Dr Oliver Mathematics
Advance Level Mathematics
AS Pure Mathematics: Calculator
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. The line l_1 has equation

$$2x + 4y - 3 = 0$$

and the line l_2 has equation

$$y = mx + 7,$$

where m is a constant.

Given that l_1 and l_2 are perpendicular,

(a) find the value of m. (2)

Solution
For l_1,

$$\begin{aligned} 2x + 4y - 3 = 0 &\Rightarrow 4y = -2x + 3 \quad (1) \\ &\Rightarrow y = -\tfrac{1}{2}x + \tfrac{3}{4}. \end{aligned}$$

Now, given that l_1 and l_2 are perpendicular,

$$\begin{aligned} (-\tfrac{1}{2}) \times m = -1 &\Rightarrow m = \frac{-1}{-\frac{1}{2}} \\ &\Rightarrow \underline{\underline{m = 2}}. \end{aligned}$$

The lines l_1 and l_2 meet at the point P.

(b) Find the x-coordinate of P. (2)

Solution
For l_2,

$$y = 2x + 7 \Rightarrow 4y = 8x + 28 \quad (2)$$

and we add together (2) − (1):

$$\begin{aligned} 0 = 10x + 25 &\Rightarrow 10x = -25 \\ &\Rightarrow \underline{\underline{x = -2\tfrac{1}{2}}}. \end{aligned}$$

2. Find, using algebra, all real solutions to the equation

(a) $16a^2 = 2\sqrt{a}$, (4)

Solution

$$\begin{aligned} 16a^2 = 2\sqrt{a} &\Rightarrow 16a^2 - 2a^{\frac{1}{2}} = 0 \\ &\Rightarrow 2a^{\frac{1}{2}}(8a^{\frac{3}{2}} - 1) = 0 \\ &\Rightarrow 2a^{\frac{1}{2}} = 0 \text{ or } 8a^{\frac{3}{2}} = 1 \\ &\Rightarrow a = 0 \text{ or } a^{\frac{3}{2}} = \tfrac{1}{8} \\ &\Rightarrow \underline{\underline{a = 0}} \text{ or } \underline{\underline{a = \tfrac{1}{4}}}. \end{aligned}$$

(b) $b^4 + 7b^2 - 18 = 0$. (4)

Solution

$$\left.\begin{array}{ll} \text{add to:} & +7 \\ \text{multiply to:} & -18 \end{array}\right\} -2, +9$$

$$\begin{aligned} b^4 + 7b^2 - 18 = 0 &\Rightarrow (b^2)^2 + 7(b^2) - 18 = 0 \\ &\Rightarrow (b^2 - 2)(b^2 + 9) = 0 \\ &\Rightarrow b^2 = 2 \text{ (as } b^2 \neq -9) \\ &\Rightarrow \underline{\underline{b = \pm\sqrt{2}}}. \end{aligned}$$

3. (a) Given that k is a constant, find (3)

$$\int \left(\frac{4}{x^3} + kx \right) \mathrm{d}x,$$

simplifying your answer.

Solution

$$\int \left(\frac{4}{x^3} + kx \right) \mathrm{d}x \Rightarrow \int \left(4x^{-3} + kx \right) \mathrm{d}x$$
$$= \underline{\underline{-2x^{-2} + \tfrac{1}{2}kx^2 + c}}.$$

(b) Hence, find the value of k such that (3)

$$\int_{0.5}^{2} \left(\frac{4}{x^3} + kx \right) \mathrm{d}x = 8.$$

Solution

$$\begin{aligned}\int_{0.5}^{2} \left(\frac{4}{x^3} + kx \right) \mathrm{d}x = 8 &\Rightarrow \left[-2x^{-2} + \tfrac{1}{2}kx^2 \right]_{x=0.5}^{2} = 8 \\ &\Rightarrow \left(-\tfrac{1}{2} + 2k \right) - \left(-8 + \tfrac{1}{8}k \right) = 8 \\ &\Rightarrow \tfrac{15}{8}k = \tfrac{1}{2} \\ &\Rightarrow \underline{\underline{k = \tfrac{4}{15}}}.\end{aligned}$$

4. A tree was planted in the ground.
Its height, H metres, was measured t years after planting.

Exactly 3 years after planting, the height of the tree was 2.35 metres.
Exactly 6 years after planting, the height of the tree was 3.28 metres.

Using a linear model,

(a) find an equation linking H with t. (3)

Solution

Using a linear model,

$$H = a + bt,$$

for some constant a and b. Now,

$$\begin{aligned} 2.35 &= a + 3b \quad (1) \\ 3.28 &= a + 6b \quad (2) \end{aligned}$$

Subtract (2) − (1):

$$0.93 = 3b \Rightarrow b = 0.31$$
$$\Rightarrow a = 1.42;$$

hence,

$$\underline{\underline{H = 1.42 + 0.31t}}.$$

The height of the tree was approximately 140 cm when it was planted.

(b) Explain whether or not this fact supports the use of the linear model in part (a). (2)

Solution

The linear model gives

$$t = 0 \Rightarrow H = 1.42 \text{ m} = 142 \text{ cm}$$

which is close enough to 140 cm and this supports the use of the linear model.

5. A curve has equation

$$y = 3x^2 + \frac{24}{x} + 2,\ x > 0.$$

(a) Find, in simplest form, $\frac{\mathrm{d}y}{\mathrm{d}x}$. (3)

Solution

$$y = 3x^2 + \frac{24}{x} + 2 \Rightarrow y = 3x^2 + 24x^{-1} + 2$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 6x - 24x^{-2}}}.$$

(b) Hence find the exact range of values of x for which the curve is increasing. (2)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} > 0 \Rightarrow 6x - 24x^{-2} > 0$$
$$\Rightarrow x > 4x^{-2}$$
$$\Rightarrow x^3 > 4$$
$$\Rightarrow \underline{\underline{x > \sqrt[3]{4}}}.$$

6. Figure 1 shows a sketch of a triangle ABC with $AB = 3x$ cm, $AC = 2x$ cm, and angle $CAB = 60°$.

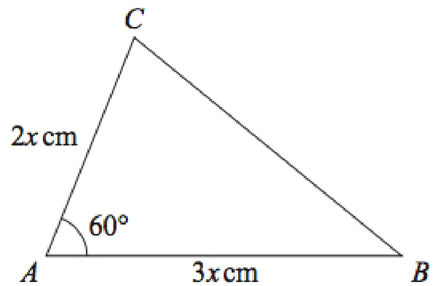


Figure 1: triangle ABC

Given that the area of triangle ABC is $18\sqrt{3}$ cm^2,

(a) show that $x = 2\sqrt{3}$. (3)

Solution

$$\tfrac{1}{2} \times 3x \times 2x \times \sin 60° = 18\sqrt{3} \Rightarrow 6x^2 = 72$$
$$\Rightarrow x^2 = 12$$
$$\Rightarrow x = \sqrt{12} \text{ (because } x \text{ is positive)}$$
$$\Rightarrow \underline{\underline{x = 2\sqrt{3}}}.$$

(b) Hence find the exact length of BC, giving your answer as a simplified surd. (3)

Solution

$$\begin{aligned} BC &= \sqrt{(6\sqrt{3})^2 + (4\sqrt{3})^2 - 2 \cdot (6\sqrt{3}) \cdot (4\sqrt{3}) \cdot \cos 60^\circ} \\ &= \sqrt{84} \\ &= \underline{\underline{2\sqrt{21}}}. \end{aligned}$$

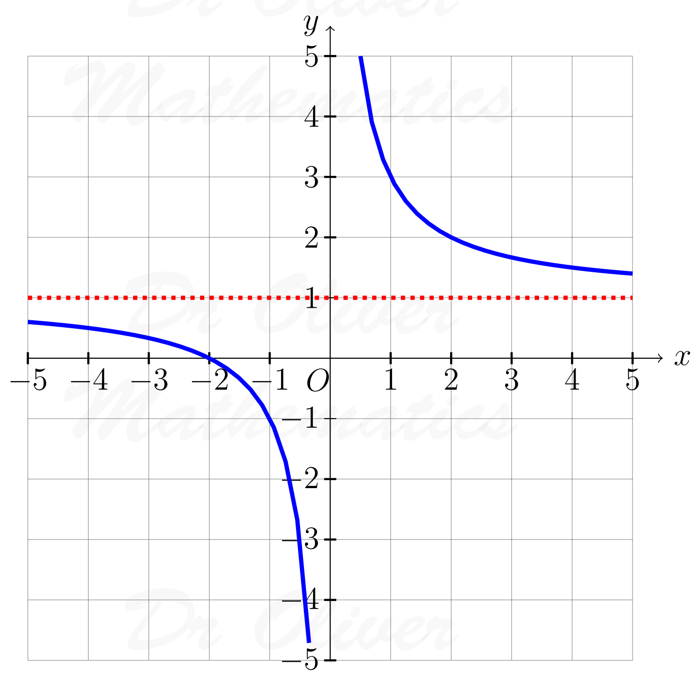
7. The curve C has equation

$$y = \frac{k^2}{x} + 1,\ x \in \mathbb{R},\ x \neq 0,$$

where k is a constant.

(a) Sketch C stating the equation of the horizontal asymptote. (3)

Solution



The horizontal asymptote is $\underline{\underline{y = 1}}$.

The line l has equation $y = -2x + 5$.

(b) Show that the x-coordinate of any point of intersection of l with C is given by a solution of the equation (2)

$$2x^2 - 4x + k^2 = 0.$$

Solution

$$\frac{k^2}{x} + 1 = -2x + 5 \Rightarrow \frac{k^2}{x} = -2x + 4$$
$$\Rightarrow k^2 = -2x^2 + 4x$$
$$\Rightarrow \underline{\underline{2x^2 - 4x + k^2 = 0}},$$

as required.

(c) Hence find the exact values of k for which l is a tangent to C. (3)

Solution

'$b^2 - 4ac = 0$':

$$(-4)^2 - 4 \times 2 \times (k^2) = 0 \Rightarrow 16 = 8k^2$$
$$\Rightarrow k^2 = 2$$
$$\Rightarrow \underline{\underline{k = \pm\sqrt{2}}}.$$

8. (a) Find the first 3 terms, in ascending powers of x, of the binomial expansion of (4)

$$\left(2 + \tfrac{3}{4}x\right)^6,$$

giving each term in its simplest form.

Solution

$$\left(2 + \tfrac{3}{4}x\right)^6 = 2^6 + \binom{6}{1}(2^5)(\tfrac{3}{4}x) + \binom{6}{2}(2^4)(\tfrac{3}{4}x)^2 + \ldots$$
$$= \underline{\underline{64 + 144x + 135x^2 + \ldots}}.$$

(b) Explain how you could use your expansion to estimate the value of 1.925^6. (1)
You do not need to perform the calculation.

Solution

We could work out

$$2 + \tfrac{3}{4}x = 1.925 \Rightarrow \tfrac{3}{4}x = -0.075$$
$$\Rightarrow \underline{\underline{x = 0.1}}$$

and so the value is

$$\underline{\underline{64 + 144(-0.1) + 135(-0.1)^2 + \ldots}}$$

9. A company started mining tin in Riverdale on 1st January 2019.

A model to find the total mass of tin that will be mined by the company in Riverdale is given by the equation

$$T = 1200 - 3(n - 20)^2,$$

where T tonnes is the total mass of tin mined in the n years after the start of mining.

Using this model,

(a) calculate the mass of tin that will be mined up to 1st January 2020, (1)

Solution

$$n = 1 \Rightarrow \underline{\underline{T = 117}}.$$

(b) deduce the maximum total mass of tin that could be mined, (1)

Solution

The total mass of tin that could be mined is $\underline{\underline{1\,200 \text{ tonnes}}}$.

(c) calculate the mass of tin that will be mined in 2023. (2)

Solution

The mass of tin that will be mined in 2023 is

$$T(5) - T(4) = [1200 - 3(5-20)^2] - [1200 - 3(4-20)^2]$$
$$= \underline{\underline{93}}.$$

(d) State, giving reasons, the limitation on the values of n. (2)

Solution
E.g., $n \leqslant 20$, the total amount mined cannot decrease.

10. A circle C has equation

$$x^2 + y^2 - 4x + 8y - 8 = 0.$$

(a) Find (3)

(i) the coordinates of the centre of C,

Solution

$$\begin{aligned} x^2 + y^2 - 4x + 8y - 8 = 0 &\Rightarrow x^2 - 4x + y^2 + 8y = 8 \\ &\Rightarrow (x^2 - 4x + 4) + (y^2 + 8y + 16) = 8 + 4 + 16 \\ &\Rightarrow (x-2)^2 + (y+4)^2 = 28; \end{aligned}$$

hence, the coordinates of the centre of C are $\underline{\underline{(2, -4)}}$.

(ii) the exact radius of C.

Solution
the exact radius of C is $\underline{\underline{2\sqrt{7}}}$.

The straight line with equation $x = k$, where k is a constant, is a tangent to C.

(b) Find the possible values for k. (2)

Solution
The possible values for k are

$$\underline{\underline{2 \pm 2\sqrt{7}}}.$$

11.

$$f(x) = 2x^3 - 13x^2 + 8x + 48.$$

(a) Prove that $(x - 4)$ is a factor of $f(x)$. (2)

Solution

4	2	−13	8	48
	↓	8	−20	−48
	2	−5	−12	0

Now, since the remainder is 0, $\underline{\underline{(x-4)}}$ is a factor of $\mathrm{f}(x)$.

(b) Hence, using algebra, show that the equation $\mathrm{f}(x) = 0$ has only two distinct roots. (4)

Solution

We use synthetic division:

$$\mathrm{f}(x) = (x-4)(2x^2 - 5x - 12)$$

$$\left.\begin{array}{ll}\text{add to:} & -5 \\ \text{multiply to:} & (+2)\times(-12) = -24\end{array}\right\} -8,\ +3$$

$$\begin{aligned} &= (x-4)(2x^2 - 8x + 3x - 12) \\ &= (x-4)[2x(x-4) + 3(x-4)] \\ &= (x-4)(2x+3)(x-4) \\ &= (x-4)^2(2x+3) \end{aligned}$$

and the equation $\mathrm{f}(x) = 0$ has only two distinct roots: $\underline{\underline{x = 4}}$ and $\underline{\underline{x = -1\frac{1}{2}}}$.

Figure 2 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$.

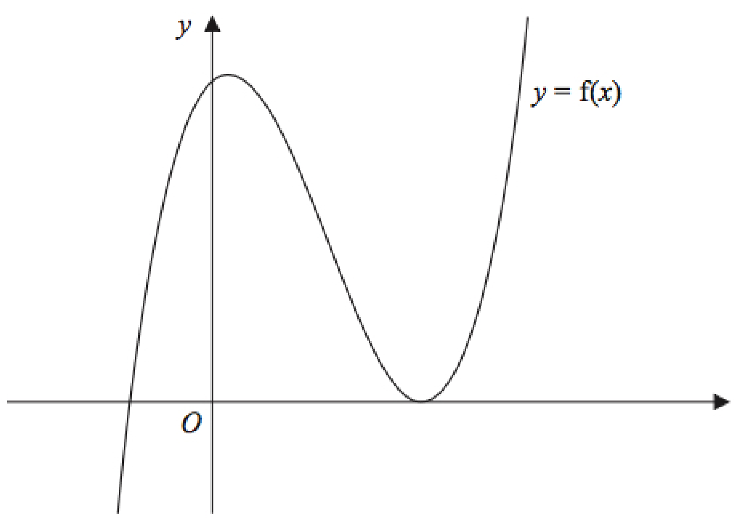


Figure 2: $y = \mathrm{f}(x)$

(c) Deduce, giving reasons for your answer, the number of real roots of the equation (2)

$$2x^3 - 13x^2 + 8x + 46 = 0.$$

Solution

$$\begin{aligned} 2x^3 - 13x^2 + 8x + 46 = 0 &\Rightarrow 2x^3 - 13x^2 + 8x + 48 = 2 \\ &\Rightarrow (x-4)^2(2x+3) = 2; \end{aligned}$$

hence, the equation

$$2x^3 - 13x^2 + 8x + 46 = 0$$

has three real roots.

Given that k is a constant and the curve with equation $y = \mathrm{f}(x + k)$ passes through the origin,

(d) find the two possible values of k. (2)

Solution

$k = 4$ and $k = -1\frac{1}{2}$.

12. (a) Show that (4)

$$\frac{10\sin^2\theta - 7\cos\theta + 2}{3 + 2\cos\theta} \equiv 4 - 5\cos\theta.$$

Solution

$$\begin{aligned} \frac{10\sin^2\theta - 7\cos\theta + 2}{3 + 2\cos\theta} &\equiv \frac{10(1 - \cos^2\theta) - 7\cos\theta + 2}{3 + 2\cos\theta} \\ &\equiv \frac{12 - 7\cos\theta - 10\cos^2\theta}{3 + 2\cos\theta} \end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -7 \\ \text{multiply to:} & (+12)\times(-10) = -120\end{array}\right\} -15,\ +8$$

$$\begin{aligned}&\equiv \frac{12 - 15\cos\theta + 8\cos\theta - 10\cos^2\theta}{3 + 2\cos\theta} \\ &\equiv \frac{3(4 - 5\cos\theta) + 2\cos\theta(4\cos\theta - 5)}{3 + 2\cos\theta} \\ &\equiv \frac{(4 - 5\cos\theta)(3 + 2\cos\theta)}{3 + 2\cos\theta} \\ &\equiv \underline{\underline{4 - 5\cos\theta}},\end{aligned}$$

as required.

(b) Hence, or otherwise, solve, for $0^\circ \leqslant x < 360^\circ$, the equation (3)

$$\frac{10\sin^2 x - 7\cos x + 2}{3 + 2\cos x} = 4 + 3\sin x.$$

Solution

$$\begin{aligned}\frac{10\sin^2 x - 7\cos x + 2}{3 + 2\cos x} = 4 + 3\sin x &\Rightarrow 4 - 5\cos x = 4 + 3\sin x \\ &\Rightarrow 3\sin x = -5\cos x \\ &\Rightarrow \tan x = -\tfrac{5}{3} \\ &\Rightarrow x = 120.963\,756\,5, 300.963\,756\,5 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 121.0, 301.0 \text{ (1 dp)}}}.\end{aligned}$$

13. Figure 3 shows a sketch of part of the curve with equation (7)

$$y = 2x^3 - 17x^2 + 40x.$$

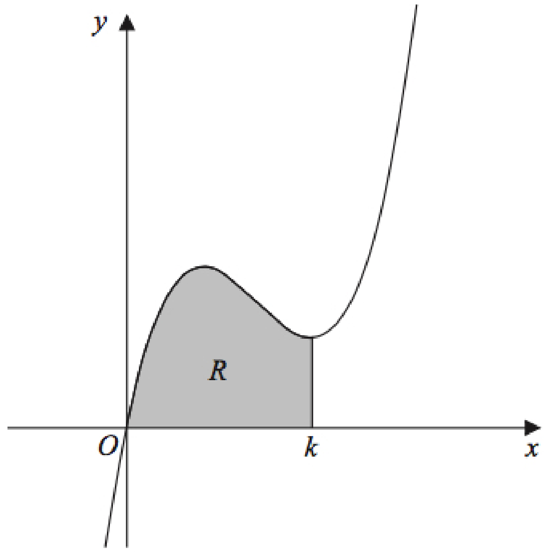


Figure 3: $y = 2x^3 - 17x^2 + 40x$

The curve has a minimum turning point at $x = k$.

The region R, shown shaded in Figure 3, is bounded by the curve, the x-axis, and the line with equation $x = k$.

Show that the area of R is $\dfrac{256}{3}$.

Solution

$$y = 2x^3 - 17x^2 + 40x \Rightarrow \frac{dy}{dx} = 6x^2 - 34x + 40$$

and

$$\frac{dy}{dx} = 0 \Rightarrow 6x^2 - 34x + 40 = 0$$

$$\left.\begin{array}{ll} \text{add to:} & -34 \\ \text{multiply to:} & (+6) \times (+40) = +240 \end{array}\right\} -24, -10$$

$$\begin{aligned} &\Rightarrow 6x^2 - 24x - 10x + 40 = 0 \\ &\Rightarrow 6x(x-4) - 10(x-4) = 0 \\ &\Rightarrow (6x-10)(x-4) = 0 \\ &\Rightarrow 6x - 10 = 0 \text{ or } x - 4 = 0 \\ &\Rightarrow x = 1\tfrac{2}{3} \text{ or } x = 4; \end{aligned}$$

hence, the minimum turning point $(4, 16)$ (why?). Now,

$$\begin{aligned}\int_0^4 (2x^3 - 17x^2 + 40x)\,\mathrm{d}x &= \left[\tfrac{1}{2}x^4 - \tfrac{17}{3}x^3 + 20x^2\right]_{x=0}^4 \\ &= \left(128 - 362\tfrac{2}{3} + 320\right) - (0 - 0 + 0) \\ &= 85\tfrac{1}{3} \\ &= \underline{\underline{\tfrac{256}{3}}},\end{aligned}$$

as required.

14. The value of a car, £V, can be modelled by the equation

$$V = 15\,700\mathrm{e}^{-0.25t} + 2\,300,\ t \in \mathbb{R},\ t \geqslant 0,$$

where the age of the car is t years.

Using the model,

(a) find the initial value of the car. (1)

Solution

$$15\,700 + 2\,300 = \underline{\underline{\text{£}18\,000}}.$$

Given the model predicts that the value of the car is decreasing at a rate of £500 per year at the instant when $t = T$,

(b) (i) show that (6)

$$3\,925\mathrm{e}^{-0.25T} = 500.$$

Solution

$$V = 15\,700\mathrm{e}^{-0.25t} + 2\,300 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t} = -3\,925\mathrm{e}^{-0.25t}$$

and

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} = -500 &\Rightarrow -3\,925\mathrm{e}^{-0.25T} = -500 \\ &\Rightarrow \underline{\underline{3\,925\mathrm{e}^{-0.25T} = 500}},\end{aligned}$$

as required.

(ii) Hence find the age of the car at this instant, giving your answer in years and months to the nearest month.

Solution

$$\begin{aligned} 3\,925\mathrm{e}^{-0.25T} = 500 &\Rightarrow \mathrm{e}^{-0.25T} = \tfrac{20}{157} \\ &\Rightarrow -0.25T = \ln \tfrac{20}{157} \\ &\Rightarrow T = -4 \ln \tfrac{20}{157} \\ &\Rightarrow T = 8.242\,054\,127 \text{ (FCD)} \\ &\Rightarrow T = 8 \text{ years } 2.904\,649\,526 \text{ months (FCD)} \\ &\Rightarrow \underline{\underline{T = 8 \text{ years } 3 \text{ months (nearest month)}}}. \end{aligned}$$

The model predicts that the value of the car approaches, but does not fall below, £A.

(c) State the value of A. (1)

Solution

£2 300.

(d) State a limitation of this model. (1)

Solution

E.g., there are other factors (such as condition or mileage) that affect the price.

15. Given $n \in \mathbb{N}$, prove that (4)

$$n^3 + 2$$

is not divisible by 8.

Solution

If n is odd, then n^3 is odd, and $n^3 + 2$ is odd.
If n is even, then n^3 is a multiple of 8, and $n^3 + 2$ cannot be a multiple of 8.
So, $n \in \mathbb{N}$, $n^3 + 2$ is not divisible by 8.

16. Two non-zero vectors, $\mathbf{a}$ and $\mathbf{b}$, are such that

$$|\mathbf{a} + \mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|.$$

(a) Explain, geometrically, the significance of this statement. (1)

Solution

It means that **a** and **b** lie in the same direction.

Two different vectors, **m** and **n**, are such that

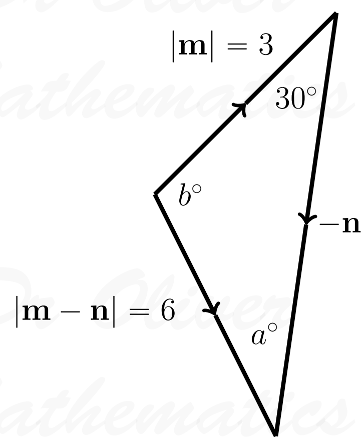
$$|\mathbf{m}| = 3 \text{ and } |\mathbf{m} - \mathbf{n}| = 6.$$

The angle between vector **m** and vector **n** is 30°.

(b) Find the angle between vector **m** and vector $\mathbf{m} - \mathbf{n}$, giving your answer, in degrees, to one decimal place. (4)

Solution

So, let us see what drawing might bring:



Now,

$$\frac{\sin a^\circ}{3} = \frac{\sin 30^\circ}{6} \Rightarrow \sin a^\circ = \tfrac{1}{4}$$
$$\Rightarrow a = 14.477\,512\,19 \text{ (FCD)}$$
$$\Rightarrow b = 135.522\,487\,8 \text{ (FCD)}$$
$$\Rightarrow b = 135.5 \text{ (1 dp)}.$$