 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 2\sec^2 t$$
$$y = 2\sec^2 t + 3 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 4\sec t(\sec t\tan t) = 4\sec^2 t\tan t.$$

Now,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{4\sec^2 t\tan t}{2\sec^2 t} \\ &= 2\tan t.\end{aligned}$$

Next,

$$t = \tfrac{1}{4}\pi \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2\tan(\tfrac{1}{4}\pi) = 2.$$

Now,

$$\begin{aligned}x &= 2\tan(\tfrac{1}{4}\pi) + 1 = 3, \\ y &= 2\sec^2(\tfrac{1}{4}\pi) + 3 = 4 + 3 = 7, \\ m_{\text{normal}} &= -\tfrac{1}{2}.\end{aligned}$$

Finally, an equation for l is

$$\begin{aligned}y - 7 = -\tfrac{1}{2}(x - 3) &\Rightarrow y - 7 = -\tfrac{1}{2}x + \tfrac{3}{2} \\ &\Rightarrow \underline{\underline{y = -\tfrac{1}{2}x + \tfrac{17}{2}}},\end{aligned}$$

as required

(b) Show that all points on C satisfy the equation (2)

$$y = \tfrac{1}{2}(x - 1)^2 + 5.$$

Solution

Well,

$$x = 2\tan t + 1 \Rightarrow x - 1 = 2\tan t$$
$$\Rightarrow \tan t = \frac{x-1}{2}$$

and

$$y = 2\sec^2 t + 3 \Rightarrow y - 3 = 2\sec^2 t$$
$$\Rightarrow \sec^2 t = \frac{y-3}{2}.$$

Now,

$$\sec^2 t = 1 + \tan^2 t \Rightarrow \frac{y-3}{2} = 1 + \left(\frac{x-1}{2}\right)^2$$
$$\Rightarrow \frac{y-3}{2} = 1 + \tfrac{1}{4}(x-1)^2$$
$$\Rightarrow y - 3 = 2 + \tfrac{1}{2}(x-1)^2$$
$$\Rightarrow \underline{\underline{y = \tfrac{1}{2}(x-1)^2 + 5}},$$

as required.

The straight line with equation

$$y = \tfrac{1}{2}x + k,$$

where k is a constant, intersects C at two distinct points.

(c) Find the range of possible values for k. (5)

Solution